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Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual Effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals

Supplementary Appendix

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Asymptotic Properties of Estimators for the Linear Panel Regression Model with Individual effects and Serially Correlated Errors: The Case of Stationary and Non-Stationary Regressors and Residuals:

Supplementary Appendix

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Abstract

This paper studies the asymptotic properties of standard panel data estimators in a simple panel regression model with error component disturbances. Both the regressor and the remainder disturbance term are assumed to be autoregressive and possibly non-stationary. Asymptotic distributions are derived for the standard panel data estimators including ordinary least squares, fixed effects, first-difference, and generalized least squares (GLS) estimators when both T and n are large. We show that all the estimators have asymptotic normal distributions and have different convergence rates dependent on the non-stationarity of the regressors and the remainder disturbances. We show using Monte Carlo experiments that the loss in efficiency of the OLS, FE and FD estimators relative to true GLS can be substantial.

Key Words: *Panel Data, OLS, Fixed-Effect, First-Difference, GLS.*

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Appendix

A Proof of Theorem 1

The following lemmas are needed to prove Theorem 1. All limits are taken as $T \to \infty$ and followed by $n \to \infty$ sequentially, $(n,T) \overset{\text{seq}}{\to} \infty$.

Lemma 1 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda| < 1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$.
2. If $\lambda = 1$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{2}$.

Proof. Consider (1). For a fixed n, it is clear to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2\right]-\left(\overline{\overline{x}}\right)^2 \\
\xrightarrow{p}\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}\right]-\frac{1}{n}\sum_{i=1}^{n}E\left(x_{it}\right) \\
=\ & \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}
\end{aligned}$$

as $T \to \infty$ because

$$\begin{aligned}
E\left(x_{it}^2\right) &= E\left(\sum_{j=0}^{\infty}\lambda^j\varepsilon_{i(t-j)}\right)^2 \\
&= \sum_{j=0}^{\infty}E\left(\lambda^j\varepsilon_{i(t-j)}\right)^2+\sum_{j=0}^{\infty}\sum_{k=0,k\neq j}^{\infty}E\left(\lambda^{j+k}\varepsilon_{i(t-j)}\varepsilon_{i(t-k)}\right) \\
&= \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}
\end{aligned}$$

and $\overline{\overline{x}} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{it} \xrightarrow{p} \frac{1}{n}\sum_{i=1}^{n}E\left(x_{it}\right) = 0$ as $T \to \infty$. Note $E\left(x_{it}\right) = 0$ for all i and t since there is no non-zero drift in (**??**). Then obviously,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). Similarly for a fixed n

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)^2 \\
= \quad & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}^2\right]-\frac{1}{T}\overline{\overline{x}}^2 \\
= \quad & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_{it}}{\sqrt{T}}\right)^2\right]-\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}}\right)\right]^2 \\
\Rightarrow \quad & \frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int W_i^2\right)-\left[\frac{1}{n}\sum_{i=1}^{n}\varpi_\varepsilon\int W_i\right]^2
\end{aligned}$$

as $T\to\infty$ because $\frac{1}{T}\sum_{t=1}^{T}\left(\frac{x_{it}}{\sqrt{T}}\right)^2 \Rightarrow \varpi_\varepsilon^2\int W_i^2$ and $\frac{1}{T}\sum_{t=1}^{T}\frac{x_{it}}{\sqrt{T}} \Rightarrow \varpi_\varepsilon\int W_i$. Then

$$\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int W_i^2\right)-\left[\frac{1}{n}\sum_{i=1}^{n}\varpi_\varepsilon\int W_i\right]^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{2}.$$

as $n\to\infty$ by a law of large numbers (LLN). This is because

$$E\int W_i^2=\frac{1}{2}$$

and

$$E\int W_i=0.$$

This proves (2). ■

Lemma 2 *If Assumptions* $1-3$ *hold, then*

1. If $|\rho|<1$ and $|\lambda|<1$,
   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$
   (b)

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
\Rightarrow \quad & N\left(0,\frac{\sigma_\mu^2\varpi_\varepsilon^2}{(1-\lambda)^2}+\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{(1-\rho\lambda)^2}\right)
\end{aligned}$$

where $\psi_{0r} = E\left(\varepsilon_{i(t-r)}^2 e_{it}^2\right), \psi_{r0} = E\left(\varepsilon_{it}^2 e_{i(t-r)}^2\right), \psi_{00} = E\left(\varepsilon_{it}^2 e_{it}^2\right).$

2. If $\rho = 1$ and $|\lambda| < 1$,
   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}$
   (b) $\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} - \sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\varpi_{e\varepsilon}\nu_{i(t-1)}e_{it}\right)+\delta_{e\varepsilon}}{1-\lambda}\right] \Rightarrow N\left(0, \frac{\varpi_{\varepsilon.e}\varpi_e^2}{2(1-\lambda)^2}\right).$

3. If $|\rho| < 1$ and $\lambda = 1$,
   (a) $\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} 0$
   (b) $\frac{1}{\sqrt{n}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} \Rightarrow N\left(0, \frac{\sigma_\mu^2\varpi_\varepsilon^2}{3}\right).$

4. If $\rho = 1$ and $\lambda = 1$,
   (a) $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$
   (b) $\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} - \sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2} + \delta_{\varepsilon e}\right] \Rightarrow N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_\varepsilon^2}{6}\right).$

Proof. :

Consider (1). For part (a), we note

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right) u_{it} \\
= & \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)\left(\mu_i + \nu_{it}\right) \\
= & \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) \\
& + \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right] - \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\nu_{it}\right) \\
= & I - II + III - IV.
\end{aligned}$$

Consider I. It is easy to see that for a fixed n,

$$\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]=\frac{1}{n}\sum_{i=1}^{n}Z_i\mu_i$$

as $T\to\infty$ where

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\Rightarrow Z_i\sim N\left(0,\frac{\varpi_\varepsilon^2}{(1-\lambda)^2}\right)$$

by a central limit theorem (CLT) since $E(x_{it})=0$. Then

$$\frac{1}{n}\sum_{i=1}^{n}Z_i\mu_i=o_p(1)$$

as $n\to\infty$ by a LLN and the assumption that μ_i and x_{it} are uncorrected as in (**??**). Hence $I=\frac{1}{\sqrt{T}}o_p(1)$. Consider II. For a fixed n,

$$\begin{aligned}&\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\\ =\;&\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\end{aligned}$$

as $T\to\infty$. Then clearly

$$\left(\frac{1}{n}\sum_{i=1}^{n}Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)\xrightarrow{p}0$$

as $n\to\infty$ because $E(\mu_i)=0$. This proves that

$$II=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right)=o_p(1).$$

Next we consider III. Clearly

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E(x_{it}\nu_{it})+o_p(1)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Also it is easy to see that

$$IV=o_p(1).$$

Collecting $I-IV$ we then prove (a).

For part (b), for a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) \\
& +\left[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right] \\
& -\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right) \\
= & I-II+III-IV.
\end{aligned}$$

For I,

$$I=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]\Rightarrow\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_i\mu_i$$

as $T\to\infty$, and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}Z_i\mu_i\Rightarrow N\left(0,\frac{\sigma_\mu^2\varpi_\varepsilon^2}{\left(1-\lambda\right)^2}\right)$$

as $n\to\infty$ by a CLT. For III,

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right) \\
= & \sqrt{nT}\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}\nu_{it}-\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\right) \\
\Rightarrow & N\left(0,\frac{\psi_{i00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{i0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{ir0}}{\left(1-\rho\lambda\right)^2}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, which is based on Lemma A0 in Choi(1999). It is easy to see that

$$II=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)=o_p\left(1\right)O_p\left(1\right).$$

$$IV=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)=\frac{1}{\sqrt{T}}o_p\left(1\right)O_p\left(1\right)=o_p\left(1\right).$$

Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\\
\Rightarrow\ & N\left(0,\frac{\sigma_{\mu}^{2}\varpi_{\varepsilon}^{2}}{\left(1-\lambda\right)^{2}}+\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^{2}}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$. This proves (b).

Consider (2). For part (a), note

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\\
=\ &\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_{i}+\nu_{it}\right)\\
=\ &\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_{i}\right]-\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_{i}\right)\\
&+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\frac{\nu_{it}}{\sqrt{T}}\right)\\
=\ & I-II+III-IV.
\end{aligned}$$

First we consider III. Note

$$\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\Rightarrow\frac{1}{1-\lambda}\left[\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int V_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int V_{i}dV_{i}\right)+\delta_{e\varepsilon}\right]$$

as $T\rightarrow\infty$. The above is taken from Lemma 1(a) in Kao and Chiang (2000). Then

$$\begin{aligned}
III &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]\\
&= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{1-\lambda}\left[\varpi_{e}\varpi_{\varepsilon.e}^{1/2}\left(\int V_{i}dW_{i}\right)+\varpi_{e\varepsilon}\left(\int V_{i}dV_{i}\right)+\delta_{e\varepsilon}\right]\\
&\xrightarrow{p}\frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}
\end{aligned}$$

as $n\rightarrow\infty$ because $E\left(\int V_{i}dW_{i}\right)=0$ and $E\left(\int V_{i}dV_{i}\right)=-\frac{1}{2}$. It is clear to see that

$$I=\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\mu_{i}\right]=\frac{1}{\sqrt{T}}o_{p}\left(1\right),$$

$$II = \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1)$$

and

$$IV = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} \frac{\nu_{it}}{\sqrt{T}} \right) = o_p(1)$$

because $\frac{1}{T^{3/2}} \sum_{t=1}^{T} \nu_{it} \Rightarrow \int V_i$ and $E\left(\int V_i\right) = 0$. This proves (a).

For part (b),

$$\begin{aligned}
& \frac{1}{\sqrt{n}T} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}} \right) u_{it} \\
= \; & \frac{1}{\sqrt{n}T} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}} \right) \left(\mu_i + \nu_{it} \right) \\
= \; & \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] - \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mu_i \right) \\
& + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \right] - \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} \nu_{it} \right) \\
= \; & I - II + III - IV.
\end{aligned}$$

First consider III,

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \right] - \sqrt{n} \left[\frac{\left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} \varpi_{e\varepsilon} \nu_{i,t-1} e_{it} \right) + \delta_{e\varepsilon}}{1-\lambda} \right] \\
= \; & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{1-\lambda} \left[\varpi_e \varpi_{\varepsilon.e}^{1/2} \left(\int V_i dW_i \right) + \varpi_{e\varepsilon} \left(\int V_i dV_i \right) + \delta_{e\varepsilon} \right] \\
& - \sqrt{n} \left[\frac{\frac{1}{n} \sum_{i=1}^{n} \left(\varpi_{e\varepsilon} \int V_i dV_i \right) + \delta_{e\varepsilon}}{1-\lambda} \right] + o_p(1) \\
= \; & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{1-\lambda} \left[\varpi_e \varpi_{\varepsilon.e}^{1/2} \left(\int V_i dW_i \right) \right] + o_p(1) \\
\Rightarrow \; & N\left(0, \frac{\varpi_{\varepsilon.e} \varpi_e^2}{2\left(1-\lambda\right)^2} \right),
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. It is easy to see that

$$I = \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] = \frac{1}{\sqrt{T}} O_p(1),$$

$$II = \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mu_i \right) = \frac{1}{\sqrt{T}} o_p(1) O_p(1)$$

and

$$IV = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} \nu_{it} \right) = o_p(1) O_p(1).$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}} \right) u_{it} - \sqrt{n} \left[\frac{\left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} \varpi_{e\varepsilon} \nu_{i(t-1)} e_{it} \right) + \delta_{e\varepsilon}}{1 - \lambda} \right] \\ \Rightarrow \quad & N\left(0, \frac{\varpi_{\varepsilon.e} \varpi_e^2}{2(1-\lambda)^2} \right), \end{aligned}$$

as $(n, T) \overset{\text{seq}}{\to} \infty$.

Consider (3). For part (a), it is easy to see that

$$\begin{aligned} & \frac{1}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}} \right) u_{it} \\ = \quad & \frac{1}{nT^{3/2}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{\overline{x}} \right) \left(\mu_i + \nu_{it} \right) \\ = \quad & \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] - \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mu_i \right) \\ & + \frac{1}{\sqrt{T}} \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{1}{T} \sum_{t=1}^{T} x_{it} \nu_{it} \right) \right] - \frac{1}{\sqrt{T}} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \right) \\ = \quad & I - II + III - IV. \end{aligned}$$

For a fixed n,

$$\begin{aligned} I &= \frac{1}{n} \sum_{i=1}^{n} \left[\left(\frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \mu_i \right] \\ &\Rightarrow \frac{1}{n} \sum_{i=1}^{n} \left[\left(\varpi_\varepsilon \int W_i \right) \mu_i \right] \end{aligned}$$

as $T \to \infty$ by a CLT. As $n \to \infty$ by a LLN and the assumption that μ_i and x_{it} are uncorrected as in (**??**), we have $I = o_p(1)$. It is easy to show that

$$II = \left(\frac{1}{n} \sum_{i=1}^{n} \frac{1}{T^{3/2}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{n} \sum_{i=1}^{n} \mu_i \right) = o_p(1) O_p(1),$$

$$III=\frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right)\right]=\frac{1}{\sqrt{T}}o_p\left(1\right)$$

and

$$IV=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)=\frac{1}{\sqrt{T}}o_p\left(1\right)o_p\left(1\right)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$ because $\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\Rightarrow\varpi_{\varepsilon}\int W_i$, $E\left[\varpi_{\varepsilon}\int W_i\right]=0$, $E\left[\mu_i\right]=0$, $\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\Rightarrow\varpi_{\varepsilon}\varpi_{e}\int W_i dV_i$, $E\left[\varpi_{\varepsilon}\varpi_{e}\int W_i dV_i\right]=0$, and $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\Rightarrow N\left(0,\frac{\varpi_e^2}{(1-\rho)^2}\right)$. Hence, we have

$$\frac{1}{nT^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\xrightarrow{p}0$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), note that

$$\begin{aligned}
&\frac{1}{n^{1/2}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}\\
=\;&\frac{1}{n^{1/2}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)\left(\mu_i+\nu_{it}\right)\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]-\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)\\
&+\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\\
=\;&I-II+III-IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
I\;&=\;\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]\\
&\Rightarrow\;\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\varpi_{\varepsilon}\int W_i\right)\mu_i\right]
\end{aligned}$$

as $T\to\infty$, and

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\varpi_{\varepsilon}\int W_i\right)\mu_i\right]\Rightarrow N\left(0,\frac{\sigma_{\mu}^2\varpi_{\varepsilon}^2}{3}\right)$$

as $n \to \infty$ by a CLT with $E\left[\varpi_\varepsilon \int W_i\right] = 0$ and $Var\left[\varpi_\varepsilon \int W_i\right] = \frac{1}{3}\varpi_\varepsilon^2$. It is easy to show that

$$II = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) = o_p\left(1\right)O_p\left(1\right),$$

$$III = \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right] = \frac{1}{\sqrt{T}}O_p\left(1\right),$$

and

$$IV = \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right) = \frac{1}{\sqrt{T}}o_p\left(1\right)O_p\left(1\right)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Hence, we have

$$\frac{1}{n^{1/2}T^{3/2}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)u_{it} \Rightarrow N\left(0, \frac{\sigma_\mu^2\varpi_\varepsilon^2}{3}\right)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

Consider (4). For part (a), it is easy to see that

$$\begin{aligned}
& \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)u_{it} \\
= & \frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)\left(\mu_i + \nu_{it}\right) \\
= & \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) \\
& + \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it}\right] - \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) \\
= & I - II + III - IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
III &= \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it}\right] \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\int W_iV_i + \varpi_{\varepsilon e}\left(\int W_i^2\right) + \delta_{\varepsilon e}\right]
\end{aligned}$$

as $T \to \infty$ by a CLT. We then have

$$\frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_i V_i + \varpi_{\varepsilon e}\left(\int W_i^2\right) + \delta_{\varepsilon e}\right] \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}.$$

as $n \to \infty$. So

$$III \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Again

$$I = \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\mu_i\right] = \frac{1}{\sqrt{T}}o_p\left(1\right)$$

$$II = \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\mu_i\right) = \frac{1}{\sqrt{T}}o_p\left(1\right)o_p\left(1\right)$$

and

$$IV = \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) = o_p\left(1\right)o_p\left(1\right)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)u_{it} \xrightarrow{p} \delta_{\varepsilon e} + \frac{\varpi_{\varepsilon e}}{2}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

For part (b),

$$\begin{aligned}
& \frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)u_{it} \\
= \quad & \frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it} - \overline{\overline{x}}\right)\left(\mu_i + \nu_{it}\right) \\
= \quad & \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\mu_i\right] - \frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right) \\
& + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T} x_{it}\nu_{it} - \left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T} x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right) \\
= \quad & I - II + III - IV.
\end{aligned}$$

For a fixed n,

$$\begin{aligned} III &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it} \\ &\Rightarrow \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i+\varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right] \end{aligned}$$

as $T\to\infty$ by a CLT. As $(n,T)\overset{\text{seq}}{\to}\infty$, we have

$$\begin{aligned} &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}x_{it}\nu_{it}-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^{2}}+\delta_{\varepsilon e}\right] \\ \Rightarrow\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i+\varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon e}\left(\int W_i^2\right)+\delta_{\varepsilon e}\right] \\ =\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\int W_iV_i\right] \\ \Rightarrow\ &N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^{2}}{6}\right). \end{aligned}$$

Also it is easy to see that

$$I=\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\mu_i\right]=\frac{1}{\sqrt{T}}O_p\left(1\right),$$

$$II=\frac{1}{\sqrt{T}}\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mu_i\right)=\frac{1}{\sqrt{T}}o_p\left(1\right)O_p\left(1\right)$$

and

$$IV=\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{T^{3/2}}\sum_{t=1}^{T}\nu_{it}\right)=o_p\left(1\right)O_p\left(1\right)$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Hence, we have

$$\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{\overline{x}}\right)u_{it}-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}x_{i(t-1)}\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^{2}}+\delta_{\varepsilon e}\right]\Rightarrow N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^{2}}{6}\right),$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. ■

B Proof of Theorem 1

Proof. The proof is straightforward by using lemmas 1 and 2. ■

C Proof of Theorem 2

The following lemmas will be used to prove Theorem 2.

Lemma 3 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda|<1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{(1-\lambda)^2(1+\lambda)}$.
2. If $\lambda=1$, $\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$.

Proof. Consider (1). For a fixed n, it is clear to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\\
=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2\right]-\frac{1}{n}\sum_{i=1}^{n}\left[\overline{x}_i^2\right]\\
\xrightarrow{p}\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}\right]\\
=\;&\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}
\end{aligned}$$

as $T\to\infty$ because $\overline{x}_i=\frac{1}{T}\sum_{t=1}^{T}x_{it}\xrightarrow{p}E\left(x_{it}\right)=0$. Hence,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\xrightarrow{p}\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{\left(1-\lambda\right)^2\left(1+\lambda\right)}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). Note for a fixed n

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\\
=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\right]\\
\Rightarrow\;&\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int\tilde{W}_i^2\right)
\end{aligned}$$

as $T\to\infty$. As $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^{n}\left(\varpi_\varepsilon^2\int\tilde{W}_i^2\right)\xrightarrow{p}\frac{\varpi_\varepsilon^2}{6}.$$

by a LLN since

$$E\left(\int \tilde{W}_i^2\right) = \frac{1}{6}.$$

Hence,

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{\varpi_\varepsilon^2}{6}$$

as $(n,T) \overset{\text{seq}}{\rightarrow} \infty$.This proves (2). ■

Lemma 4 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho|<1$ and $|\lambda|<1$,
   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$
   (b) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} - \sqrt{nT}\lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)$
   $\Rightarrow N\left(0, \frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{(1-\rho\lambda)^2}\right)$
   where $\psi_{0r}=E\left(\varepsilon_{t-r}^2 e_t^2\right)$, $\psi_{r0}=E\left(\varepsilon_t^2 e_{t-r}^2\right)$, and $\psi_{00}=E\left(\varepsilon_t^2 e_t^2\right)$.

2. If $\rho=1$ and $|\lambda|<1$,
   (a) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon}+\delta_{e\varepsilon}}{1-\lambda}$
   (b) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(\nu_{it}-\bar{\nu}_i)e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_e^2}+\delta_{e\varepsilon}}{1-\lambda}\right] \Rightarrow$
   $N\left(0, \frac{\varpi_{\varepsilon.e}\varpi_e^2}{6(1-\lambda)^2}\right)$.

3. If $|\rho|<1$ and $\lambda=1$,
   (a) $\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \Rightarrow \frac{1}{1-\rho}\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int \widetilde{W}_i dW_i'\right)+\delta_{\varepsilon e}\right]$,
   (b) $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho}$,
   (c) $\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\overline{x}_i)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\delta_{\varepsilon e}}{1-\rho}\right] \Rightarrow$
   $N\left(0, \frac{\varpi_{e.\varepsilon}\varpi_\varepsilon^2}{6(1-\rho)^2}\right)$.

4. If $\rho=1$ and $\lambda=1$,
   (a) $\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it} \Rightarrow \varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\varpi_{\varepsilon e}\left(\int \widetilde{W}_i^{\,2}\right)+\delta_{\varepsilon e}$,
   (b) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it}\right] \xrightarrow{p} \frac{\varpi_{\varepsilon e}}{6}+\delta_{\varepsilon e}$,

(c) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)u_{it}\right]-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right]\Rightarrow N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{90}\right).$

Proof.

Consider (1). For part (a), note that

$$\begin{aligned}&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]-\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right]\end{aligned}$$

Because

$$\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\nu_{it}\right]=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)+o_p\left(1\right)$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$. Also it is easy to see that

$$\frac{1}{T}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right]=o_p\left(1\right).$$

Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}=\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)+o_p\left(1\right).$$

This proves (a).

For part (b), for a fixed n,

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\right)\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\nu_{it}\right)\right]\end{aligned}$$

Because

$$\begin{aligned}&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_{it}\nu_{it}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}\nu_{it}\right)\\\Rightarrow\;&N\left(0,\frac{\psi_{00}+\sum_{r=1}^{\infty}\lambda^{2r}\psi_{0r}+\sum_{r=1}^{\infty}\rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^2}\right)\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\rightarrow} \infty$, where $\psi_{0r} = E\left(\varepsilon_{t-r}^2 e_t^2\right)$, $\psi_{r0} = E\left(\varepsilon_t^2 e_{t-r}^2\right)$, $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$. Also it is easy to see that

$$\frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \right) \right] = o_p(1)$$

Hence, we have

$$\begin{aligned} & \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{x}_i\right) \nu_{it} - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} E\left(x_{it}\nu_{it}\right) \\ = & \left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it}\nu_{it} \right) - \sqrt{nT} \lim \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} E\left(x_{it}\nu_{it}\right) \right] \\ & - \frac{1}{\sqrt{T}} \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_{it} \right) \left(\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \nu_{it} \right) \right] \\ \Rightarrow & N\left(0, \frac{\psi_{00} + \sum_{r=1}^{\infty} \lambda^{2r}\psi_{0r} + \sum_{r=1}^{\infty} \rho^{2r}\psi_{r0}}{\left(1-\rho\lambda\right)^2}\right). \end{aligned}$$

Consider (2). For part (a), for a fixed n, note that

$$\begin{aligned} & \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{x}_i\right) \nu_{it} \\ = & \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} x_{it} \left(\nu_{it} - \overline{\nu}_i\right) \right] \\ \Rightarrow & \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\varpi_e \varpi_{\varepsilon.e}^{1/2} \left(\int \widetilde{V}_i dW_i\right) + \varpi_{e\varepsilon} \left(\int \widetilde{V}_i dV_i\right) + \delta_{e\varepsilon}}{1-\lambda} \right] \end{aligned}$$

as $T \to \infty$, by a CLT because

$$\begin{aligned} & \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\varpi_e \varpi_{\varepsilon.e}^{1/2} \left(\int \widetilde{V}_i dW_i\right) + \varpi_{e\varepsilon} \left(\int \widetilde{V}_i dV_i\right) + \delta_{e\varepsilon}}{1-\lambda} \right] \\ & \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1-\lambda}, \end{aligned}$$

as $(n,T) \overset{\text{seq}}{\rightarrow} \infty$. Hence, we have

$$\frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - \overline{x}_i\right) \nu_{it} \xrightarrow{p} \frac{-\frac{1}{2}\varpi_{e\varepsilon} + \delta_{e\varepsilon}}{1-\lambda}$$

as $(n,T) \overset{\text{seq}}{\rightarrow} \infty$.

For part (b), for a fixed n,

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}\left(\nu_{it}-\overline{\nu}_i\right)\right]\\
\Rightarrow\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_i dW_i\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_i dV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]
\end{aligned}$$

as $T\to\infty$, by a CLT. Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(\nu_{it}-\bar{\nu}_i\right)e_{it}\right)\frac{\varpi_{e\varepsilon}}{\varpi_e^2}+\delta_{e\varepsilon}}{1-\lambda}\right]\\
\Rightarrow\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_i dW_i\right)+\varpi_{e\varepsilon}\left(\int\widetilde{V}_i dV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{e\varepsilon}\left(\int\widetilde{V}_i dV_i\right)+\delta_{e\varepsilon}}{1-\lambda}\right]\\
=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e\varpi_{\varepsilon.e}^{1/2}\left(\int\widetilde{V}_i dW_i\right)}{1-\lambda}\right]\\
\Rightarrow\;&N\left(0,\frac{\varpi_{\varepsilon.e}\varpi_e^2}{6\left(1-\lambda\right)^2}\right).
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Consider (3). For part (a), note that for a fixed n,

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\\
=\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\right]\\
\Rightarrow\;&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_i dW_i\right)+\delta_{\varepsilon e}}{1-\rho}\right]
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. And

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_i dW_i\right)+\delta_{\varepsilon e}}{1-\rho}\right]\\
&\xrightarrow{p}\frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho},
\end{aligned}$$

as $n \to \infty$. Hence, we have

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}\xrightarrow{p}\frac{-\frac{1}{2}\varpi_{\varepsilon e}+\delta_{\varepsilon e}}{1-\rho}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), for a fixed n,

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}\\
=\quad&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}\right]\\
\Rightarrow\quad&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_{i}dV_{i}\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_{i}dW_{i}\right)+\delta_{\varepsilon e}}{1-\rho}\right],
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\nu_{it}-\sqrt{n}\left[\frac{\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_{i}\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^{2}}+\delta_{\varepsilon e}}{1-\rho}\right]\\
\Rightarrow\quad&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_{i}dV_{i}\right)+\varpi_{\varepsilon e}\left(\int\widetilde{W}_{i}dW_{i}\right)+\delta_{\varepsilon e}}{1-\rho}\right]\\
&-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon e}\left(\int\widetilde{W}_{i}dW_{i}\right)+\delta_{\varepsilon e}}{1-\rho}\right]\\
=\quad&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int\widetilde{W}_{i}dV_{i}\right)}{1-\rho}\right]\\
\Rightarrow\quad&N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^{2}}{6\left(1-\rho\right)^{2}}\right).
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

If $\rho=1$ and $\lambda=1$,

Consider (4). For part (a), note that

$$\begin{aligned}
&\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} \\
=\;& \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\right] \\
\Rightarrow\;& \frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W}_i^{\,2}\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right]
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. And

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W}_i^{\,2}\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right] \\
&\xrightarrow{p} \frac{\varpi_{\varepsilon e}}{6}+\delta_{\varepsilon e}
\end{aligned}$$

as $n\to\infty$. Hence, we have

$$\frac{1}{nT^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\xrightarrow{p}\frac{\varpi_{\varepsilon e}}{6}+\delta_{\varepsilon e}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), for a fixed n,

$$\begin{aligned}
&\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it} \\
=\;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}\right] \\
\Rightarrow\;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W}_i^{\,2}\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right],
\end{aligned}$$

as $T\to\infty$, by a central limit theorem. Hence, we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}T^2}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)\nu_{it}-\sqrt{n}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T^2}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\delta_{\varepsilon e}\right] \\
\Rightarrow\;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)+\left(\int \widetilde{W}_i^{\,2}\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right]-\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\int \widetilde{W}_i^{\,2}\right)\varpi_{\varepsilon e}+\delta_{\varepsilon e}\right] \\
=\;& \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int \widetilde{W}_i\tilde{V}_i\right)\right] \\
\Rightarrow\;& N\left(0,\frac{\varpi_{e.\varepsilon}\varpi_{\varepsilon}^2}{90}\right).
\end{aligned}$$

as $(n, T) \overset{\text{seq}}{\to} \infty$. ■

D Proof of Theorem 2

Proof. Te proof of Theorem 2 is straightforward with above lemmas. ■

E Proof of Theorem 3

The following lemmas will be used to prove Theorem 3.

Lemma 5 *If Assumptions* $1-2$ *hold, then*

1. If $|\lambda|<1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2 \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}$,
2. If $\lambda=1$, $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2 \xrightarrow{p} \sigma_\varepsilon^2$.

Proof. Consider (1). If $|\lambda|<1$, for a fixed n, it is clear to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2\\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)^2\right]\\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}x_{it}^2+\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2-\frac{2}{T}\sum_{t=1}^{T}x_{it}x_{it-1}\right]\\
&\xrightarrow{p} \frac{1}{n}\sum_{i=1}^{n}\left[2\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)-2\left[\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}\right]\right]\\
=\ & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]\\
&\xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}
\end{aligned}$$

as $T\to\infty$ because $\frac{1}{T}\sum_{t=1}^{T}x_{it}^2 \xrightarrow{p} E\left(x_{it}^2\right)=\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}$ and $\frac{1}{T}\sum_{t=1}^{T}x_{it}x_{it-1} \xrightarrow{p} E\left(x_{it}x_{it-1}\right)=\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}$. Hence,

$$\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-\overline{x}_i\right)^2 \xrightarrow{p} \frac{2\sigma_\varepsilon^2}{1+\lambda}+\frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}$$

holds for all n and hence it holds for a large n as well. This proves (1).

Next we consider (2). If $\lambda = 1$,

$$
\begin{aligned}
& \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - x_{it-1} \right)^2 \\
= & \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} \left(x_{it} - x_{it-1} \right)^2 \right] \\
= & \frac{1}{n} \sum_{i=1}^{n} \left[\frac{1}{T} \sum_{t=1}^{T} \varepsilon_{it}^2 \right] \xrightarrow{p} \sigma_{\varepsilon}^2
\end{aligned}
$$

as $(n, T) \overset{\text{seq}}{\to} \infty$.This proves (2). ■.

Lemma 6 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho| < 1$ and $|\lambda| < 1$,
   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right] \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)$
   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{nT}\lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)$
   $\Rightarrow N\left(0, \frac{(2-\lambda-\rho)^2\psi_{00}+\sum_{r=1}^{\infty}\left(-\rho^{r-1}+2\rho^{r}-\rho^{r+1}\right)^2\psi_{0r}+\sum_{r=1}^{\infty}\left(-\lambda^{r-1}+2\lambda^{r}-\lambda^{r+1}\right)^2\psi_{r0}}{(1-\rho\lambda)^2}\right)$
   where $\psi_{0r} = E\left(\varepsilon_{t-r}^2 e_t^2\right)$, $\psi_{r0} = E\left(\varepsilon_t^2 e_{t-r}^2\right)$, $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,
   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right] \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}$
   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it} \Rightarrow$
   $N\left(0, \frac{2\psi_{00}}{1+\lambda}\right)$, where $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$

3. If $|\rho| < 1$ and $\lambda = 1$,
   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right] \xrightarrow{p} \lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)$
   (b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T} \varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \Rightarrow$
   $N\left(0, \frac{2\psi_{00}}{1+\rho}\right)$, where $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$.

4. If $\rho = 1$ and $\lambda = 1$,
   (a) $\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right] \xrightarrow{p} \sigma_{\varepsilon e}$

(b) $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{nT}\sigma_{\varepsilon e}\Rightarrow N\left(0,\varpi_{e}^{2}\varpi_{\varepsilon}^{2}\right).$

Proof. Consider (1). For part (a), it is easy to see that

$$\begin{aligned}&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\&\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), note that

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)\,x_{it-1}\right)\left(e_{it}+(\rho-1)\,\nu_{it-1}\right)\right]\\=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)\,\varepsilon_{it-1}+\lambda(\lambda-1)\,\varepsilon_{it-2}+\lambda^{2}(\lambda-1)\,\varepsilon_{it-3}+\cdots\right)\\&\left(e_{it}+(\rho-1)\,e_{it-1}+\rho(\rho-1)\,e_{it-2}+\rho^{2}(\rho-1)\,e_{it-3}+\cdots\right)]\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\=\;&\frac{1}{\sqrt{n}}\sum_{i=1}^{n}[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)\,\varepsilon_{it-1}+\lambda(\lambda-1)\,\varepsilon_{it-2}+\lambda^{2}(\lambda-1)\,\varepsilon_{it-3}+\cdots\right)\\&\left(e_{it}+(\rho-1)\,e_{it-1}+\rho(\rho-1)\,e_{it-2}+\rho^{2}(\rho-1)\,e_{it-3}+\cdots\right)]\\&-\sqrt{nT}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\\Rightarrow\;&N\left(0,\frac{(2-\lambda-\rho)^{2}\psi_{00}+\sum_{r=1}^{\infty}\left(-\rho^{r-1}+2\rho^{r}-\rho^{r+1}\right)^{2}\psi_{0r}+\sum_{r=1}^{\infty}\left(-\lambda^{r-1}+2\lambda^{r}-\lambda^{r+1}\right)^{2}\psi_{r0}}{(1-\rho\lambda)^{2}}\right)\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, where $\psi_{0r}=E\left(\varepsilon_{t-r}^{2}e_{t}^{2}\right)$, $\psi_{r0}=E\left(\varepsilon_{t}^{2}e_{t-r}^{2}\right)$, $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right).$

Consider (2). For part (a), it is easy to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\
=\ &\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\right]\\
&\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), using Lemma A0 in Choi (1999), we have

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\\
=\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\\
=\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+\left(\lambda-1\right)\varepsilon_{it-1}+\lambda\left(\lambda-1\right)\varepsilon_{it-2}+\lambda^{2}\left(\lambda-1\right)\varepsilon_{it-3}+\cdots\right)e_{it}\right]\right]\\
&-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}E\left(\varepsilon_{it}+\left(\lambda-1\right)x_{it-1}\right)e_{it}\\
\Rightarrow\ &\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)\right]=N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, where $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$.

Consider (3). For part (a), it is easy to see that

$$\begin{aligned}
&\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)\\
=\ &\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\right]\\
&\xrightarrow{p}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b),

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right)-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\left(e_{it}+\left(\rho-1\right)\nu_{it-1}\right)\right]-\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\left[\varepsilon_{it}\left(e_{it}+\left(\rho-1\right)e_{it-1}+\rho\left(\rho-1\right)e_{it-2}+\rho^{2}\left(\rho-1\right)e_{it-3}+\cdots\right)\right]\right] \\
& -\sqrt{n}\lim\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\varepsilon_{it}\left(\nu_{it}-\nu_{it-1}\right) \\
\Rightarrow & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\rho}\right)\right]=N\left(0,\frac{2\psi_{00}}{1+\rho}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$, where $\psi_{00}=E\left(\varepsilon_{t}^{2}e_{t}^{2}\right)$.

Consider (4). For part (a),

$$\begin{aligned}
& \frac{1}{nT}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}e_{it}\right] \\
& \xrightarrow{p}\sigma_{\varepsilon e}
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

For part (b), note that

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}\sum_{i=1}^{n}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)\left(\nu_{it}-\nu_{it-1}\right) \\
= & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it}\right]
\end{aligned}$$

For a fixed n,

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right) \left(\nu_{it} - \nu_{it-1}\right) - \sqrt{nT} \sigma_{\varepsilon e} \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} \varepsilon_{it} e_{it} \right] - \sqrt{nT} \sigma_{\varepsilon e} \\
\Rightarrow & \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i
\end{aligned}$$

as $T \to \infty$, by a central limit theorem, where $Z_i \sim N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$.

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} Z_i \Rightarrow N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$$

as $n \to \infty$. Hence, we have

$$\frac{1}{\sqrt{nT}} \sum_{i=1}^{n} \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right) \left(\nu_{it} - \nu_{it-1}\right) - \sqrt{nT} \sigma_{\varepsilon e} \Rightarrow N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)$$

as $(n, T) \overset{\text{seq}}{\to} \infty$. ■

Proof of Theorem 3:

Proof. By $\widehat{\beta}_{FD} - \beta = \frac{\sum_{i=1}^{n} \sum_{t=1}^{T} (x_{it} - x_{it-1})(\nu_{it} - \nu_{it-1})}{\sum_{i=1}^{n} \sum_{t=1}^{T} (x_{it} - x_{it-1})^2}$, the proof of Theorem 3 is straightforward with above lemmas. ■

F Proof of Theorem 4

Define $\mathbf{z} = [\boldsymbol{\iota}_{nT}, \mathbf{x}]$, then

$$\begin{aligned}
\begin{pmatrix} \hat{\alpha}_{GLS} \\ \widehat{\beta}_{GLS} \end{pmatrix} &= \left(\mathbf{z}'\Phi^{-1}\mathbf{z}\right)^{-1} \left(\mathbf{z}'\Phi^{-1}\mathbf{y}\right) \\
&= \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT} & \mathbf{x} \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} \boldsymbol{\iota}_{nT}' \\ \mathbf{x}' \end{bmatrix} \Phi^{-1}\mathbf{y} \right) \\
&= \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x} \\ \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} & \mathbf{x}'\Phi^{-1}\mathbf{x} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} \\ \mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix} \\
&= \begin{bmatrix} F_{11}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{12}\mathbf{x}'\Phi^{-1}\mathbf{y} \\ F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}\mathbf{x}'\Phi^{-1}\mathbf{y} \end{bmatrix}
\end{aligned}$$

where

$$\begin{aligned}
F_{11} &= \left[\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} - \boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\left(\mathbf{x}'\Phi^{-1}\mathbf{x}\right)^{-1}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right]^{-1}, \\
F_{12} &= -\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}, \\
F_{21} &= -\left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}, \\
F_{22} &= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
\widehat{\beta}_{GLS} &= F_{21}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y} + F_{22}'\mathbf{x}'\Phi^{-1}\mathbf{y} \\
&= \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}\right]^{-1}\left[\mathbf{x}'\Phi^{-1}\mathbf{y} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{y}\right]
\end{aligned}$$

and

$$\widehat{\beta}_{GLS} - \beta = G_1^{-1}G_2,$$

where $G_1 = \mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{x}$ and $G_2 = \mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}$.

With the definition of Φ, we have

$$\mathbf{x}'\Phi^{-1}\mathbf{x} = \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}_i^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}_i^{-1}\mathbf{x}_i\right)$$

$$\begin{aligned}
\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}_i^{-1}\mathbf{x}_i\right) \\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\theta\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) \\
&= \frac{1}{\varpi_e^2 + \theta\sigma_\mu^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right),
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right)\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\theta - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\theta^2\right)\\
&= \frac{1}{\varpi_e^2}n\theta\left(1 - \frac{\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\right)\\
&= \frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2},
\end{aligned}$$

$$\begin{aligned}
\mathbf{x}'\Phi^{-1}\mathbf{u} &= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left(\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'A^{-1}\mathbf{u}_i\right)\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\Bigg[\left(\mu_i\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \Bigg(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-}\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]\\
&= \frac{1}{\varpi_e^2}\sum_{i=1}^{n}\left[\frac{\mu_i\sigma_e^2}{\sigma_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T + \left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right],
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{u}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{u}_i\right)\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\Bigg[\left(\mu_i\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T - \frac{\mu_i\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) + \Bigg(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left[\left(\mu_i\theta - \frac{\mu_i\theta^2\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\right) + \left(\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\theta\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]\\
&= \frac{1}{\sigma_e^2}\sum_{i=1}^{n}\left[\frac{\mu_i\theta\varpi_e^2}{\varpi_e^2+\theta\sigma_\mu^2} + \frac{\theta\varpi_e^2}{\varpi_e^2+\theta\sigma_\mu^2}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right]\\
&= \sum_{i=1}^{n}\left[\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}$$

because $\mathbf{u}_i = \mu_i\boldsymbol{\iota}_T + \boldsymbol{\nu}_i$.

When $|\rho| < 1$, $\mathbf{A} = \frac{1}{1-\rho^2}\begin{bmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{T-1}\\ \rho & 1 & \rho & \cdots & \rho^{T-2}\\ \rho^2 & \rho & 1 & \cdots & \rho^{T-3}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \cdots & 1\end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 1 & -\rho & 0 & 0 & \cdots & 0\\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0\\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & -\rho & 1+\rho\\ 0 & 0 & 0 & \cdots & 0 & -\rho \end{bmatrix}$

It can be shown $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$, where $\mathbf{C} = \begin{bmatrix} \sqrt{1-\rho^2} & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -\rho & 1 & 0 \\ 0 & 0 & 0 & 0 & -\rho & 1 \end{bmatrix}$ is the Prais-Winsten (PW) transformation matrix suggested in Baltagi and Li (1991). Hence

$$\mathbf{C}\mathbf{x}_i = \begin{bmatrix} \sqrt{1-\rho^2}x'_{i1} \\ x'_{i2} - \rho x'_{i1} \\ x'_{i3} - \rho x'_{i2} \\ \vdots \\ x'_{iT-1} - \rho x'_{iT-2} \\ x'_{iT} - \rho x'_{iT-1} \end{bmatrix}, \ \mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \sqrt{1-\rho^2}\nu'_{i1} \\ \nu'_{i2} - \rho \nu'_{i1} \\ \nu'_{i3} - \rho \nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \rho \nu'_{iT-2} \\ \nu'_{iT} - \rho \nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} \sqrt{1-\rho^2}e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix},$$

$\mathbf{C}\boldsymbol{\iota}_T = \begin{bmatrix} \sqrt{1-\rho^2} \\ 1-\rho \\ 1-\rho \\ \vdots \\ 1-\rho \\ 1-\rho \end{bmatrix}$. And

$$\mathbf{x}'_i\mathbf{A}^{-1}\mathbf{x}_i = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\mathbf{x}_i \approx \sum_{t=1}^{T} (x_{it} - \rho x_{it-1})^2$$

$$\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\nu}_i = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\boldsymbol{\nu}_i \approx \sum_{t=1}^{T} (x_{it} - \rho x_{it-1})\, e_{it}$$

$$\mathbf{x}'_i\mathbf{A}^{-1}\boldsymbol{\iota}_T = \mathbf{x}'_i\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T \approx (1-\rho) \sum_{t=1}^{T} (x_{it} - \rho x_{it-1})$$

$$\boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\nu}_i = \boldsymbol{\iota}'_T\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T\boldsymbol{\nu}_i \approx (1-\rho) \sum_{t=1}^{T} e_{it}$$

$$\theta = \boldsymbol{\iota}'_T\mathbf{A}^{-1}\boldsymbol{\iota}_T = \boldsymbol{\iota}'_T\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T \approx \sum_{t=1}^{T} (1-\rho)^2 = O\left((1-\rho)^2 T\right).$$

When $\rho = 1$, $\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 2 & 3 & \cdots & T \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

It can be shown $\mathbf{A}^{-1} = \mathbf{C}\mathbf{C}'$, where $\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$.

Hence

$$\mathbf{C}\mathbf{x}_i = \begin{bmatrix} x'_{i1} \\ x'_{i2} - x'_{i1} \\ x'_{i3} - x'_{i2} \\ \vdots \\ x'_{iT-1} - x'_{iT-2} \\ x'_{iT} - x'_{iT-1} \end{bmatrix}, \mathbf{C}\boldsymbol{\nu}_i = \begin{bmatrix} \nu'_{i1} \\ \nu'_{i2} - \nu'_{i1} \\ \nu'_{i3} - \nu'_{i2} \\ \vdots \\ \nu'_{iT-1} - \nu'_{iT-2} \\ \nu'_{iT} - \nu'_{iT-1} \end{bmatrix} = \begin{bmatrix} e'_{i1} \\ e'_{i2} \\ e'_{i3} \\ \vdots \\ e'_{iT-1} \\ e'_{iT} \end{bmatrix}, \mathbf{C}\boldsymbol{\iota}_T = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}.$$

And

$$\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\mathbf{x}_i \approx \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right)^2$$

$$\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\boldsymbol{\nu}_i \approx \sum_{t=1}^{T} \left(x_{it} - x_{it-1}\right) e_{it}$$

$$\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \mathbf{x}_i'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T = x'_{i1}$$

$$\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \boldsymbol{\iota}_T'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T\boldsymbol{\nu}_i = \nu'_{i1}$$

$$\theta = \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \boldsymbol{\iota}_T'\mathbf{C}'\mathbf{C}\boldsymbol{\iota}_T = 1.$$

The following lemmas will be used to prove Theorem 4.

Lemma 7 *If Assumptions* $1-2$ *hold, then*

1. If $|\rho| < 1$ and $|\lambda| < 1$,

    (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2} + \frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]$,

    (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]$,

  (c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)$.

2. If $\rho = 1$ and $|\lambda| < 1$,
   - (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]$,
   - (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]$,
   - (c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^n\frac{1}{T}\sum_{t=1}^T E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \frac{2\psi_{00}}{1+\lambda}\right)$.

3. If $|\rho| < 1$ and $\lambda = 1$,
   - (a) $\frac{1}{nT^2}G_1 \xrightarrow{p} \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2}$,
   - (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{1}{\varpi_e^2}\left[(1-\rho)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]$,
   - (c) $\frac{1}{\sqrt{nT}}G_2 - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^n(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^T\left(x_{it}-\bar{x}_i\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \Rightarrow$ $N\left(0, \frac{(1-\rho)^2\varpi_\varepsilon^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right)$,

4. If $\rho = 1$ and $\lambda = 1$,
   - (a) $\frac{1}{nT}G_1 \xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}$,
   - (b) $\frac{1}{nT}G_2 \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}$,
   - (c) $\frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \Rightarrow \frac{1}{\varpi_e^2}N\left(0, \varpi_e^2\varpi_\varepsilon^2\right)$.

Proof.

Note that

$$\frac{1}{nT}G_1 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n}$$

First consider

$$\begin{aligned} & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\ = \; & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^n\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T}\right)\end{aligned}$$

For a fixed n,

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)^2 \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}x_{it}^2+\rho^2\frac{1}{T}\sum_{t=1}^{T}x_{it-1}^2-\rho\frac{1}{T}\sum_{t=1}^{T}2x_{it-1}x_{it} \\
& \xrightarrow{p}\left(1+\rho^2\right)\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)-2\rho\left[\lambda\left(\frac{\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\lambda\gamma_\varepsilon^2}{1-\lambda}\right)+\frac{\gamma_\varepsilon^2}{1-\lambda}\right] \\
= \quad & \frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}
\end{aligned}$$

by a CLT and also

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T=(1-\rho)\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)\xrightarrow{p}0$$

as $T\to\infty$. So, with $\theta=O\left((1-\rho)^2T\right)$ when $|\rho|<1$,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i-\frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T}\right) \\
& \xrightarrow{p}\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right] \\
= \quad & \frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

holds for all n and hence it holds for a large n as well.

Also because

$$\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_e^2+\theta\sigma_\mu^2}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right)\xrightarrow{p}0$$

because $\theta=O(T)$ and $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$. And

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}=\frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x}-\frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n}\\ &\xrightarrow{p}\frac{1}{\varpi_e^2}\left[\frac{\left(1-2\rho\lambda+\rho^2\right)\sigma_\varepsilon^2}{1-\lambda^2}+\frac{2\left(\lambda-2\rho\lambda^2+\rho^2\lambda-\rho\right)\gamma_\varepsilon^2}{1-\lambda}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Note that

$$\frac{1}{nT}G_2=\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider

$$\begin{aligned}&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =\;&\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\right].\end{aligned}$$

Because $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$, which has been proved in part (a), and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i=\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)e_{it}\xrightarrow{p}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right],$$

and

$$\frac{1}{T}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i=(1-\rho)\frac{1}{T}\sum_{t=1}^{T}e_{it}\xrightarrow{p}0$$

as $(n,T)\overset{\text{seq}}{\to}\infty$. Hence,

$$\begin{aligned}&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =\;&\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{T}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\right]\\ &\xrightarrow{p}\frac{1}{T}o(1)+\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\\ =\;&\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Also because

$$\frac{1}{nT}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}=\frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\left(\frac{\mu_i}{T}+\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}\right)\xrightarrow{p}0$$

and $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}\\ &\xrightarrow{p}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\rightarrow}\infty$.

Note that

$$\frac{1}{\sqrt{nT}}G_2=\frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}}{\sqrt{nT}}$$

First consider

$$\begin{aligned}&\sqrt{nT}\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}\\ =\;&\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\end{aligned}$$

For a fixed n, $\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\xrightarrow{p}0$, which has been proved in part (a), and

$$\begin{aligned}&\sqrt{T}\left[\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =\;&\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)e_{it}-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =\;&\sqrt{T}\left[\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}+\left(\lambda-\rho\right)x_{it-1}\right)e_{it}-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right]\\ =\;&\sqrt{T}\{\frac{1}{T}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+\left(\lambda-\rho\right)\varepsilon_{it-1}+\lambda\left(\lambda-\rho\right)\varepsilon_{it-2}+\lambda^2\left(\lambda-\rho\right)\varepsilon_{it-3}+\cdots\right)e_{it}\right]\\ &-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\}\\ =\;&N\left(0,\frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)\end{aligned}$$

where $\psi_{00} = E\left(\varepsilon_t^2 e_t^2\right)$ and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = (1-\rho)\frac{1}{\sqrt{T}}\sum_{t=1}^{T} e_{it} \Rightarrow (1-\rho)\varpi_e N(0,1)$$

as $T \to \infty$. Hence,

$$\begin{aligned}
&\sqrt{nT}\left[\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right]\right] \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
& -\frac{1}{\varpi_e^2}\sqrt{nT}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}N\left(0,\frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right) \\
= \quad & \frac{1}{\varpi_e^2}N\left(0,\frac{\left(1-2\rho\lambda+\rho^2\right)\psi_{00}}{1-\lambda^2}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Also consider that

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
&= \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\mu_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}
\end{aligned}$$

It is easy to see that

$$\frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\mu_i = \frac{1}{\sqrt{T}}o_p(1)$$

Also, for a fixed n,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)\right]$$

as $T\to\infty$ because $\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i \Rightarrow (1-\rho)\varpi_e N(0,1)$, which has been proved in part (e). Also,

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)\right] \Rightarrow \frac{(1-\rho)\varpi_e}{\sigma_\mu^2}N(0,1)$$

as $n \to \infty$. Therefore,

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\theta}{\sigma_e^2 + \theta \sigma_\mu^2} \frac{\boldsymbol{\iota}_T' \mathbf{A}^{-1} \boldsymbol{\nu}_i}{\sqrt{T}} \Rightarrow \frac{(1-\rho)\varpi_e}{\sigma_\mu^2} N(0,1)$$

as $(n,T) \overset{\text{seq}}{\to} \infty$. So,

$$\frac{1}{\sqrt{nT}} \boldsymbol{\iota}_{nT}' \Phi^{-1} \mathbf{u} \Rightarrow \frac{(1-\rho)\varpi_e}{\sigma_\mu^2} N(0,1)$$

Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} E\left[(x_{it} - \rho x_{it-1}) e_{it} \right] \\
= \; & \frac{1}{\sqrt{nT}} \mathbf{x}' \Phi^{-1} \mathbf{u} - \sqrt{nT} \frac{1}{\varpi_e^2} \lim \frac{1}{n} \sum_{i=1}^{n} \frac{1}{T} \sum_{t=1}^{T} E\left[(x_{it} - \rho x_{it-1}) e_{it} \right] \\
& - \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}_{NT}' \Phi^{-1} \mathbf{u}}{\sqrt{nT}} \\
\Rightarrow \; & \frac{1}{\sigma_e^2} N\left(0, \frac{\left(1 - 2\rho\lambda + \rho^2\right) \psi_{00}}{1 - \lambda^2} \right).
\end{aligned}$$

Note that

$$\frac{1}{nT} G_1 = \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} - \frac{1}{T} \frac{\mathbf{x}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \boldsymbol{\iota}_{nT}}{n} \right)^{-1} \frac{\boldsymbol{\iota}_{nT}' \Phi^{-1} \mathbf{x}}{n}$$

First consider that

$$\begin{aligned}
& \frac{1}{nT} \mathbf{x}' \Phi^{-1} \mathbf{x} \\
= \; & \frac{1}{\varpi_e^2} \frac{1}{n} \sum_{i=1}^{n} \left(\frac{\mathbf{x}_i' \mathbf{A}^{-1} \mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta \sigma_\mu^2} \frac{\mathbf{x}_i' \mathbf{A}^{-1} \boldsymbol{\iota}_T}{\sqrt{T}} \frac{\boldsymbol{\iota}_T' \mathbf{A}^{-1} \mathbf{x}_i}{\sqrt{T}} \right)
\end{aligned}$$

for a fixed n,

$$\begin{aligned}
& \frac{1}{T} \mathbf{x}_i' \mathbf{A}^{-1} \mathbf{x}_i \\
= \; & \frac{1}{T} \sum_{t=1}^{T} \left(x_{it} - x_{it-1} \right)^2 \\
= \; & \frac{1}{T} \sum_{t=1}^{T} x_{it}^2 + \frac{1}{T} \sum_{t=1}^{T} x_{it-1}^2 - \frac{1}{T} \sum_{t=1}^{T} 2 x_{it-1} x_{it} \\
& \xrightarrow{p} 2 \left(\frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda \gamma_\varepsilon^2}{1 - \lambda} \right) - 2 \left[\lambda \left(\frac{\sigma_\varepsilon^2}{1 - \lambda^2} + \frac{2\lambda \gamma_\varepsilon^2}{1 - \lambda} \right) + \frac{\gamma_\varepsilon^2}{1 - \lambda} \right] \\
= \; & \frac{2\sigma_\varepsilon^2}{1 + \lambda} + \frac{2\left(-2\lambda^2 + 2\lambda - 1 \right) \gamma_\varepsilon^2}{1 - \lambda}
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0,$$

as $T \to \infty$. Hence

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \; & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right) \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right] \\
= \; & \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

holds for all n and hence it holds for a large n as well.

Also because that

$$\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_e^2+\theta\sigma_\mu^2}\left(X_i'A^{-1}\iota_T\right) \xrightarrow{p} 0$$

since $\theta = 1$, and

$$\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n} \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\frac{2\sigma_\varepsilon^2}{1+\lambda} + \frac{2\left(-2\lambda^2+2\lambda-1\right)\gamma_\varepsilon^2}{1-\lambda}\right]
\end{aligned}$$

as $(n,T) \overset{\text{seq}}{\to} \infty$.

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider that

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \; & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \xrightarrow{p} 0$ which is proved in part (a), and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{T}\sum_{t=1}^{T}\left(x_{it} - x_{it-1}\right)e_{it} \xrightarrow{p} \lim \frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it} - x_{it-1}\right)e_{it}\right],$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}\nu_{i1}' \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2 + \theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\lim \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it} - x_{it-1}\right)e_{it}\right]
\end{aligned}$$

as $n \to \infty$.

Also because that

$$\begin{aligned}
& \frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2 + \theta\sigma_\mu^2}\left(\mu_i + \iota_T'A^{-1}\nu_i\right) \\
= \quad & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2 + \sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
& \xrightarrow{p} 0
\end{aligned}$$

and $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 \quad = \quad & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT} \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\lim \frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it} - x_{it-1}\right)e_{it}\right]
\end{aligned}$$

as $(n, T) \overset{\text{seq}}{\to} \infty$.

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{nT}$$

First consider that

$$
\begin{aligned}
& \sqrt{nT}\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}-\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}
$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \xrightarrow{p} 0$ which is proved in part (a), and

$$
\begin{aligned}
& \sqrt{T}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i-\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]\right) \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it}-x_{it-1}\right)e_{it}-\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(\varepsilon_{it}+(\lambda-1)x_{it-1}\right)e_{it}-\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left[\left(\varepsilon_{it}+(\lambda-1)\varepsilon_{it-1}+\lambda(\lambda-1)\varepsilon_{it-2}+\lambda^2(\lambda-1)\varepsilon_{it-3}+\cdots\right)e_{it}\right] \\
& -\sqrt{T}\lim\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
\Rightarrow \quad & N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}
$$

where $\psi_{00}=E\left(\varepsilon_t^2e_t^2\right)$, and

$$
\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i=\frac{1}{\sqrt{T}}x_{i1}\nu_{i1}'\xrightarrow{p}0,
$$

as $T \to \infty$. Hence,

$$\begin{aligned}
& \sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right]\right) \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\right) \\
& +\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
& -\sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
\Rightarrow \quad & o_p(1) + \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[N\left(0,\frac{2\psi_{00}}{1+\lambda}\right) - o_p(1)\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}N\left(0,\frac{2\psi_{00}}{1+\lambda}\right)
\end{aligned}$$

as $(n,T)\overset{\text{seq}}{\to}\infty$.

Also because that

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i + \iota_T'A^{-1}\nu_i\right) \\
\Rightarrow \quad & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
\Rightarrow \quad & \frac{1}{\varpi_e^2+\sigma_\mu^2}N\left(0,1\right)
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}G_2 - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-x_{it-1}\right)e_{it}\right] \\
= \quad & \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \sqrt{nT}\frac{1}{\varpi_e^2}\lim\frac{1}{n}\sum_{i=1}^{n}\frac{1}{T}\sum_{t=1}^{T}E\left[\left(x_{it}-\rho x_{it-1}\right)e_{it}\right] \\
& -\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{nT}} \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}N\left(0,\frac{2\psi_{00}}{1+\lambda}\right).
\end{aligned}$$

Note that

$$\frac{1}{nT^2}G_1 = \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n\sqrt{T}}$$

First consider that

$$\frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} = \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T\sqrt{T}}\right)$$

For a fixed n, because $\theta = O\left((1-\rho)^2 T\right)$ when $|\rho| < 1$ and

$$\begin{aligned}
& \frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i \\
= \; & \frac{1}{T^2}\sum_{t=1}^{T}(x_{it}-\rho x_{it-1})^2 \\
= \; & \frac{1}{T^2}\sum_{t=1}^{T}((1-\rho)\,x_{it-1}+\varepsilon_{it})^2 \\
= \; & (1-\rho)^2\frac{1}{T^2}\sum_{t=1}^{T}x_{it-1}^2 + \frac{1}{T}\left[(1-\rho)\frac{1}{T}\sum_{t=1}^{T}2x_{it-1}\varepsilon_{it} + \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2\right] \\
\Rightarrow \; & (1-\rho)^2\varpi_\varepsilon^2\int W_i^2 + \frac{1}{T}o_p(1),
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \\
= \; & (1-\rho)\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}(x_{it}-\rho x_{it-1}) \\
= \; & (1-\rho)\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}((1-\rho)\,x_{it-1}+\varepsilon_{it}) \\
= \; & (1-\rho)^2\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}x_{it-1} + \frac{1}{\sqrt{T}}\left[(1-\rho)\frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}\right] \\
\Rightarrow \; & (1-\rho)^2\varpi_\varepsilon\int W_i + \frac{1}{\sqrt{T}}o_p(1),
\end{aligned}$$

as $T \to \infty$. Hence

$$\begin{aligned}\frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} &= \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{T\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{T\sqrt{T}}\right) \\ &\xrightarrow{p} \frac{1}{\varpi_e^2}E\left[(1-\rho)^2\varpi_\varepsilon^2\int W_i^2 - (1-\rho)^2\varpi_\varepsilon^2\left(\int W_i\right)^2\right] \\ &= \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2}\end{aligned}$$

as $n \to \infty$.

And because

$$\begin{aligned}\frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{T}{\varpi_e^2+\theta\sigma_\mu^2}\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right) \\ &\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sigma_\mu^2}(1-\rho)^2\varpi_\varepsilon\int W_i\right] \\ &\xrightarrow{p} 0\end{aligned}$$

because $\theta = O(T)$ and $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \Rightarrow (1-\rho)^2\varpi_\varepsilon\int W_i + o_p(1)$, which is proved in (a). And because

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}\frac{1}{nT^2}G_1 &= \frac{1}{nT^2}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n\sqrt{T}} \\ &\xrightarrow{p} \frac{(1-\rho)^2\varpi_\varepsilon^2}{6\varpi_e^2}\end{aligned}$$

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n\sqrt{T}}$$

First consider that

$$\begin{aligned}&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\ =\ &\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right) + \left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T} - \frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\end{aligned}$$

For a fixed n, because $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T \Rightarrow (1-\rho)^2\,\varpi_\varepsilon \int W_i + o_p(1)$, which is proved in (a) and

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)e_{it} \\
= \quad & \frac{1}{T}\sum_{t=1}^{T}\left(\left(1-\rho\right)x_{it-1}+\varepsilon_{it}\right)e_{it} \\
= \quad & \left(1-\rho\right)\frac{1}{T}\sum_{t=1}^{T}\left(x_{it-1}e_{it}\right)+\frac{1}{T}\sum_{t=1}^{T}\left(\varepsilon_{it}e_{it}\right) \\
\Rightarrow \quad & \left(1-\rho\right)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e},
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \left(1-\rho\right)\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it} \Rightarrow \left(1-\rho\right)\varpi_e V_i\left(1\right),$$

as $T \to \infty$. Hence,

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2 T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\{\left(1-\rho\right)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i dV_i\right)+\varpi_{\varepsilon e}\left(\int W_i dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e} \\
& -\frac{1}{\left(1-\rho\right)^2}\left(1-\rho\right)^3\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int W_i\right)V_i\left(1\right)+\varpi_{\varepsilon e}\left(\int W_i\right)W_i\left(1\right)\right]\}+o_p(1) \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\{\left(1-\rho\right)\left[\varpi_\varepsilon\varpi_{e.\varepsilon}^{1/2}\left(\int \tilde{W}_i dV_i\right)+\varpi_{\varepsilon e}\left(\int_i \tilde{W} dW_i\right)+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\} \\
& \xrightarrow{p} \frac{1}{\varpi_e^2}\left[\left(1-\rho\right)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]
\end{aligned}$$

as $n \to \infty$.

And because

$$\begin{aligned}
\frac{1}{n\sqrt{T}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\\
&= \frac{1}{\sqrt{T}}\frac{1}{n}\sum_{i=1}^{n}\left[\left(\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\right)\mu_i\right]+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right)\right]\\
&\Rightarrow o_p\left(1\right)+\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(1-\rho\right)\varpi_e}{\sigma_\mu^2}V_i\left(1\right)\right]\\
&\xrightarrow{p} 0
\end{aligned}$$

and also because $\frac{1}{n\sqrt{T}}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\xrightarrow{p}\frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n\sqrt{T}}\\
&\xrightarrow{p}\frac{1}{\varpi_e^2}\left[\left(1-\rho\right)\left[-\frac{1}{2}\varpi_{\varepsilon e}+\gamma_{\varepsilon e}\right]+\sigma_{\varepsilon e}\right]
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{n}T}G_2-\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(1-\rho\right)\left[\left(\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\bar{x}_i\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right]$$

First consider that

$$\begin{aligned}
&\frac{1}{\sqrt{n}T}\mathbf{x}'\Phi^{-1}\mathbf{u}\\
=\;&\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\left(\frac{\varpi_e^2\mu_iT}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{T\sqrt{T}}\right)+\left(\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{T}-\frac{\sigma_\mu^2T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right)\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{T\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\Rightarrow\left(1-\rho\right)^2\varpi_\varepsilon\int W_i+o_p(1)$, which is proved in (a) and

$$\begin{aligned}
& \frac{1}{T^2}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i \\
= \quad & \frac{1}{T^2}\left[(1-\rho)\sum_{t=1}^{T}\left(x_{it}-\rho x_{it-1}\right)\right]\left[(1-\rho)\sum_{t=1}^{T}e_{it}\right] \\
= \quad & (1-\rho)^2\,\frac{1}{T^2}\left[\sum_{t=1}^{T}\left((1-\rho)\,x_{it-1}+\varepsilon_{it}\right)\right]\left[\sum_{t=1}^{T}e_{it}\right] \\
= \quad & (1-\rho)^3\left[\frac{1}{T\sqrt{T}}\sum_{t=1}^{T}x_{it-1}\right]\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}e_{it}\right]+(1-\rho)^2\,\frac{1}{\sqrt{T}}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}\right]\left[\frac{1}{T}\sum_{t=1}^{T}e_{it}\right] \\
\Rightarrow \quad & (1-\rho)^3\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int W_i\right)V_i\left(1\right)+\varpi_{\varepsilon e}\left(\int W_i\right)W_i\left(1\right)\right]+\frac{1}{\sqrt{T}}o_p(1)
\end{aligned}$$

as $T\to\infty$. Hence,

$$\begin{aligned}
& \frac{1}{\sqrt{n}T}\mathbf{x}'\Phi^{-1}\mathbf{u}-\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(1-\rho\right)\left[\left(\frac{1}{T}\sum_{t=1}^{T}\left(x_{it}-\bar{x}_i\right)\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_{\varepsilon}^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\left(1-\rho\right)\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int_i\tilde{W}dV_i\right)+\varpi_{\varepsilon e}\left(\int_i\tilde{W}dW_i\right)\right]+\sigma_{\varepsilon e}\right] \\
& -\frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(1-\rho\right)\left[\varpi_{\varepsilon e}\left(\int_i\tilde{W}dW_i\right)+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
= \quad & \frac{1-\rho}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\varpi_{\varepsilon}\varpi_{e.\varepsilon}^{1/2}\left(\int_i\tilde{W}dV_i\right)\right] \\
\Rightarrow \quad & N\left(0,\frac{\left(1-\rho\right)^2\varpi_{\varepsilon}^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right)
\end{aligned}$$

as $n\to\infty$.

And because

$$\begin{aligned}
\frac{1}{\sqrt{nT}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \quad &= \quad \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\sqrt{T}}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i+\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
&= \quad \frac{1}{\sqrt{T}}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\mu_i+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}} \\
&\Rightarrow \quad o_p\left(1\right)+\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\left(1-\rho\right)\varpi_e}{\sigma_\mu^2}V_i\left(1\right)\right] \\
&\Rightarrow \quad \frac{\left(1-\rho\right)\varpi_e}{\sigma_\mu^2}N\left(0,1\right)
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}'_{NT}\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}}G_2 - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\,\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
= \; & \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(1-\rho)\left[\left(\frac{1}{T}\sum_{t=1}^{T}(x_{it}-\bar{x}_i)\,\varepsilon_{it}\right)\frac{\varpi_{\varepsilon e}}{\varpi_\varepsilon^2}+\gamma_{\varepsilon e}+\frac{\sigma_{\varepsilon e}}{1-\rho}\right] \\
& -\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n\sqrt{T}}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{u}}{\sqrt{nT}} \\
\Rightarrow \; & N\left(0, \frac{(1-\rho)^2\,\varpi_\varepsilon^2\varpi_{e.\varepsilon}}{6\varpi_e^4}\right).
\end{aligned}$$

Note that

$$\frac{1}{nT}G_1 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}'_{nT}\Phi^{-1}\mathbf{x}}{n}$$

First consider that

$$\begin{aligned}
& \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \; & \frac{1}{\sigma_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right)
\end{aligned}$$

For a fixed n, because

$$\begin{aligned}
& \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i \\
= \; & \frac{1}{T}\sum_{t=1}^{T}(x_{it}-x_{it-1})^2 \\
= \; & \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}^2 \\
\Rightarrow \; & \sigma_\varepsilon^2,
\end{aligned}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} \\
= \quad &\frac{1}{\sigma_e^2}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\mathbf{x}_i - \frac{\sigma_\mu^2}{\sigma_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}}\frac{\boldsymbol{\iota}_T'\mathbf{A}^{-1}\mathbf{x}_i}{\sqrt{T}}\right) \\
&\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}.
\end{aligned}$$

as $n \to \infty$.

Also because

$$\begin{aligned}
\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\right) \\
&\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}x_{i1} \\
&\xrightarrow{p} 0
\end{aligned}$$

because $\theta = 1$. And

$$\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT} = \frac{1}{n}\frac{n\theta}{\varpi_e^2+\theta\sigma_\mu^2} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$$

Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_1 &= \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{x} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{x}}{n} \\
&\xrightarrow{p} \frac{\sigma_\varepsilon^2}{\varpi_e^2}
\end{aligned}$$

Note that

$$\frac{1}{nT}G_2 = \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n}$$

First consider that

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \quad &\frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{1}{\sqrt{T}}\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right]
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0$, which is proved in part (a) and

$$\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{T}\sum_{t=1}^{T}\varepsilon_{it}e_{it} \xrightarrow{p} \sigma_{\varepsilon e}$$

and

$$\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}x_{i1}\nu_{i1}' \xrightarrow{p} 0,$$

as $T \to \infty$. Hence,

$$\begin{aligned}
&\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} \\
= \ & \frac{1}{\varpi_e^2}\frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{\sqrt{T}}\frac{\varpi_e^2\mu_i}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{1}{\sqrt{T}}\frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i}{\sqrt{T}}\right] \\
& \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

as $n \to \infty$.

And because

$$\begin{aligned}
&\frac{1}{n}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= \ & \frac{1}{n}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2+\theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
= \ & \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\varpi_e^2+\sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
& \xrightarrow{p} 0
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
\frac{1}{nT}G_2 \ = \ & \frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{T}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{n} \\
& \xrightarrow{p} \frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

Note that

$$\frac{1}{\sqrt{nT}}G_2 = \frac{1}{\sqrt{nT}}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{1}{\sqrt{T}}\frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1}\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{n}}$$

First consider that

$$\begin{aligned}
&\sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right) \\
= \ & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e^2\mu_i T}{\varpi_e^2+\theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2+\theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right] - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2}
\end{aligned}$$

For a fixed n, because $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T = \frac{1}{\sqrt{T}}x_{i1} \xrightarrow{p} 0$, which is proved in part (a) and

$$\begin{aligned}
& \sqrt{T}\left(\frac{1}{T}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \sigma_{\varepsilon e}\right) \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\left(x_{it} - x_{it-1}\right)e_{it} + o(1) - \sqrt{T}\sigma_{\varepsilon e} \\
= \quad & \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\varepsilon_{it}e_{it} + o(1) - \sqrt{T}\sigma_{\varepsilon e} \\
\Rightarrow \quad & N\left(0, \varpi_e^2\varpi_\varepsilon^2\right),
\end{aligned}$$

and $\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i = \frac{1}{\sqrt{T}}x_{i1}\nu_{i1}' \xrightarrow{p} 0$, which is proved in part (a), as $T \to \infty$. Hence,

$$\begin{aligned}
& \sqrt{nT}\left(\frac{1}{nT}\mathbf{x}'\Phi^{-1}\mathbf{u} - \frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right) \\
= \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[\frac{\varpi_e^2\mu_i T}{\varpi_e^2 + \theta\sigma_\mu^2}\frac{\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T}{\sqrt{T}} + \frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\nu}_i - \frac{\sigma_\mu^2}{\varpi_e^2 + \theta\sigma_\mu^2}\frac{1}{\sqrt{T}}\mathbf{x}_i'\mathbf{A}^{-1}\boldsymbol{\iota}_T\boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right] - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left[o_p(1) + N\left(0, \varpi_e^2\varpi_\varepsilon^2\right) - o_p(1)\right] \\
\Rightarrow \quad & \frac{1}{\varpi_e^2}N\left(0, \varpi_e^2\varpi_\varepsilon^2\right)
\end{aligned}$$

as $n \to \infty$.

Also because

$$\begin{aligned}
& \frac{1}{\sqrt{n}}\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u} \\
= \quad & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\theta}{\varpi_e^2 + \theta\sigma_\mu^2}\left(\mu_i + \boldsymbol{\iota}_T'\mathbf{A}^{-1}\boldsymbol{\nu}_i\right) \\
\Rightarrow \quad & \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{1}{\varpi_e^2 + \sigma_\mu^2}\left(\mu_i + \nu_{i1}'\right) \\
\Rightarrow \quad & \frac{\varpi_e + \sigma_\mu}{\sigma_\mu^2}N\left(0, 1\right)
\end{aligned}$$

and also because $\frac{1}{n}\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} 0$ and $\frac{1}{n}\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT} \xrightarrow{p} \frac{1}{\sigma_\mu^2}$, which are proved in part (a). Hence, we have

$$\begin{aligned}
& \frac{1}{\sqrt{nT}} G_2 - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2} \\
= & \left[\frac{1}{\sqrt{nT}} \mathbf{x}'\Phi^{-1}\mathbf{u} - \sqrt{nT}\frac{\sigma_{\varepsilon e}}{\varpi_e^2}\right] \\
& - \frac{1}{\sqrt{T}} \frac{\mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n} \left(\frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\boldsymbol{\iota}_{nT}}{n}\right)^{-1} \frac{\boldsymbol{\iota}_{nT}'\Phi^{-1}\mathbf{u}}{\sqrt{n}} \\
\Rightarrow & \frac{1}{\varpi_e^2} N\left(0, \varpi_e^2 \varpi_\varepsilon^2\right)
\end{aligned}$$

■

Proof of Theorem 4:

Proof. By $\widehat{\beta}_{GLS} - \beta = G_1^{-1} G_2 = \left[\mathbf{x}'\Phi^{-1}\mathbf{x} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{x}\right]^{-1}$ $\left[\mathbf{x}'\Phi^{-1}\mathbf{u} - \mathbf{x}'\Phi^{-1}\boldsymbol{\iota}_{nT}\left(\boldsymbol{\iota}_{NT}'\Phi^{-1}\boldsymbol{\iota}_{nT}\right)^{-1}\boldsymbol{\iota}_{NT}'\Phi^{-1}\mathbf{u}\right]$, the proof of Theorem 4 is straightforward with above lemmas. ■