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Managing Longevity Risk: Tontines vs. Annuities

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Introduction

Annuity providers face systematic mortality risk:

- Solvency regulations force insurers to set aside capital
- Possible consequences: High annuity/reinsurance premiums, solvency risk when capital requirements are not sufficient, . . .

Measures taken:

- Risk transfer to other parties (e.g. Swaps) or policyholders

Objectives

- Derive optimal payouts for expected-utility-maximizers
- Fairness restrictions
- Analyze risks borne by providers
- Calculation of risk-adequate loadings ($\rightarrow$ Solvency II)
- Multiple perspectives

Tontines: Past and present

- Early suggestion by Tonti (17 th century)
- Collection of money in the UK
- Popular product in the US - now forbidden
- Milevsky, Salisbury (2015): Optimal Retirement Tontines

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Relevant quantities

- Tontine contract:
  - Provider pays a fixed amount to a group of policyholders
  - alive policyholders share the payout
- Annuity contract:
  - Provider pays a fixed amount to each alive individual

Contract payoffs

At time $t > 0$

- an individual tontine-policyholder receives

$$b^{\bullet}(t) := \mathbb{1}_{\{\zeta > t\}} \frac{nd(t)}{N(t)}, \tag{1}$$

- an annuitant receives

$$b^{\circ}(t) := \mathbb{1}_{\{\zeta > t\}} c(t). \tag{2}$$

where ζ is the residual lifetime of the individual and $N(t)$ is the number of policyholders at time t.

Value of a tontine

$$
\begin{aligned}
P^{\bullet}\left(._{}p_x^*, d(\cdot), n\right) &:= \mathbb{E}\left[\int_0^\infty e^{-rt} b^{\bullet}(t)\mathrm{d}t\right] \\
&= \int_0^\infty e^{-rt}{}_tp_x^* \sum_{k=0}^{n-1} \binom{n-1}{k} ({}_tp_x^*)^k (1 - {}_tp_x^*)^{n-1-k} \frac{nd(t)}{k+1}\,\mathrm{d}t \\
&= \int_0^\infty e^{-rt}\left(1 - (1 - {}_tp_x^*)^n\right) d(t)\,\mathrm{d}t\,. \qquad (3)
\end{aligned}
$$

Value of an annuity

$$P^{\circ}\left({}_{\cdot}p_x^{*}, c(\cdot), n\right) := \mathbb{E}\left[\int_0^{\infty} e^{-rt} b^{\circ}(t)\mathrm{d}t\right] = \int_0^{\infty} e^{-rt}\,{}_t p_x^{*} c(t)\,\mathrm{d}t\,. \quad (4)$$

Expected utility - Policyholder perspective

Assume an investor with; Power utility with constant relative risk aversion (CRRA)

$$u(X) = \frac{X^{1-\gamma}}{1-\gamma}$$

Expected utility of a tontine policyholder:

$$U^{\bullet}\left({}_{\cdot}p_x^*, d(\cdot), n\right) := \mathbb{E}\left[\int_0^\infty \mathbb{1}_{\{\zeta > t\}} e^{-rt} u\left(\frac{nd(t)}{N(t)}\right) \mathrm{d}t\right]$$

$$= \int_0^\infty e^{-rt} \sum_{k=0}^{n-1} \binom{n-1}{k} u\left(\frac{nd(t)}{k+1}\right) \left({}_tp_x^*\right)^{k+1} \left(1 - {}_tp_x^*\right)^{n-1-k} \mathrm{d}t\,. \tag{5}$$

Optimal tontine payout

$$d^*(t) := \max_{d(t)} U^{\bullet}\left({}_{\cdot}p_x^*, d(\cdot), n\right), \tag{6}$$

$$s.t. \int_0^{\infty} e^{-rt} d(t) \left(1 - (1 - {}_t p_x^*)^n\right) \mathrm{d}t \leq 1 \,.$$

Solution:

$$d^*(t) = \left(\frac{\sum_{k=0}^{n-1} \binom{n-1}{k} \left(\frac{n}{k+1}\right)^{1-\gamma} ({}_t p_x^*)^{k+1} (1 - {}_t p_x^*)^{n-1-k}}{\lambda^* \left(1 - (1 - {}_t p_x^*)^n\right)} \right)^{\frac{1}{\gamma}}, \tag{7}$$

Optimal annuity payout

$$c^*(t) := \max_{c(t)} U^\circ\left({}_{\cdot}p_x^*, c(\cdot), n\right) = \max_{c(t)} \int_0^\infty e^{-rt}\,{}_tp_x^*\, u\left(c(t)\right) \mathrm{d}t, \quad (8)$$

$$s.t. \quad \int_0^\infty e^{-rt} c(t)\,{}_tp_x^* \,\mathrm{d}t \leq 1\,.$$

Solution:

$$c(t) = \left(\int_0^\infty e^{-rt}\,{}_tp_x^* \,\mathrm{d}t \right)^{-1}. \quad (9)$$

Mortality assumptions

- Gompertz law
- Binomial distribution for number of survivors up to time t
- life tables with mortality shock: ${}_tp_x^{new} = ({}_tp_x)^{1-\epsilon}$, where ϵ is the (random) magnitude of a longevity shock

Risk Margin

Calculation of the risk margin (see e.g. Börger (2010))

- Solvency II: Technical Provisions = Best Estimate Liabilities + Risk Margin
- In numerical illustrations: Fair Premium = Technical Provision
- Risk Margin $= CoC \sum_{t \geq 0} \frac{SCR_t}{(1+r)^t}$
- Simplifications allowed, e.g. $SCR(t) = \frac{BEL_t}{BEL_0} SCR_0$
- CoC = 6%
- $SCR = argmin_x \left\{ P\left(\frac{BEL_1 - CF_1}{1+r} - BEL_0 > x \right) \leq 0.005 \right\}$

Future losses

At time $t > 0$ the losses generated by a longevity shock can be calculated as

$$L_\epsilon^\circ(t, {}_\cdot p_x^*, d^*(\cdot)) := \int_t^\infty e^{-rs}\Big(({}_s p_x^*)^{1-\epsilon} - {}_s p_x^*\Big) c^*(s)\,\mathrm{d}s\,, \tag{10}$$

$$L_\epsilon^\bullet(t, {}_\cdot p_x^*, c^*(\cdot)) := \int_t^\infty e^{-rs}\Big(\big(1-({}_s p_x^*)^{1-\epsilon}\big)^n - \big(1-({}_s p_x^*)\big)^n\Big) d^*(s)\,\mathrm{d}s\,, \tag{11}$$

Switching from tontine to annuity

Fix a switching time t^*, at time t a policyholder receives:

$$\mathbb{1}_{\{0\leq\zeta<t^*\}}\frac{nd(t)}{N(t)}+\mathbb{1}_{\{\zeta\geq t^*\}}c, \tag{12}$$

Fair value:

$$\int_0^{t^*} e^{-rt}(1-(1-{}_tp_x^*)^n)d(t)\,\mathrm{d}t+e^{-rt^*}{}_{t^*}p_x^*\int_{t^*}^{\infty} e^{-r(t-t^*)}{}_{t-t^*}p_{x+t^*}^*c\,\mathrm{d}t=1 \tag{13}$$

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Tontine vs Annuity: Loss distribution

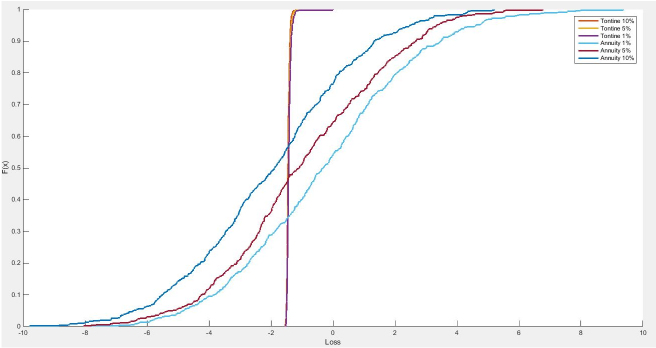

Figure: loss distribution: age 65, r=4%

Tontine vs Annuity:Risk margin

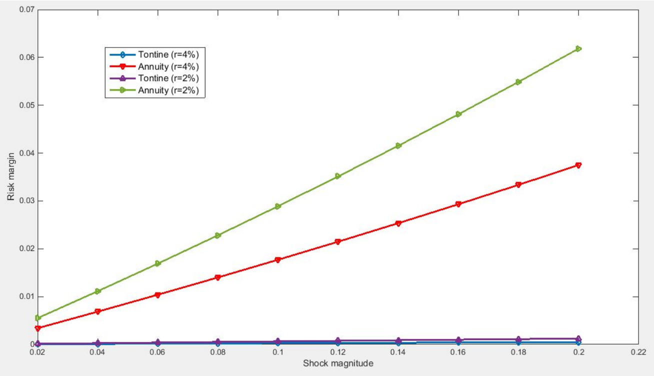

Figure: Risk margins: age 65, different longevity shock magnitudes

Annuity: Risk margin

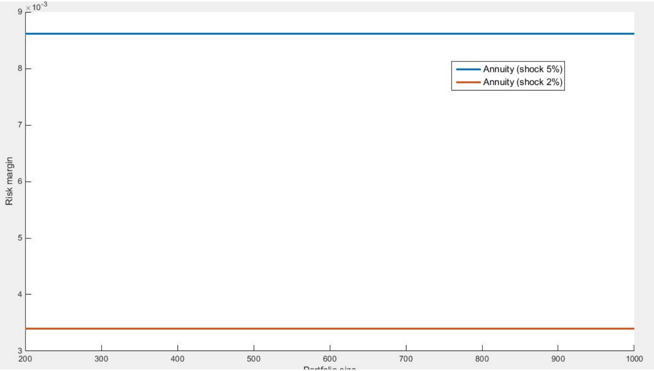

Figure: Risk margin: age 65, various portfolio sizes at inception

Tontine: risk margin

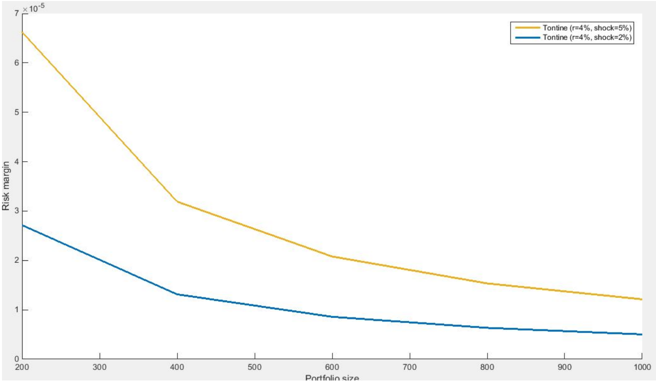

Figure: Risk margin: age 65, various portfolio sizes at inception

Expected utility - risk loading

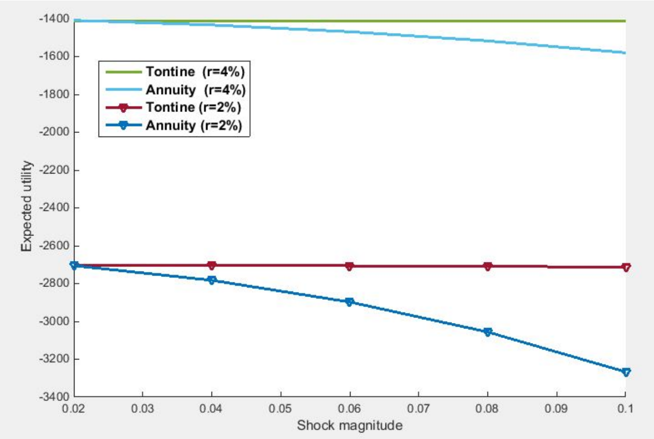

Figure: Expected utility with risk-based loading: varying interest and shock magnitude

Switching times - Solvency Capital Requirement

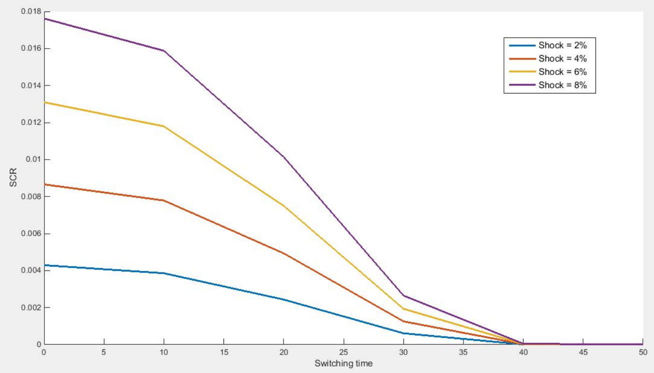

Figure: SCR for deferred payout: age 65

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Conclusions

- Fairness restrictions
- Products with different risk structures: take into account compensation for risk transfer
- New products: multiple perspectives have to be analyzed

Outlook/Paper

- Mortality models
- Detailed proofs
- Sensitivity analyses
- ...

Thank you for your attention!

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Selected references

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