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Testing for Instability in Factor Structure of Yield Curves

ABSTRACT

A widely relied upon but a formally untested consideration is the issue of stability in factors underlying the term structure of interest rates. In testing for stability, practitioners as well as academics have employed ad-hoc techniques such as splitting the sample into a few sub-periods and determining whether the factor loadings have appeared to be similar over all sub-periods. Various authors have found mixed evidence on stability in the factors. In this paper we develop formal tests in order to evaluate the factor structure stability of the US zero coupon yield term structure. We find the factor structure of level to be unstable over the sample period considered. The slope and curvature factor structures are however found to be stable. We corroborate the literature that variances (volatility) explained by the level, slope, and curvature factors are unstable over time. We find evidence of the presence of common economic shocks affecting the level and slope factors, unlike slope and curvature factors that responded differently to economic shocks and were unaffected by any common instabilities.

JEL classification: C12; C13; C14; C51

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1 Introduction

Modelling the dynamics of interest rates is vital in trading fixed income securities that are sensitive to movements in interest rates. The main interest for both practitioners and academics is to fit the interest rates data within a framework that is able to capture the future evolution of the term structure of interest rates. Understanding the process governing the interest rate movements is crucial to analyse and alter the risk exposures at a given point of time. Statistical models using factor decomposition techniques such as principal component analysis (PCA), where the yield curve dynamics can be summarized by a few estimated principal factors have been highly favored in extracting the yield curve factors. Litterman and Scheinkman (1991) show that the three principal factors, explaining around 99 percent of the changes in treasury bond yields, could be interpreted to be the level (or parallel movement component), slope (or slope oscillation component), and curvature component. The level factor or the parallel movement component alone was the most important factor that accounted for an average of 89 percent of the variations observed in the yield changes data.

Bliss and Smith (1997) argued that model selection and stability of the parameters underlying the process are closely related. By critically examining the main findings in Chan et al. (1992), the paper showed that the unaccounted structural break, due to the Fed change in the monetary policy, biased the main findings of the paper. Structural changes in term structures have been modelled in a regime switching framework. Following the seminal work of Hamilton (1989) on modelling short rates using a regime switching process, Lewis (1991), Evans and Lewis (1995), Garcia and Perron (1996), Gray (1996), Ang and Bekaert (2002), and Audrino (2006) studied regime switches in interest rate models. Empirical evidence suggest that not only the short rates but also the whole term structure of interest rates might experience shifts in regimes due to business cycle expansions and contractions, changes in monetary policies and regime changes in economic variables such as consumption and inflation. Bansal and Zhou (2002) showed that term structure models incorporating regime shifts provide considerable improvements over multifactor models.

The presence of instabilities in the short and long term yields can also seep into the estimated factor structures governing these yields. Though there is a widespread use of factor analysis for term structure of interest rates, very little attention has been given to evaluate the factor structure stability of interest rates. Usually authors have assumed that the principal factors driving the evolution of interest rates have a stable relationship through time. This inherently would mean that the latent factors of the yield curves, generally extracted using PCA rotations, are robust to structural changes observed in yields themselves. Some authors use split-sample analysis in order to investigate factor structure stability. For instance, Bliss (1997) divided the sample period January 1970 - December 1995 into three sub-periods of arbitrary lengths and investigated the change in the factor loadings. Since the factor loadings patterns in the different sub-periods seemed similar in the case of all three factors, the factors were concluded to be stable. However, the factor volatilities were found to fluctuate over the sub-periods considered. On the basis of evidence on time-varying nature of volatility associated with factors, Perignon and Villa (2006) account for a time-varying covariance matrix when estimating the factor structure of interest rates. Using the U.S. term structure data between January 1960 and December 1999, Perignon and Villa observed that the factor structure (factor loadings) remained quite stable across sub-periods considered but the volatility (eigenvalues) of the factors varied through time. Reisman and Zohar (2004) used the yield to maturity data of US discount bonds from 1982. They found that the first two principal components were quite stable; the third component was marginally stable; and the fourth component was unstable.

The stability analyses on factors were carried out by graphically plotting the factor loadings and by weighing the similarity in results over time. The standard procedure implemented in this regard was to divide the data into sub-periods and to identify the factor loading for the corresponding periods. If the explanatory power of the factor loadings appeared to be similar over all periods, then the factors were concluded to be stable over time. The first and the only formal test (to the best of our knowledge) in evaluating stability of level, slope, and curvature factors governing interest rates was introduced in Audrino et al. (2005) who considered a three-factor model with conditional heteroskedastic factors. The paper found contradicting conclusions that the factor loadings of US discount bond yields were in fact unstable over the period January 1986 to May 1995. The paper used independent filtered innovations in order to find the principal factors for the different sub-periods considered and then using a regression framework on the filtered innovations, tested the hypothesis that the regression coefficients (factor loadings) in the different sub-periods are indeed equal. Since the authors constructed factors on the filtered innovations, the instability detected could not be interpreted as instability of the level, slope, or curvature factors.

The main contribution of this paper is to introduce a testing framework that would enable us to formally investigate the instability present in the factor structure of level, slope, and curvature of the yield curves. More precisely, this paper develops a formal stability test on all the eigenspace variables associated with the level, slope, and curvature factors of the US zero coupon yield term structure. In particular, we consider instabilities in factors that are associated with instabilities in eigenvalues, instabilities in eigenvectors, and instabilities in the factor loadings governing the system. These eigenspace variables are estimated using PCA. We formalize a series of hypotheses in order to test for instabilities present in the eigenspace variables of the factors. To anticipate some results, we find instability in the factor structure of level but stability in the factor structures of slope and curvature. We find the eigenvalues (volatility) of level, slope, and curvature factors are unstable over the sample period considered. The instability in the volatility of level is due to structural changes common to all interest rate maturities. We find that the volatility of slope factor is sensitive to shocks affecting the short rates and the volatility of curvature factor is sensitive to shocks affecting the medium and long rates. Investigating for common structural changes in factors, we find evidence of the presence of common economic shocks affecting the level and slope factors. The slope and curvature factors are however unaffected by any common instabilities.

The remainder of this paper is structured as follows. Section 2 presents the principal component factor analysis framework for term structure of yield curves. We provide the asymptotic properties of the estimated eigenspace variables for the three factors, which is applied into developing the stability testing procedure in the subsequent section. In Section 3 we formulate six hypotheses for statistically evaluating the stability in the eigenspace variables governing the level, slope, and curvature factors of the yield curves and device the test statistics for evaluating each hypothesis. Section 4 describes the dataset used, graphical analysis of the evolution of eigenspace variables (eigenvalues, eigenvectors, and factor loadings), and presents the results of the testing procedure developed in Section 3. Section 5 tests for stability in factor structures of alternative commonly used yield curve datasets (Federal Reserve and Fama-Bliss term structures). Section 6 concludes.

2 Framework and Estimation of the Eigenspace

In this section, we present the estimation framework and the inferential theory developed for the eigenspace variables (i.e. eigenvalues, eigenvectors, and factor loadings) estimated via PCA. The limiting distributions of the eigenspace variables developed in this section allows to construct the asymptotic test statistics for evaluating the presence of instability in the eigensystem.

Consider the stationary representation for term structure panel with finite cross-sectional (N) and time-series (T) dimension and with (r) factors:

$$Y_t = \boldsymbol{\gamma}' F_t + \varepsilon_t \qquad t = 1, 2, ..., T \tag{1}$$

Let $Y_t = (Y_{1t}, ..., Y_{Nt})'$ be an $N \times 1$ vector of cross-sectional observations from the panel data structure at time period t, $\boldsymbol{\gamma}$ is an $r \times N$ matrix of the factor loadings, F_t is the $r \times 1$ vector of latent common factors for all cross-sectional units at time period t, and $\varepsilon_t = (\varepsilon_{1t}, ..., \varepsilon_{Nt})'$ is the $N \times 1$ vector of idiosyncratic *i.i.d.* disturbances, allowed to have limited correlation among units. Estimation of such panel factor models enables us to capture the main sources of variations and covariations among the N independent random variables in a panel framework. This framework falls under 'stationary approximate factor models', particularly useful in financial applications.

Since the latent factors (F_t) are unknown and estimated usually via PCA, the errors in estimation of the factor loadings ($\boldsymbol{\gamma}$) are dependent on the errors in estimation of the factors. In this case, it is well known that estimation methodologies such as least squares provide inconsistent estimates of the factor loadings. The inconsistency arises due to the fact that there is limited information available to estimate the factors when the number of cross sectional units is fixed and not large.

We estimate the factor loadings via PCA eigen decomposition technique, where we consider the optimization problem

$$\begin{aligned} & \underset{\beta_i}{max}\ \beta_i' \Sigma \beta_i \qquad (2) \\ \text{subject to } \beta_i' \beta_i &= 1 \text{ and } \beta_i' Y' \perp \beta_j' Y' \quad \text{ for } i, j = 1, ..., N \text{ and } i < j \end{aligned}$$

where Σ the covariance matrix of a stationary $N \times T$ panel Y. The estimated matrix $\boldsymbol{\beta} = (\beta_1, \beta_2, ., \beta_i, .., \beta_N)$ is such that each vector $\beta_i' Y'$ $(i = 1, ..., N)$ is the directional vector that captures the maximum variability in Y and are orthogonal to each other.

Since the factor loadings matrix ($\boldsymbol{\gamma}$) by definition are a function of eigenvalues and eigenvectors, $\boldsymbol{\gamma}$ can be computed as the unit length eigenvectors matrix ($\boldsymbol{\beta}$) multiplied by its singular value, which is the square-root of eigenvalues. Thus $\boldsymbol{\gamma}$ characterizes the unit length eigenvectors in its true size and encompasses in them the information of direction as well as magnitude.

When Σ is unknown, we estimate the sample variance covariance matrix whose elements at position i, j is given as

$$\left[\hat{\Sigma}\right]_{i,j} = \frac{1}{T-1}\sum_{t=1}^{T}\left(y_{it} - \mu_{y_i}\right)\left(y_{jt} - \mu_{y_j}\right) \qquad i, j = 1, ..., N \tag{3}$$

where $(y_{i1}, ..., y_{iT})$ for $i = 1, ..., N$ are each independent and identically distributed. The PCA framework is summarized in the Appendix A. In Appendix B, we report the limiting distribution of the eigenvalues and eigenvectors estimated from a covariance matrix, which is Wishart distributed.

The following theorem provides the rate of convergence and the limiting distribution of the factor loadings for the case of interest rate panels with large T.

Theorem. *(Limiting distribution of factor loadings) Consider* $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_N$ *and* $\beta_1, \beta_2, ..., \beta_N$ *as the first* N *ordered eigenvalues and their corresponding eigenvectors of* Σ *respectively. Define* $\gamma_i = \beta_i \lambda_i^{1/2}$ *as the* i^{th} *factor loading vector where* $\gamma_i = (\gamma_{i1}, \gamma_{i2}, .., \gamma_{iN})'$ *and* $\boldsymbol{\gamma} = (\gamma_1, ..., \gamma_N)$ *. Since* $\hat{\lambda}_i - \lambda_i$ *is independent of* $\hat{\beta}_i - \beta_i$ *it holds that*

$$(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}) = O_p(T^{-1/2}) \tag{4}$$

$$\sqrt{T}\left(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}\right) \xrightarrow{d} N(0, \Psi) \tag{5}$$

where $\Psi = \sum_{i=1}^{N}\sum_{j=1}^{N}\left(U_{ij} \otimes \Psi_{ij}\right)$ *where* U_{ij} *is an* $N \times N$ *matrix that has 1 in the* ij^{th} *position and 0's elsewhere. The asymptotic covariance matrix*

$$\Psi_{ij} = \begin{cases} \lambda_i \Theta_{ij} + \frac{1}{2}\lambda_i \beta_j \beta_i^{'} & \textit{for } i = j \\ \left(\lambda_i \lambda_j\right)^{1/2} \Theta_{ij} & \textit{for } i \neq j \end{cases}$$

Proof: *see Appendix C*

Let $\hat{\Psi}$ be the estimated covariance matrix of the factor loadings. According to the continuous mapping theorem, as $T \longrightarrow \infty$, $\hat{\Psi} \xrightarrow{p} \Psi$. $\hat{\Psi}$ is consistent since it is a continuous function of the estimated eigenvalues and eigenvectors that are consistent. Since the factor loadings matrix, $\hat{\boldsymbol{\gamma}}$ is estimated in the classical PCA framework, we find consistent estimates for the factor loadings. This is because the factors loadings, defined in terms of the eigenvalues and eigenvectors obtained via eigen decomposition, are consistent for panels with large T.

3 Testing for Instability in the Eigensystem

In this section, we formulate a series of hypotheses that will enable us to evaluate stability among the eigenspace variables of the yield curves. Since we are primarily concerned with the level, slope, and curvature factors governing the yield curves, we investigate stability in the eigenspace variables of the first three principle factors.

We examine instability by testing the null hypothesis of no change point against the alternative of at least one change point happening at the unknown time, τ. We define τ as a fraction of the sample space T such that $\tau = [T\epsilon]$ where $\epsilon = (0, 1)$. We define the eigenvalues (Λ), eigenvectors (β_i), and the factor loadings (γ_i) for the sample split around the unknown time point τ as

$$\Lambda = \begin{cases} \Lambda^a & for\ t = 1, ..., \tau \\ \Lambda^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

$$\beta_i = \begin{cases} \beta_i^a & for\ t = 1, ..., \tau \\ \beta_i^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

$$\gamma_i = \begin{cases} \gamma_i^a & for\ t = 1, ..., \tau \\ \gamma_i^b & for\ t = \tau + 1, ..., T \end{cases} \quad for\ some\ \tau$$

We test the following hypotheses in order to gather inference on the instability in the underlying eigensystem of the yield curves:

I. $H_0 : \Lambda^a = \Lambda^b$

$H_1 : \Lambda^a \neq \Lambda^b$

II. $H_0 : \lambda_i^a = \lambda_i^b$

$H_1 : \lambda_i^a \neq \lambda_i^b$ for $i = 1, 2, 3$

III. $H_0 : \beta_i^a = \beta_i^b$

$H_1 : \beta_i^a \neq \beta_i^b$ for $i = 1, 2, 3$

IV. $H_0 : \beta_{ip}^a = \beta_{ip}^b$

$H_1 : \beta_{ip}^a \neq \beta_{ip}^b$ for $i = 1, 2, 3$, and $p = 1, 2, ..., N$

V. $H_0 : \gamma_i^a = \gamma_i^b$

$H_1 : \gamma_i^a \neq \gamma_i^b$ for $i = 1, 2, 3$

VI. $H_0 : \gamma_i^a = \gamma_i^b$ and $\gamma_j^a = \gamma_j^b$ for $i, j = 1, 2, 3$ and $i \neq j$

$H_1 : \gamma_i^a \neq \gamma_i^b$ or $\gamma_j^a \neq \gamma_j^b$

The main aim of testing the series of hypotheses formulated above is to study the economic shocks causing structural changes and their impact on the eigensystem of the yield curves. Since the risks associated with the yield curves can be sufficiently summarized in the first three factors, we investigate the impact of these economic shocks to structural changes in level risks, slope risks, and curvature risks.

Hypothesis I tests for stability in the overall eigensystem of the yield curves. The results from this test would indicate whether there is a persistent statistically significant structural change in the volatility governing the factor structure of the yield curves. Further, we investigate which factors (if any) might have caused the instability and which eigenspace variables might have incurred significant changes. Hypotheses II, III, and V test for instabilities present in the magnitude, direction, and loading respectively of the three factors. The results from these tests would indicate whether the instabilities have been induced by level breaks, slope breaks, or rather curvature breaks. Hypothesis IV relates to testing for instability in each factor, and understanding which interest rates maturities have experienced structural changes, causing the instability in the factor. Hypothesis VI tests for common structural changes in factors. Since the level, slope, and curvature factors are correlated, the test captures change points in one factor that might ripple into the other factors causing common change points in all factors.

In what follows, we develop the stability test statistics for evaluating the six hypothees formulated above. Define $\Pi = (\Lambda, \beta, \boldsymbol{\gamma})$ as the parameter space. Let $\hat{\Pi}^a$ and $\hat{\Pi}^b$ be the consistent estimators of Π^a and Π^b. The limiting distribution of Π for the restricted sample space before the break and after the break, given the change point τ, is

$$\sqrt{T}\left(\hat{\Pi}^a - \Pi\right) = \sqrt{T}\begin{pmatrix} \hat{\Lambda}^a - \Lambda \\ \hat{\boldsymbol{\beta}}^a - \boldsymbol{\beta} \\ \hat{\boldsymbol{\gamma}}^a - \boldsymbol{\gamma} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} N\left(0, \bar{\Upsilon}^a\right) \\ N\left(0, \bar{\Theta}^a\right) \\ N\left(0, \bar{\Psi}^a\right) \end{pmatrix},$$

and

$$\sqrt{T}\left(\hat{\Pi}^b-\Pi\right)=\sqrt{T}\begin{pmatrix}\hat{\Lambda}^b-\Lambda\\ \hat{\boldsymbol{\beta}}^b-\boldsymbol{\beta}\\ \hat{\boldsymbol{\gamma}}^b-\boldsymbol{\gamma}\end{pmatrix}\xrightarrow{d}\begin{pmatrix}N\left(0,\bar{\Upsilon}^b\right)\\ N\left(0,\bar{\Theta}^b\right)\\ N\left(0,\bar{\Psi}^b\right)\end{pmatrix}$$

where the superscript a and b denote estimation from restricted sample before and after the break respectively; $\bar{\Upsilon}^a=\frac{\Upsilon^a}{\epsilon}, \bar{\Upsilon}^b=\frac{\Upsilon^b}{1-\epsilon}, \bar{\Theta}^a=\frac{\Theta^a}{\epsilon}, \bar{\Theta}^b=\frac{\Theta^b}{1-\epsilon}, \bar{\Psi}^a=\frac{\Psi^a}{\epsilon}$ and $\bar{\Psi}^b=\frac{\Psi^b}{(1-\epsilon)}$ are the associated covariance weighting structure; and the covariance matrices Υ, Θ, and Ψ are as defined in Appendix B.

In testing the Hypothesis I, when τ is fixed, the Wald test statistic under the null hypothesis of no structural change in Λ against the alternative of at least one structural change in Λ can be constructed as below:

$$\begin{aligned} W_I(\tau) &= \left(\left(\hat{\Lambda}^a-\Lambda\right)-\left(\hat{\Lambda}^b-\Lambda\right)\right)'\left[\left(\bar{\Upsilon}^a\right)+\left(\bar{\Upsilon}^b\right)\right]^{-1}\left(\left(\hat{\Lambda}^a-\Lambda\right)-\left(\hat{\Lambda}^b-\Lambda\right)\right)\\ &\xrightarrow{d} Z'\left[\left(\bar{\Upsilon}^a\right)+\left(\bar{\Upsilon}^b\right)\right]^{-1}Z \end{aligned}$$

where $Z\sim N\left(0,\bar{\Upsilon}^a+\bar{\Upsilon}^b\right)$.

Define $\hat{\boldsymbol{\Upsilon}}=\bar{\Upsilon}^a+\bar{\Upsilon}^b$ where $\hat{\boldsymbol{\Upsilon}}$ is positive definite. Using Cholesky decomposition, we have $\hat{\boldsymbol{\Upsilon}}=LL'$ and $\hat{\boldsymbol{\Upsilon}}^{-1}=L^{-1}L^{-1\prime}$ where L is a lower triangular matrix with strictly positive diagonal entries. Premultiplying Z by the inverse of L,

$$\begin{aligned} L^{-1}Z &\sim N\left(0,L^{-1}\hat{\boldsymbol{\Upsilon}}L^{-1\prime}\right)\\ &= N\left(0,L^{-1}LL'L^{-1\prime}\right)\\ &= N\left(0,I_r\right). \end{aligned}$$

Using this result, we can show that asymptotically

$$W_I(\tau)\xrightarrow{d} Z'\hat{\boldsymbol{\Upsilon}}^{-1}Z=Z'L^{-1'}L^{-1}Z=Q(\tau) \tag{6}$$

where for a given $\tau = [T\epsilon], Q(\tau) \sim \chi^2(q)$ with the degrees of freedom q corresponding to the number of restrictions being tested. Thus the test statistic $W_I(\tau)$ under the null is asymptotically pivotal for τ fixed.

Since the eigenvectors and the factor loadings are also asymptotically normal (see Appendix B), we may test all the other five hypotheses using Wald statistics, which when normalized with their respective asymptotic variances, converges to a chi-squared as above. The form of the Wald statistics corresponding to the five hypotheses is given below:

$$W_{II}(i,\tau) = \frac{\left(\hat{\lambda}_i^a - \hat{\lambda}_i^b\right)^2}{\left[\left(2\left(\hat{\lambda}_i^a\right)^2/\epsilon\right) + \left(2\left(\hat{\lambda}_i^b\right)^2/(1-\epsilon)\right)\right]} \tag{7}$$

$$W_{III}(i,\tau) = \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right)' \left[\bar{\Theta}_{ii}^a + \bar{\Theta}_{ii}^b\right]^{-1} \left(\hat{\beta}_i^a - \hat{\beta}_i^b\right) \tag{8}$$

$$W_{IV}(i,p,\tau) = \frac{\left(\hat{\beta}_{ip}^a - \hat{\beta}_{ip}^b\right)^2}{\left[\left(\bar{\Theta}_{ii,pp}^a\right) + \left(\bar{\Theta}_{ii,pp}^b\right)\right]} \text{ for } \bar{\Theta}_{ii,pp}^s \text{ is } pp^{th} \text{ position of matrix } \bar{\Theta}_{ii}^s, s = a, b \tag{9}$$

$$W_V(i,\tau) = \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right)' \left[\bar{\Psi}_{ii}^a + \bar{\Psi}_{ii}^b\right]^{-1} \left(\hat{\gamma}_i^a - \hat{\gamma}_i^b\right) \tag{10}$$

$$W_{VI}(i,j,\tau) = \begin{pmatrix} \hat{\gamma}_i^a - \hat{\gamma}_i^b \\ \hat{\gamma}_j^a - \hat{\gamma}_j^b \end{pmatrix}' \begin{bmatrix} \left(\bar{\Psi}_{ii}^a + \bar{\Psi}_{ii}^b\right) & \left(\bar{\Psi}_{ij}^a + \bar{\Psi}_{ij}^b\right) \\ \left(\bar{\Psi}_{ij}^a + \bar{\Psi}_{ij}^b\right) & \left(\bar{\Psi}_{jj}^a + \bar{\Psi}_{jj}^b\right) \end{bmatrix}^{-1} \begin{pmatrix} \hat{\gamma}_i^a - \hat{\gamma}_i^b \\ \hat{\gamma}_j^a - \hat{\gamma}_j^b \end{pmatrix} \tag{11}$$

When the date of the structural change is unknown but known to fall within a finite range, to test for a break occurring at time τ we use the *Sup*, *Exp*, and *Avg* Wald-type

(W) defined as:

$$
\begin{aligned}
SupW &= \max_{t_1<\tau<t_2} W && (12)\\
AvgW &= \frac{1}{t_2-t_1+1}\sum_{\tau=t_1}^{t_2} W && (13)\\
ExpW &= \ln\left[\frac{1}{t_2-t_1+1}\sum_{\tau=t_1}^{t_2}\exp\left(\frac{1}{2}W\right)\right] && (14)
\end{aligned}
$$

where W corresponds to one of the equations (7)-(11). The breakpoint τ lies between t_1 and t_2 such that $t_1 = [T\epsilon_1]$, $t_2 = [T\epsilon_2]$, $t_1 \neq t_2, \epsilon_2 = 1-\epsilon_1$, and t_1 is bounded away from zero and t_2 is bounded away from T; this condition is required since the proposed test statistic is unbounded in limit at the boundaries. Following Andrews (1993) and Andrews and Ploberger (1994), we use the restricted interval $t_1 = 0.15T$ and $t_2 = 0.85T$ such that ϵ_1 and ϵ_2 lies in the interval $[0.15, 0.85]$.

Under the null of no structural change, from the continuous mapping theorem, the asymptotic distributions of the test statistics converge to:

$$
\begin{aligned}
SupW &\xrightarrow{d} \max_{\epsilon_1<\epsilon<\epsilon_2} Q(\epsilon) && (15)\\
AvgW &\xrightarrow{d} \int_{\epsilon_1}^{\epsilon_2} Q(\epsilon)d\epsilon && (16)\\
ExpW &\xrightarrow{d} \ln\left[\int_{\epsilon_1}^{\epsilon_2}\exp\left(\frac{1}{2}Q(\epsilon)\right)d\epsilon\right] && (17)
\end{aligned}
$$

where if we know the break point fraction $\epsilon, Q(\epsilon)$ will be $\chi^2(q)$ with the degrees of freedom q corresponding to the number of restrictions being tested. Since the break fractions are not identified, the tests would have non-standard distributions.

Andrews (1993) and Andrews and Ploberger (1994) provide the asymptotic critical values for Sup, Avg,and Exp of optimal tests based on a regression type framework. Where the least squares problem minimizes the vertical distances between the data points, the PCA framework minimizes the orthogonal distances between the data points. In providing inference on the eigensystem stability, we rely upon the bootstrapped critical values of the test statistics. In this, we bootstrap the space vector of N maturities by resampling across time.

We construct the bootstrap distribution of the test statistics as follows:

1. For a given value of the break fraction ϵ, we randomly draw the vector of maturities from the $T \times N$ term structure data in order to construct the $T \times N$ bootstrapped data.

2. We construct the covariance matrix for the bootstrapped data and conduct the PCA in order to estimate the eigenspace variables $\hat{\Lambda}, \hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\gamma}}$.

3. We compute the Wald statistics $W_k(., \tau)$ for $k = I, II, ..., VI$ and calculate the weighted measures Sup, Avg,and Exp of the Wald statistics.

4. We repeat steps 1 through 3 for BR number of bootstrap replications.

The procedure generates BR number of bootstrap statistics of Sup, Avg,and Exp of $W_k(., \tau)$. For $\epsilon = 0.15$, we conduct 1000 iterations ($BR = 1000$) and in each iteration we resample the term structure panel, which is of 1923 by 21 dimension. Note that since the Wald test statistic is asymptotically pivotal, the asymptotic distribution of the test statistics does not depend on a particular data generating process under the null. Therefore bootstrap distribution can consistently estimate the asymptotic distribution of the test statistics and provide more reliable inference than asymptotically based inferences by removing the finite sample biases. Davidson and MacKinnon (1999) find that for asymptotically pivotal test statistics, using critical values from the bootstrap will produce smaller size distortions (reduced by an order of $T^{-1/2}$) than when using the critical values obtained from the first order asymptotics. Using the bootstrapped critical values, one may be able to mimic the skewness and kurtosis of the empirical distribution that is not captured by the first order limiting distribution.

A set of Monte Carlo experiments are conducted in order to study the finite sample properties of the tests proposed. The simulation results with 5000 Monte Carlo runs and 500 bootstraps show good size (at 5% significance level) and power properties for varied cross sectional dimension panels ($N = 5, 10, 20$), with three possible change points ($\tau = T/3, T/2, 2T/3$), and three possible break sizes generated from different intervals of the uniform distribution. Overall, we find that the empirical size of the bootstrap tests is very close to the nominal size in the case of *Sup*, *Avg*, and *Exp* of almost all of the six Wald type test statistics. In the case of testing the curvature factors, we find under-sizing for small N. However, we see substantial size improvements as N increase. In terms of power performance, we find that the test statistics $W_I(\cdot) - W_{III}(\cdot)$ and $W_{VI}(\cdot)$ show power essentially close to one. The test statistic $W_{VI}(\cdot)$ however show low power in evaluating the curvature factor for small N and small structural change magnitudes. There is however power gain as the magnitude of structural change increase. The full set of results is not reported in this paper but available upon request.

4 Empirical Results

4.1 Data

We use the term structure of US zero coupon bond yields obtained from Datastream. The term structure of zeros are extremely useful in fixed income applications such as pricing bonds, swaps, and other fixed income derivatives; financial engineering the interest rates exposures; obtaining the forward rate curves, par yield curves; and so on. Table 1, summarizes the datasets used in the previous studies that have evaluated the term structure stability.

The term structure of US zero coupon bond yields from Datastream include 21 maturities of 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132 and 144 months. The matrix plot in Figure 1 depicts the relationship among the yields of various interest rate maturities considered. The sample period extends from 11 Jan 1999 to 31 May 2006, with daily frequency (1927 observations). The data period covers both the period of downturn (during the technology stock boom in 2001) and upswings where the risk aversions of the investors are high causing gains in the bond markets. The bond yields data for maturities less than 3 months were filtered out in order to reduce the market microstructure effects and avoid liquidity issues. On the same note, we use the five day weekly change in yields in order to perform the eigen decomposition on its covariance structure as recommended by Lardic et al. (2003) and as commonly used in factor analysis literature of term structure of interest rates.

[Insert Table 1 here]

[Insert Figure 1 here]

4.2 A First Examination of Factor Structure Instability

In this section, we undertake some graphical analyses for the term structure of US zero coupon bond yields in order to identify the instability risk present in the factor structure of the yield curve.

First, we arbitrarily split the seven and half year's bond yield data into three approximately equal, two and half year subsample periods; January '99 - June '01, July '01 - Dec '03, and Jan '04 - May '06 and graphically investigate whether the eigensystem has remained stable over the three subperiods. We perform the PCA on the 5 year holding period returns data for the three subsamples, in order to extract the level, slope, and curvature factors that drive the evolution of change in interest rates data. Following Litterman and Scheinkman (1991), we consider the first three principal factors in explaining the evolution of term structure of interest rates. In order to extract the three principal factors using PCA, we perform the following steps:

1. We form the covariance matrix from the change in panel of yields panel for the three subperiods considered.

2. We compute the eigenvalues and the corresponding eigenvectors from the covariance matrix for each period using the eigen decomposition. The eigenvectors are the principal components and the eigenvalues present the explanatory power of the corresponding eigenvectors.

Second, we graphically investigate instability along the short end, medium term, and long end of the yield curve separately over the three subsample periods considered. For this, we draw the direction of the principal axes (the eigenvectors), along with the scatter plot of the original yield changes data for the three subsample periods. In order to visualize the direction of the eigenvectors, we have to limit our analysis to the two dimensional plots. We use the three month and six month rate as a proxy for the short end of the curve; the five year and seven year rate as a proxy for the middle (medium term) of the curve; and the ten and twelve year rate as a proxy for the long end of the curve.

Third, in order to examine the evolution of the entire eigenspace, we conduct recursive PCA by expanding the estimation window at every run by including one new observation and then record the evolution of the eigenvalues, eigenvectors, and factor loadings. We undertake two recursive schemes, namely Forward Recursive Scheme (FRS) and Backward Recursive Scheme (BRS). The two schemes allow us to evaluate stability in an informal way. The FRS allows us to visually gauge the impact of adding one extra observation at each recursion and the BRS allows us to visually gauge the impact of removing one observation at each recursion. The instability can be seen as the abrupt increase in variability at a point in time in the case of the FRS and a reduction in variability at a point in time in the case of the BRS. This FRS and BRS patterns can also be used to check if there are more than one changes affecting the variability in the recursion.

Figures 2 to *8* present the results towards the preliminary study of the issue of instability. Figure 2 plots the three principal components determined over the three subsamples. We observe that, in all the three subsample periods, considering the first three principal components would be sufficient in explaining the dynamics of the term structure. Though the three factors vary in detail, the term structure responsiveness to these factors has remained stable over time. This stability result concurs with that recorded by Bliss (1997), Perignon and Villa (2006), and others. However, the column charts of Figure 2 show that the shocks to the term structure varied during the subperiods considered. The level risks, captured by the first principal component, were the highest in the third subsample period, corresponding to upward shifts in the yield curve. The slope risks, explained by the second principal component and the curvature risks, explained by the third principal component, were the highest in the second subsample, corresponding to the flattening of the yield curve observed during the bear market (2000-03).

Figures *3*, *4*, and *5* plot the short run, medium term, and long run principal axes (directional vectors) for the three subsample period considered. The two directional vectors are orthogonal to each other by construction. The plot shows how well the principal axes explain the variability in yields. Table *2* records the eigenvalues (volatility), eigenvectors, and the percentage of variances explained by the two principal components. For the case of short rates, if we compare the direction of the principal axes across the three subsample period, we find that the first principal axis differ across the three subsamples and by the orthogonality condition, so does the second principal axis. Further, we observe that the sample data for the short rates are dispersed distinctly across the three subsample periods. This means there exist different volatility patterns in the three subperiods and support the argument allowing for distinct time-varying covariance matrices. Therefore considering a constant covariance matrix decomposition of principal components may induce instability in the components. For the case of medium term and long term rates governing the yield curves (Figures *4*, and *5* respectively), we find that the two eigenvalues have similar directional vectors for the three subsample periods, with around 99% explanatory power of the variances.

[Insert Figures 2 - 5 here]

[Insert Table 2]

Further consider the recursive plots of the eigenvalues, eigenvectors, and factor loadings reported in Figures 6, 7, and 8. The plots obtained from the recursion clearly show endurance of instability in the eigensystem. In the case of eigenvalues governing the factors (Figure *6*), we can clearly see that the dynamics have not remained the same over time even though the percentage variation explained by the eigenvalues has remained the same. The eigenvalues for the level and curvature factors seems to have one prominent change but the eigenvalues governing the slope seems to have more than one abrupt change. Looking at the recursion patterns for eigenvectors (Figure *7*), the level and curvature eigenvectors show two prominent patterns and the slope eigenvector shows three prominent patterns suggesting possible structural changes in the eigenvectors. In the case of factor loadings (Figure *8*), the FRS suggest one possible pattern change in the case of level, and two pattern changes in the case of slope and curvature. However, if we also consider the BRS, we can see there exist one possible intermittent blip in the level, slope, and curvature factor loadings. The observations of pattern changes surely corroborate the time-varying nature of the eigensystem, which may have caused possible structural breaks in the series.

[Insert Figures 6 - 8 here]

To summarise, we find that the shocks contributing to the level, slope, and curvature instability risks have varied during the three arbitrary identified subsamples. We find that the directional axes of the short end interest rates have also varied over time. The forward and backward recursive plots of the eigenspace variables indicate the possible presence of instabilities.

In the following section, we formally test for the instability present in the eigenvalues, eigenvectors and factor loadings, using the testing framework we developed in the paper.

4.3 Stability Testing Results

Table 3 records the results from implementing the *Sup*, *Avg*, and *Exp* test statistics for the six hypotheses formulated above. We test the linear restrictions of equality in eigenspace variables for a given change point occurring at time τ, using the Wald test. In practice since we do not know this change point τ, we calculate the weighted statistics *Sup*, *Avg*, and *Exp* for all possible change points within the restricted sample period. The tests are evaluated for significant structural changes within the restricted sample period $[0.15T, 0.85T]$.[1] The conclusions are drawn based on results from all the three weighted measures *Sup*, *Avg*, and *Exp* that concur. We use the 5% significance level to drawn inferences.

[Insert Table 3 here]

4.3.1 ***Investigating stability in the overall eigensystem***

Evaluating the weighted test statistics for $W_I(\tau)$, we reject the null in favor of the alternative that $\Lambda^a \neq \Lambda^b$. Thus, *Sup*, *Avg*, and *Exp* test statistics of $W_I(\tau)$ infer that significant changes persist in the eigensystem of the yield curves. Instability in the vector of eigenvalues would mean structural instability in the variance process governing the factors. Bliss (1997), Audrino et al. (2005), among others, detected the same instability.

It is worth mentioning that the conclusions on instability of the factors governing the volatility are indeed different to the conclusions drawn in this paper where we evaluated the volatility governing the factors. The distinction lies within the fact that the information extracted (using eigen decomposition) from the covariance matrix of the yields are different than the information summarized in the covariance matrix of unobserved volatility. In regard to the latter, Perignon and Villa (2006) document the time-varying

[1] We avoid the boundaries since the test statistics produce unstable results at the boundaries as documented by Andrews (1993).

nature of the volatility governing the factors and Bliss (1997) reported instability present in the factor volatility structures using graphical methods.

4.3.2 *Investigating stability in eigensystem of the level factor*

Evaluating the weighted test statistics for $W_{II}(1,\tau)$, $W_{III}(1,\tau)$, and $W_V(1,\tau)$ we reject the null in favor of the alternative that $\lambda_1^a \neq \lambda_1^b, \beta_1^a \neq \beta_1^b$, and $\gamma_1^a \neq \gamma_1^b$ respectively. Thus according to all the three weighted measures (Sup, Avg, and Exp) for the various hypotheses, we can conclude that all the three eigenspace variables (eigenvalue, eigenvector, and factor loading) governing the level factor has statistically significant structural changes inducing instability. The result differs to the graphical inferences gathered by several authors such as Reisman and Zohar (2004) who has drawn stability conclusions for the level factor of discount bond yields.

In order to gauge which interest rate maturities have contributed to structural instability in the level factor, we evaluated the weighted test statistics for $W_{IV}(1,\tau)$. According to all the three weighted measures Sup, Avg, and Exp we conclude that the structural instability was common and evident in all the 21 interest rate maturities governing the level factor. This means that the structural change in the level factor has been caused by economic shocks that eminently influenced the whole yield curve (short end as well as the long end maturities).

4.3.3 *Investigating stability in eigensystem of the slope factor*

In the case of the slope factor, we find that the eigenvalue or volatility governing the factor has incurred structural changes. Using all the three weighted statistics for $W_{II}(2,\tau)$, we reject the null in favor of the alternative of $\lambda_2^a \neq \lambda_2^b$. However, by evaluating $W_{III}(2,\tau)$ and $W_V(2,\tau)$ we find that the eigenvectors and the factor loadings governing the slope factor have remained stable over time. By evaluating the weighted test statistic of $W_{IV}(2,\tau)$ for the slope factor, we can find that the short term interest rates (3 months - 1 year) governing the factor were unstable whereas the medium and long term interest rates (2 years - 12 years) governing the factor were tested to be stable over time. Thus the volatility of the slope factor is sensitive to shocks affecting the short rates, but the slope factor structure has remained stable over time. The test results for the slope factor concur with Reisman and Zohar (2004) who conclude stability of the slope factor.

4.3.4 *Investigating stability in eigensystem of the curvature factor*

In the case of testing for instability in the eigenspace variables of the curvature factor, we find similar results to that of the slope factor. Using the *Sup*, *Avg*, and *Exp* for $W_{II}(3,\tau)$, $W_{III}(3,\tau)$, and $W_V(3,\tau)$ we find that the curvature eigenvalue (volatility) has been subject to statistically significant structural changes but the corresonding eigenvector and factor loading have remained stable through time. By evaluating stability in the interest rates governing the curvature factor (using $W_{IV}(3,\tau)$), we find that the medium and long term rates (2 years - 12 years) have contributed to the structural change in the volatility of the curvature factor. Unlike the slope factor, we find that the short term interest rates (3 months - 1 year) were stable through time.

Thus we can conclude that, as in the case of the slope factor structure, the volatility governing the curvature factor has incurred statistically significant structural changes. However, the variance explained by the curvature factor is sensitive to movements and shocks affecting only the long rates. We find that the factor structure of curvature has remained stable over the sample period considered. Reisman and Zohar (2004) documented marginal stability of the curvature factor structure using graphical analysis.

4.3.5 *Investigating common instability in factor loadings*

Since we have found that the eigenspace variables for the level, slope, and curvature factors have incurred instability and since the three factors are correlated with each other, the economic shocks affecting one factor could also have affected the other. Therefore we investigate the presence of common structural changes due to common shocks in factors. By evaluating the weighted test statistics of $W_{VI}(1,2,\tau)$ we do not reject the null of presence of common structural changes in level and slope factor loadings. Thus we can conclude that there exist statistically significant change points common to the level and slope factors. Combining this result with the instability conclusions found for the level and slope eigenvectors, we can identify the common sources of instability within the level and slope factors as the economic shocks that have caused structural changes in the short term interest rates (3 months - 10 months). Since the Sup, Avg, and Exp for $W_{VI}(1,3,\tau)$ provide variant conclusions from testing common instabilities in level and curvature factor loadings, we cannot infer any presence of common structural changes. In the case of evaluating common instabilities present in the slope and curvature factor loadings, we reject the weighted test statistics of $W_{VI}(2,3,\tau)$ in favor of the alternative that no common structural changes exist between the slope and curvature factor loadings. Thus we can conclude that the slope and curvature factors behave dissimilarly to economic shocks that may have caused structural instabilities in them separately. This result corroborates with the above findings that the slope and curvature factors are sensitive to economic shocks influencing different ends of the yield curve.

5 Stability of Alternative Bond Yield Term Structures

We investigate stability in several zero coupon bond yield term structures commonly used in literature. We test for the factor's eigensystem stability using the procedure outlined above and we present the results in this section.

5.1 Federal Reserve Constant Maturity Zero Coupon Bond Yields

The Federal Reserve constant maturity term structure dataset includes the US yields with maturities 3, 6, 12, 24, 36, 60, 84, and 120 months over the years January 4, 1982 to Febuary 4, 2008. We use the dataset with daily frequency (6806 observations) as well as the monthly frequency (314 observations). This is the same dataset with monthly and weekly frequencies are used in the paper by Reisman and Zohar (2004) that extracts the principal factors using PCA and graphically validates (in)stability of the factor loadings. The paper concludes that the factor loadings of level and slope are quite stable, but the curvature factor changes rapidly over time. We conduct a statistical test on the eigenspace variables and report the results below.

[Insert Table 4 here]

Table 4 provides the stability test results for the term structure with daily frequency. We test the six hypotheses on the eigensystem variables and calculate the weighted measures *Sup*, *Avg*, and *Exp* of tests and present the value of the test statistic with its p-value. Using a significance level of 5%, we gather inference on the stability of the term structure variables. Testing for stability in the vector of eigenvalues of level, slope and curvature, we find rejection of the null hypothesis. This means that the overall eigensystem explaining the variance process of the term structure has remained unstable over time. Investigating stability in the eigensystem variables of the individual factors, we find instability in the eigenvalues, eigenvectors and factor loading matrices of level and curvature. In the case of slope factor, the eigenvector is stable and the factor loading seems to be marginally stable (pvalue around 0.05). Testing for stability in the interest rates governing the three individual factors; we find all interest rate maturities of level factor unstable. For the slope factor, the short end maturities up to 12 months have been unstable, but the long end rates were stable. Testing for common instability points in the factors, we find that the factors have common dates of instability.

[Insert Table 5 here]

Table 5 records the stability results in the case of Fed monthly term structure of yields. We find the results are almost the same with the overall eigensystem unstable, all three eigensystem variables of the level factor unstable, presence of common break points among the three factors, and short rates unstable for the slope factor. Unlike the daily data, we now find all the three eigensystem variables of the slope factor unstable. Reisman and Zohar (2004) found the slope factor graphically stable over time. When using lower frequency, we find the long rates of the level factor have become stable.

5.2 Fama-Bliss Constant Maturity Zero Coupon Bond Yields

The Fama-Bliss term structure of unsmoothed US zero coupon bond yields include 21 maturities of 3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, 120, 180, 240, 300, and 360 months. The dataset extends over the period January 1970 to December 2003 (408 observations). The dataset is widely used in term structure empirical papers for the purpose of forecasting (see Diebold and Li (2006)). Bliss (1997) and Perignon and Villa (2006) use this dataset in order to study the time-varying movements of factors in yield curves. Bliss (1997) evaluate the stability patterns of level, slope, and curvature factors for the period January 1970 to December 1995 and find that though the factors vary in detail, the cumulative explanatory power of the factors have remained same over the entire period. Perignon and Villa (2006) studied the sources of time variation in the covariance matrices of bond yields. The paper found the variances (eigenvalues) have significant time variations but the factor loadings remain same over time. Conducting a statistical test for stability in eigenvalues, eigenvectors, and factor loadings, we summarize the results below.

[Insert Table 6 here]

Table 6 presents the stability results for the Fama-Bliss term structure. We find that the overall eigensystem, measured by the eigenvalues of level, slope, and curvature factors, have been unstable. Testing for stability in the individual three factors, we find all three factors have unstable eigenvalues, eigenvectors, and factor loadings with the exception of the slope factor. The slope factor has stable factor loadings, as in the case of using monthly term structure from the Federal Reserve database. We find that the long end maturities governing the slope factor have been stable over time. This result corroborates the results found using other datasets. Further, we find presence of common structural changes among all the three factors of the Fama-Bliss term structure.

6 Conclusion

This paper explores the important question of whether the yield curve factor structure is stable through time. Several authors have either assumed stability or relied upon graphical analysis to make inferences. We propose a formal testing procedure and evaluate its asymptotic properties. We formulate six hypotheses for statistically evaluating the stability in the eigenspace variables (eigenvalues, eigenvectors, and factor loadings) governing the level, slope, and curvature factors of the yield curves. We then formally test for stability of the US zero coupon bond yield factor structures between January 1999 and May 2006.

We find that the overall variance process governing the first three factors of the yield curves were unstable over time. Previous literature documents the time-varying volatility of yield factors; see for example, Perignon and Villa (2006). Our results corroborate that but further, we also find abrupt fluctuations (instabilities) present in the factors, captured by the Wald-type testing procedure. We find that even if the volatility (eigenvalues) of factors were unstable, the linear relationship (factor loadings) of slope and curvature were stable.

To summarize the results: for the level factor, we find structural instability in all the eigenspace variables. Structural changes affecting all the interest rate maturities in the term structure panel fostered instability in the factor structure as well as the volatility explained by the factor. In the case of the slope and curvature factors, we find that the variances accounted by the factors incur structural instabilities. However, we find the eigenvectors and loadings have remained stable through time. Therefore we can conclude that the slope and curvature factor structures have remained stable; though the volatility associated with the factors are unstable over time. The instability in the volatility of the slope factor is caused by instability affecting only the short term maturities (3 months - 1 year) whereas in the case of the curvature factor, the instability in the volatility of the factor is caused by instability affecting only the medium and long term rates (2 years - 12 years). In investigating the presence of common structural changes in factors, we find statistically significant breaks common to level and slope factors and no statistically significant common breaks in the slope and curvature factors.

Further, testing for stability in eigenspace variables of factors governing the Fama-Bliss and Federal Reserve term structures, we find similar results of presence of instability in factors. While this paper evaluated the stability of PCA factors - level, slope, and curvature, for future work one can study the stability issues in factors estimated using the popular function-based Nelson-Siegel factor model as parameterized by Diebold and Li (2006).

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Appendices

A Principal Component Analysis Framework

In principal component analysis, we estimate the eigenvalues $\lambda_1 \geqq \lambda_2 \geqq ... \geqq \lambda_N$ of the matix Σ satisfying the equality

$$|\Sigma - \Lambda I| = 0 \tag{18}$$

where $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ and their corresponding vectors $\beta_1, \beta_2, ..., \beta_N$ satisfying the two conditions

$$\Sigma\beta_i = \lambda_i\beta_i \tag{19}$$

$$\beta_i'\beta_i = 1 \tag{20}$$

The conditions ensure that the characteristic vectors β_i for $i = 1, 2, ..., N$ are orthogonal to each other and are of unit length.

The estimated vectors $\beta_1, \beta_2, ., \beta_i, .., \beta_N$ are such that the vector $\beta_i'Y$ is the directional vector that captures the maximum variability in Y. Therefore the estimation of β_i can be seen as solution to the optimization problem

$$\begin{aligned} & max\ E(\beta_i'Y\ Y'\beta_i) \\ = & \ max\ \beta_i'\Sigma\beta_i \end{aligned}$$

subject to the conditions $\beta_i'\beta_i = 1$ and $\beta_i Y' \perp \beta_j Y'$ for $i < j$. The orthogonality condition between the characteristic vectors means that

$$0 = E\left[\left(\beta_j'Y'\right)\left(\beta_i'Y'\right)'\right] = E\left(\beta_j'Y'Y\beta_i\right) = \beta_j'\Sigma\beta_i.$$

The lagrangian equation to be maximized is therefore

$$L_j = \beta_j' \Sigma \beta_j - \xi(\beta_j' \beta_j - 1) - 2 \sum_{i=1}^{j-1} \phi_i \beta_j' \Sigma \beta_i$$

where ξ and $\phi = (\phi_1, ..., \phi_{j-1})$ are the lagrange multipliers and $j = 1, 2, ..., N$. The solution to this optimization problem satisfies the equation (19) and (18) and therefore the eigenvalues λ_i summarize the amount of variability captured by the corresponding eigenvector β_i.

B Asymptotic Properties of the Eigenspace Variables

We provide the inferential theory for the eigenvalues, and eigenvectors that are estimated using the classical PCA. Let $Z = (z_1', ..., z_T')$ be $N \times T$ matrix such that $ZZ' = (T-1)\hat{\Sigma}$ in equation (3). Therefore

$$\hat{\Sigma} = \frac{1}{T-1} \sum_{t=1}^{T} z_t z_t'$$

where $z_t = (y_t - \bar{y})$ is the demeaned vector and $z_t \sim N_N(0, \Sigma)$.

Definition. *($N-$variate wishart distribution) Let $x_1, ..., x_k$ be k-independent $N-$vectors. Suppose each $x_i \sim N_N(0, \Sigma)$. Let $U = x_1 x_1' + x_2 x_2' + ... + x_k x_k'$. Then U is said to have a $N-$ variate Wishart Distribution with k degrees of freedom and covariance matrix Σ. That is,*

$$U \sim W_N(\Sigma, k)$$

According to the above definition, $\hat{\Sigma}(T-1) = \sum_{t=1}^{T} z_t z_t' = \sum_{t=1}^{T} y_{it} \cdot y_{jt} \sim W_N(\Sigma, T-1)$. Therefore

$$\hat{\Sigma} \sim W_N((T-1)^{-1}\Sigma, T-1) \tag{21}$$

The density function of matrix $\hat{\Sigma}$ is

$$f\left(\hat{\Sigma}\right) = \frac{\left((T-1)^{-N}\left|\hat{\Sigma}\right|\right)^{\frac{1}{2}(T-N-2)} e^{-\frac{1}{2(T-1)}tr\left(\hat{\Sigma}\Sigma^{-1}\right)}}{2^{\frac{1}{2}N(T-1)}\pi^{\frac{1}{4}N(N-1)}\left|\Sigma\right|^{\frac{1}{2}(T-1)}\prod_{i=1}^{N}\Gamma\left[\frac{1}{2}(T-i)\right]}$$

where $\Gamma(.)$ is the gamma function.

The following proposition provides the rate of convergence and the limiting distribution of the eigenvalues and eigenvectors decomposed from a covariance matrix $\hat{\Sigma}$.

Proposition. *(Limiting distribution of eigenvalues and eigenvectors) Let* $y_1, ..., y_T$ *be independently distributed, each being an* $N-$*vector of* $N_N(0, \Sigma)$. *Define* $\Lambda = (\lambda_1, \lambda_2, ..., \lambda_N)'$ *a* $N \times 1$ *vector of independent eigenvalues and* $\beta = (\beta_1, \beta_2, ..., \beta_N)$ *a* $N \times N$ *matrix of orthogonal eigenvectors. The sample covariance matrix* $\hat{\Sigma}$ *is such that* $\hat{\Sigma} \sim W_N\left((T-1)^{-1}\Sigma, T-1\right)$. *Then as* $T \to \infty$,

$$\left(\hat{\Lambda} - \Lambda\right) = O_p(T^{-1/2}) \tag{22}$$

$$\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) = O_p(T^{-1/2}) \tag{23}$$

where the sequence $\left(\hat{\Lambda} - \Lambda\right)$ *and* $\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right)$ *are independent to each other. The limiting distribution is given by*

$$\sqrt{T}\left(\hat{\Lambda} - \Lambda\right) \xrightarrow{d} N(0, \Upsilon) \tag{24}$$

where $\Upsilon = diag\left(2\lambda_1^2, 2\lambda_2^2, ..., 2\lambda_N^2\right)$ *and*

$$\sqrt{T}\left(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}\right) \xrightarrow{d} N(0, \Theta) \tag{25}$$

where $\Theta = \sum_{i=1}^{N} \sum_{j=1}^{N} (U_{ij} \otimes \Theta_{ij})$ *with*

$$\Theta_{ij} = \begin{cases} \lambda_i \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} \beta_k \beta_k' & \text{for } i = j \\ -\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_j \beta_i' & \text{for } i \neq j \end{cases}$$

and U_{ij} *is an* $N \times N$ *matrix that has 1 in the* ij^{th} *position and 0's elsewhere.*

The results mentioned in this proposition have been proved almost simultaneously by Girshick (1939), Hsu (1939), Fisher (1939), Roy (1939), Mood (1951), Anderson (1963) and widely known in multivariate statistics literature. For the proof, we refer the reader to any of the above papers or book by Anderson 2003 pp.546.

C Proof of the Theorem

We know from the Proposition that as $T \to \infty$,

$$\sqrt{T} \left(\hat{\lambda}_i - \lambda_i \right) \xrightarrow{d} N(0, 2\lambda_i^2)$$

and

$$\sqrt{T} \left(\hat{\beta}_i - \beta_i \right) \xrightarrow{d} N(0, \Theta_{ii})$$
$$\sqrt{T} \left(\left(\hat{\beta}_i - \beta_i \right) \left(\hat{\beta}_j - \beta_j \right) \right) \xrightarrow{d} N(0, \Theta_{ij})$$

where

$$\Theta_{ij} = \begin{cases} \lambda_i \sum_{\substack{k=1 \\ k \neq i}}^{N} \frac{\lambda_k}{(\lambda_i - \lambda_k)^2} \beta_k \beta_k' & \text{for } i = j \\ -\frac{\lambda_i \lambda_j}{(\lambda_i - \lambda_j)^2} \beta_j \beta_i' & \text{for } i \neq j \end{cases}$$

We define the error in estimation of the eigenvalues $\left(\hat{\lambda}_i - \lambda_i \right)$ as ε_{λ_i} and the error in estimation of the eigenvectors $\left(\hat{\beta}_i - \beta_i\right)$ as ε_{β_i}, with $\left\{\varepsilon_{\lambda_i}, \varepsilon_{\beta_i}\right\}$ mutually independent across i.

Let $\hat{\lambda}_i^{1/2} = (\lambda_i + \varepsilon_{\lambda_i})^{1/2} = \lambda_i^{1/2}\left(1 + \frac{\varepsilon_{\lambda_i}}{\lambda_i}\right)^{1/2}$. Using Taylor expansion up to the first order,

$$\hat{\lambda}_i^{1/2} = \lambda_i^{1/2}\left(1 + \frac{1}{2}\frac{\varepsilon_{\lambda_i}}{\lambda_i}\right) + o_p(1).$$

Therefore we can write

$$\hat{\lambda}_i^{1/2} - \lambda_i^{1/2} = \frac{1}{2}\frac{\varepsilon_{\lambda_i}}{\lambda_i}.$$

Since we know the limiting distribution of the ε_{λ_i}, we have

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2} - \lambda_i^{1/2}\right) \xrightarrow{d} N\left(0, \frac{1}{2}\lambda_i\right). \tag{26}$$

We define $\hat{\lambda}_i^{1/2} - \lambda_i^{1/2} \equiv \tilde{\varepsilon}_{\lambda_i}$. Therefore we can write

$$\begin{aligned}\hat{\lambda}_i^{1/2}\hat{\beta}_i &= \left(\lambda_i^{1/2} + \tilde{\varepsilon}_{\lambda_i}\right)\left(\beta_i + \varepsilon_{\beta_i}\right) \\ &= \lambda_i^{1/2}\beta_i + \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}. \end{aligned} \tag{27}$$

Therefore

$$\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i = \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}$$

We know that, $\varepsilon_{\beta_i} = O_p(T^{-1/2}), \tilde{\varepsilon}_{\lambda_i} = O_p(T^{-1/2})$ and $\tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i} = O_p(T^{-1})$. Therefore

$$\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i = O_p\left(T^{-1/2}\right).$$

This proves equation (4).

From the above, $\sqrt{T}\left(\lambda_i^{1/2}\varepsilon_{\beta_i}\right) \xrightarrow{d} N(0, \lambda_i\Theta_{ii})$ and $\sqrt{T}\left(\beta_i\tilde{\varepsilon}_{\lambda_i}\right) \xrightarrow{d} N\left(0, \frac{1}{2}\lambda_i\beta_i\beta_i^{'}\right)$

Since $\tilde{\varepsilon}_{\lambda_i}$ and ε_{β_i} are independent, it holds that

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i\right) \stackrel{D}{=} N\left(0, \lambda_i\Theta_{ii} + \frac{1}{2}\lambda_i\beta_i\beta_i^{'}\right) + Q + o_p(1)$$

where $\frac{1}{\sqrt{T}}\left(\tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}\right) \xrightarrow{d} Q$ where Q is a distribution of the product of two mean zero independent normal variates. As $T \to \infty$, the effect of Q is $O_p\left(T^{-1/2}\right)$ and negligible. Therefore it holds that

$$\sqrt{T}\left(\hat{\lambda}_i^{1/2}\hat{\beta}_i - \lambda_i^{1/2}\beta_i\right) \xrightarrow{d} N\left(0, \lambda_i\Theta_{ii} + \frac{1}{2}\lambda_i\beta_i\beta_i^{'}\right)$$

This proves equation (5) for the case $i = j$.

The asymptotic covariance matrix for $(\hat{\gamma}_i - \gamma_i)\left(\hat{\gamma}_j - \gamma_j\right), i \neq j$:

$$\begin{aligned}
&\Psi_{ij} = Cov\left(\left(\hat{\gamma}_i - \gamma_i\right), \left(\hat{\gamma}_j - \gamma_j\right)\right) \\
= \;& E\left[\hat{\gamma}_i\left(\hat{\gamma}_j - \gamma_j\right) - \gamma_i\left(\hat{\gamma}_j - \gamma_j\right)\right] \\
= \;& E\left[\hat{\lambda}_i^{1/2}\hat{\beta}_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] - E\left[\lambda_i^{1/2}\beta_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \;& I - II
\end{aligned}$$

Substituting for the estimators of $\hat{\lambda}_l^{1/2}\hat{\beta}_l$ for $l = i, j$, we solve the two parts below:

I :

$$\begin{aligned}
&E\left[\hat{\lambda}_i^{1/2}\hat{\beta}_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \;& E\left[\lambda_i^{1/2}\beta_i + \lambda_i^{1/2}\varepsilon_{\beta_i} + \beta_i\tilde{\varepsilon}_{\lambda_i} + \tilde{\varepsilon}_{\lambda_i}\varepsilon_{\beta_i}\left(\lambda_j^{1/2}\beta_j + \lambda_j^{1/2}\varepsilon_{\beta_j} + \beta_j\tilde{\varepsilon}_{\lambda_j} + \tilde{\varepsilon}_{\lambda_j}\varepsilon_{\beta_j} - \lambda_j^{1/2}\beta_j\right)\right] \\
= \;& E\left[\lambda_i^{1/2}\lambda_j^{1/2}\varepsilon_{\beta_i}\varepsilon_{\beta_j}\right]
\end{aligned}$$

$II:$

$$\begin{aligned}
& E\left[\lambda_i^{1/2}\beta_i\left(\hat{\lambda}_j^{1/2}\hat{\beta}_j - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & E\left[\lambda_i^{1/2}\beta_i\left(\lambda_j^{1/2}\beta_j + \lambda_j^{1/2}\varepsilon_{\beta_j} + \beta_j\tilde{\varepsilon}_{\lambda_j} + \tilde{\varepsilon}_{\lambda_j}\varepsilon_{\beta_j} - \lambda_j^{1/2}\beta_j\right)\right] \\
= \; & 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
Cov\left(\left(\hat{\gamma}_i - \gamma_i\right), \left(\hat{\gamma}_j - \gamma_j\right)\right) &= E\left[\lambda_i^{1/2}\lambda_j^{1/2}\varepsilon_{\beta_i}\varepsilon_{\beta_j}\right] \\
&= \lambda_i^{1/2}\lambda_j^{1/2}\Theta_{ij}
\end{aligned}$$

This proves equation (5) for the case $i \neq j$.

Table 1: List of datasets used in studies that evaluated the issue of factor's stability

Authors	*Term Structure (rates considered)*	*Source*	*Frequency*	*Period*
Bliss (1997)	Fama-Bliss Discount bond yields (0.25, 0.5, 1, 2, 3, 5, 7, 10, 15, and 20 yrs)	CRSP	Monthly	Jan 1970 - Dec 1995
Diebold and Li (2006)	Fama-Bliss Zeros (3, 6, 9, 12, 15, 18, 21, 24, 30, 36, 48, 60, 72, 84, 96, 108, and 120 months)	CRSP	Monthly	Jan 1985 - Dec 2000
Perignon and Villa (2006)	Fama-Bliss Discount Bond Yields (0.25, 0.5, 1, 2, 6, 12 yrs)	CRSP	Monthly	Jan 1960 - Dec 1999
Reisman and Zohar (2004)	Discount bond yields (0.25, 0.5, 1, 2, 3, 5, 7, and 10 yrs)	Fed	Monthly & weekly	1982 - 2003
Audrino et al (2005)	Discount bond yields (1,...,30 yrs)	J.P. Morgan	Daily	Jan 1986 - May 1995

Table 2: Principal Component Estimates of the Subsample Periods Considered. The table contains the PCA results conducted for the full sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006. Panel A, B, and C report the estimates for eigenvalues, eigenvectors, and percentage of variations explained by corresponding eigenvalue in the case of short term (proxied by the three month and 6 month rates), medium term (proxied by the five year and seven year rates), and long term (proxied by the ten year and 12 year rates) respectively.

Panel A: Short Term Rates

	Eigenvalues	Eigenvectors		% Explained
		Vector 1	Vector 2	
Full Sample Period:	0.0015	-0.5365	-0.8439	88.5676
Jan '99 - May '06	0.0002	-0.8439	0.5365	11.4324
First Subperiod:	0.000723	-0.6509	-0.7592	91.7486
Jan '99 - Jun '01	0.000065	-0.7592	0.6509	8.2514
Second Subperiod:	0.003	-0.4808	-0.8768	87.3595
Jul '01 - Dec '03	0.0004	-0.8768	0.4808	12.6405
Third Subperiod:	0.0005558	-0.546	-0.8378	92.4486
Jan '04 - May '06	0.0000454	-0.8378	0.546	7.5514

Panel B: Medium Term Rates

	Eigenvalues	Eigenvectors		% Explained
		Vector 1	Vector 2	
Full Sample Period:	0.0018	-0.7523	-0.6589	99.3849
Jan '99 - May '06	0	-0.6589	0.7523	0.6151
First Subperiod:	0.0007643	-0.7193	-0.6947	99.3881
Jan '99 - Jun '01	0.0000047	-0.6947	0.7193	0.6119
Second Subperiod:	0.0031	-0.7609	-0.6489	99.5701
Jul '01 - Dec '03	0	-0.6489	0.7609	0.4299
Third Subperiod:	0.0014	-0.7491	-0.6625	99.1222
Jan '04 - May '06	0	-0.6625	0.7491	0.8778

Panel C: Long Term Rates

	Eigenvalues	Eigenvectors		% Explained
		Vector 1	Vector 2	
Full Sample Period:	0.0012	-0.7265	-0.6871	99.7203
Jan '99 - May '06	0	-0.6871	0.7265	0.2797
First Subperiod:	0.0007137	-0.713	-0.7011	99.2305
Jan '99 - Jun '01	0.0000055	-0.7011	0.713	0.7695
Second Subperiod:	0.0019	-0.7315	-0.6818	99.8754
Jul '01 - Dec '03	0	-0.6818	0.7315	0.1246
Third Subperiod:	0.0009108	-0.7259	-0.6878	99.8185
Jan '04 - May '06	0.0000017	-0.6878	0.7259	0.1815

Table 3: Testing Reults for Bond Yields from Datastream. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 6 - 11. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.96701	0.26218	-0.38973
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.24475	0.062922	-0.59174
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.87365	0.22575	-0.43265
	0.000	*0.000*	*0.000*
$W_V(1,\tau)$	0.70327	0.24609	-0.41435
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.34214	0.090797	-0.5643
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	249.64	10.664	118.11
	0.817	*0.793*	*0.817*
$W_V(2,\tau)$	1.9815	0.77043	0.11505
	0.466	*0.156*	*0.180*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.38012	0.10846	-0.54758
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	335.49	5.6174	160.9
	0.948	*0.963*	*0.948*
$W_V(3,\tau)$	1.9899	0.44182	-0.20286
	0.541	*0.478*	*0.572*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	3.4721	1.2964	0.62638
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	2.393	0.70712	0.063935
	0.000	*0.150*	*0.185*
$W_{VI}(2,3,\tau)$	5.4103	1.4711	0.83286
	0.119	*0.149*	*0.293*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.084479	0.008332	-0.64515
	0.001	*0.028*	*0.028*
$W_{IV}(1,2,\tau)$	0.13845	0.019008	-0.63475
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.087745	0.011164	-0.64241
	0.000	*0.002*	*0.002*
$W_{IV}(1,4,\tau)$	0.1796	0.031018	-0.62314
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.23842	0.058936	-0.59595
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.39888	0.087758	-0.56738
	0.000	*0.000*	*0.000*
$W_{IV}(1,7,\tau)$	0.52171	0.105	-0.54994
	0.000	*0.000*	*0.000*
$W_{IV}(1,8,\tau)$	0.53798	0.11548	-0.53989
	0.000	*0.000*	*0.000*
$W_{IV}(1,9,\tau)$	0.48691	0.1161	-0.53983
	0.000	*0.000*	*0.000*
$W_{IV}(1,10,\tau)$	0.42221	0.11239	-0.54391
	0.000	*0.000*	*0.000*
$W_{IV}(1,11,\tau)$	0.5422	0.078327	-0.57431
	0.000	*0.000*	*0.000*
$W_{IV}(1,12,\tau)$	0.5252	0.090457	-0.56329
	0.000	*0.000*	*0.000*
$W_{IV}(1,13,\tau)$	0.49403	0.09326	-0.56105
	0.000	*0.000*	*0.000*
$W_{IV}(1,14,\tau)$	0.41073	0.08511	-0.56987
	0.000	*0.000*	*0.000*
$W_{IV}(1,15,\tau)$	0.3606	0.08916	-0.5662
	0.000	*0.000*	*0.000*
$W_{IV}(1,16,\tau)$	0.3292	0.086432	-0.56912
	0.000	*0.000*	*0.000*
$W_{IV}(1,17,\tau)$	0.30412	0.080209	-0.57535
	0.000	*0.000*	*0.000*
$W_{IV}(1,18,\tau)$	0.27069	0.074814	-0.5807
	0.000	*0.000*	*0.000*
$W_{IV}(1,19,\tau)$	0.24245	0.06894	-0.58645
	0.000	*0.000*	*0.000*
$W_{IV}(1,20,\tau)$	0.2327	0.064932	-0.59037
	0.000	*0.000*	*0.000*
$W_{IV}(1,21,\tau)$	0.22566	0.060909	-0.5943
	0.000	*0.000*	*0.000*

Testing the IRs governing the slope factor			
	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.18589	0.033578	-0.62067
	0.000	*0.000*	*0.000*
$W_{IV}(2,2,\tau)$	0.33459	0.060201	-0.59436
	0.000	*0.000*	*0.000*
$W_{IV}(2,3,\tau)$	0.27183	0.059894	-0.59495
	0.000	*0.000*	*0.000*
$W_{IV}(2,4,\tau)$	0.79596	0.17493	-0.47831
	0.000	*0.000*	*0.000*
$W_{IV}(2,5,\tau)$	1.3499	0.27752	-0.36647
	0.000	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	1.6302	0.32541	-0.31064
	0.000	*0.002*	*0.001*
$W_{IV}(2,7,\tau)$	1.5131	0.29581	-0.34281
	0.000	*0.033*	*0.013*
$W_{IV}(2,8,\tau)$	1.5473	0.27801	-0.36074
	0.000	*0.036*	*0.021*
$W_{IV}(2,9,\tau)$	1.3915	0.23797	-0.40494
	0.000	*0.053*	*0.032*
$W_{IV}(2,10,\tau)$	1.1574	0.19343	-0.45422
	0.000	*0.077*	*0.060*
$W_{IV}(2,11,\tau)$	1.5084	0.27769	-0.37404
	0.278	*0.465*	*0.490*
$W_{IV}(2,12,\tau)$	2.5622	0.48654	-0.12241
	0.394	*0.478*	*0.514*
$W_{IV}(2,13,\tau)$	4.7042	0.87814	0.46817
	0.283	*0.443*	*0.471*
$W_{IV}(2,14,\tau)$	5.4103	1.0661	0.79033
	0.285	*0.431*	*0.426*
$W_{IV}(2,15,\tau)$	5.7059	1.2922	1.0658
	0.304	*0.384*	*0.375*
$W_{IV}(2,16,\tau)$	5.9104	1.265	0.99521
	0.201	*0.379*	*0.370*
$W_{IV}(2,17,\tau)$	5.421	1.1148	0.7536
	0.128	*0.384*	*0.379*
$W_{IV}(2,18,\tau)$	4.7358	0.95788	0.51804
	0.087	*0.397*	*0.389*
$W_{IV}(2,19,\tau)$	4.0563	0.81108	0.30857
	0.053	*0.402*	*0.397*
$W_{IV}(2,20,\tau)$	3.6786	0.75652	0.23197
	0.070	*0.400*	*0.402*
$W_{IV}(2,21,\tau)$	3.2564	0.69068	0.14226
	0.107	*0.402*	*0.406*

Testing the IRs governing the curvature factor			
	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	3.2792	0.092784	-0.50577
	0.352	*0.938*	*0.940*
$W_{IV}(3,2,\tau)$	4.8078	0.1234	-0.36822
	0.595	*0.942*	*0.940*
$W_{IV}(3,3,\tau)$	3.8986	0.13472	-0.44289
	0.468	*0.930*	*0.937*
$W_{IV}(3,4,\tau)$	2.5426	0.10303	-0.52279
	0.177	*0.922*	*0.920*
$W_{IV}(3,5,\tau)$	1.0262	0.067277	-0.58338
	0.141	*0.858*	*0.863*
$W_{IV}(3,6,\tau)$	0.15693	0.019444	-0.63427
	0.004	*0.165*	*0.163*
$W_{IV}(3,7,\tau)$	0.31938	0.009214	-0.64412
	0.029	*0.933*	*0.933*
$W_{IV}(3,8,\tau)$	1.3524	0.025804	-0.62289
	0.185	*0.949*	*0.948*
$W_{IV}(3,9,\tau)$	2.0552	0.044823	-0.59434
	0.155	*0.946*	*0.945*
$W_{IV}(3,10,\tau)$	2.0115	0.053352	-0.5844
	0.124	*0.940*	*0.939*
$W_{IV}(3,11,\tau)$	0.23199	0.06061	-0.59451
	0.000	*0.000*	*0.000*
$W_{IV}(3,12,\tau)$	0.50206	0.058314	-0.59496
	0.000	*0.000*	*0.000*
$W_{IV}(3,13,\tau)$	0.53585	0.046009	-0.60558
	0.000	*0.000*	*0.000*
$W_{IV}(3,14,\tau)$	0.077917	0.011581	-0.64201
	0.004	*0.102*	*0.102*
$W_{IV}(3,15,\tau)$	0.20318	0.038946	-0.61529
	0.000	*0.004*	*0.004*
$W_{IV}(3,16,\tau)$	0.39983	0.069968	-0.58369
	0.000	*0.000*	*0.000*
$W_{IV}(3,17,\tau)$	0.52036	0.089416	-0.56384
	0.000	*0.000*	*0.000*
$W_{IV}(3,18,\tau)$	0.56755	0.097685	-0.55579
	0.000	*0.000*	*0.000*
$W_{IV}(3,19,\tau)$	0.58345	0.10417	-0.54988
	0.000	*0.000*	*0.000*
$W_{IV}(3,20,\tau)$	0.55312	0.10467	-0.54966
	0.000	*0.000*	*0.000*
$W_{IV}(3,21,\tau)$	0.51483	0.10014	-0.55433
	0.000	*0.000*	*0.000*

Table 4: Testing Results for Federal Reserve Daily Bond Yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 6 - 11. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.21788	0.024133	-1.8342
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.080904	0.009637	-1.8837
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.23128	0.026217	-1.8272
	0.000	*0.000*	*0.000*
$W_V(1,\tau)$	0.24585	0.029767	-1.8151
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.069336	0.006697	-1.8936
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	248.12	26.206	117.61
	0.753	*0.604*	*0.753*
$W_V(2,\tau)$	1.9809	0.21744	-1.1462
	0.363	*0.052*	*0.058*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.073677	0.007799	-1.8899
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	0.093286	0.007266	-1.8917
	0.000	*0.000*	*0.000*
$W_V(3,\tau)$	0.18308	0.016498	-1.8602
	0.000	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.074561	0.006958	-1.8927
	0.000	*0.000*	*0.000*
$W_{IV}(1,2,\tau)$	0.15959	0.015836	-1.8625
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.096229	0.009853	-1.8829
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.054365	0.005447	-1.8979
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.1079	0.011748	-1.8765
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.12296	0.013145	-1.8717
	0.000	*0.000*	*0.000*
$W_{IV}(1,7,\tau)$	0.074699	0.007382	-1.8913
	0.000	*0.000*	*0.000*
$W_{IV}(1,8,\tau)$	0.046066	0.004125	-1.9024
	0.000	*0.000*	*0.000*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	2.3662	0.26583	-0.97505
	0.000	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	0.41223	0.046288	-1.7586
	0.000	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	2.182	0.2569	-1.013
	0.000	*0.000*	*0.000*

Testing the IRs governing the slope factor			
	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	3.4434	0.37308	-0.55909
	0.009	*0.005*	*0.005*
$W_{IV}(2,2,\tau)$	1.2778	0.1228	-1.4827
	0.026	*0.064*	*0.067*
$W_{IV}(2,3,\tau)$	0.28623	0.026416	-1.826
	0.002	*0.025*	*0.023*
$W_{IV}(2,4,\tau)$	0.047793	0.00537	-1.8982
	0.698	*0.225*	*0.227*
$W_{IV}(2,5,\tau)$	0.27393	0.031617	-1.8085
	0.646	*0.156*	*0.164*
$W_{IV}(2,6,\tau)$	0.85967	0.098243	-1.5771
	0.604	*0.120*	*0.143*
$W_{IV}(2,7,\tau)$	0.97655	0.11195	-1.529
	0.578	*0.111*	*0.129*
$W_{IV}(2,8,\tau)$	0.8575	0.09659	-1.5828
	0.536	*0.096*	*0.109*

Testing the IRs governing the curvature factor			
	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.014698	0.001279	-1.9121
	0.014	*0.001*	*0.001*
$W_{IV}(3,2,\tau)$	0.075935	0.004772	-1.9002
	0.000	*0.000*	*0.000*
$W_{IV}(3,3,\tau)$	0.006669	0.000508	-1.9147
	0.130	*0.050*	*0.050*
$W_{IV}(3,4,\tau)$	0.038835	0.002895	-1.9066
	0.000	*0.000*	*0.000*
$W_{IV}(3,5,\tau)$	0.03513	0.003013	-1.9062
	0.000	*0.000*	*0.000*
$W_{IV}(3,6,\tau)$	0.002986	8.39E-05	-1.9162
	0.073	*0.208*	*0.208*
$W_{IV}(3,7,\tau)$	0.014231	0.001109	-1.9127
	0.001	*0.001*	*0.001*
$W_{IV}(3,8,\tau)$	0.0285	0.002491	-1.908
	0.000	*0.000*	*0.000*

Table 5: Testing Results for Federal Reserve Monthly Bond Yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 6 - 11. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.28311	0.62693	1.257
	0.006	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.125	0.30613	1.2065
	0.002	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.28359	0.62246	1.2565
	0.081	*0.008*	*0.008*
$W_V(1,\tau)$	0.34216	0.82941	1.2888
	0.003	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.069375	0.15996	1.1836
	0.242	*0.015*	*0.015*
$W_{III}(2,\tau)$	0.51744	0.8568	1.2935
	0.006	*0.002*	*0.002*
$W_V(2,\tau)$	0.29807	0.77128	1.2797
	0.068	*0.001*	*0.001*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.10018	0.16084	1.1837
	0.044	*0.004*	*0.004*
$W_{III}(3,\tau)$	0.21639	0.48289	1.2344
	0.131	*0.008*	*0.008*
$W_V(3,\tau)$	0.24129	0.56157	1.2467
	0.036	*0.001*	*0.001*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.084724	0.14268	1.1809
	0.300	*0.120*	*0.121*
$W_{IV}(1,2,\tau)$	0.09609	0.15852	1.1834
	0.125	*0.040*	*0.040*
$W_{IV}(1,3,\tau)$	0.056443	0.078843	1.1708
	0.054	*0.026*	*0.026*
$W_{IV}(1,4,\tau)$	0.07872	0.13886	1.1803
	0.091	*0.021*	*0.021*
$W_{IV}(1,5,\tau)$	0.18757	0.33367	1.211
	0.006	*0.001*	*0.001*
$W_{IV}(1,6,\tau)$	0.087837	0.15356	1.1826
	0.143	*0.044*	*0.045*
$W_{IV}(1,7,\tau)$	0.044407	0.064474	1.1686
	0.341	*0.192*	*0.192*
$W_{IV}(1,8,\tau)$	0.021252	0.01793	1.1612
	0.737	*0.695*	*0.696*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	0.6136	1.6169	1.4127
	0.008	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	0.50214	1.3229	1.3664
	0.004	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	0.49117	1.3041	1.3634
	0.023	*0.000*	*0.000*

Testing the IRs governing the slope factor				Testing the IRs governing the curvature factor			
	Sup	*Avg*	*Exp*		*Sup*	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.059979	0.12319	1.1778	$W_{IV}(3,1,\tau)$	0.013676	0.014301	1.1607
	0.073	*0.008*	*0.008*		*0.700*	*0.514*	*0.515*
$W_{IV}(2,2,\tau)$	0.10178	0.19236	1.1887	$W_{IV}(3,2,\tau)$	0.070769	0.1332	1.1794
	0.064	*0.013*	*0.013*		*0.100*	*0.020*	*0.020*
$W_{IV}(2,3,\tau)$	0.27441	0.5948	1.2523	$W_{IV}(3,3,\tau)$	0.13232	0.10164	1.1745
	0.006	*0.000*	*0.000*		*0.002*	*0.026*	*0.026*
$W_{IV}(2,4,\tau)$	0.073562	0.12782	1.1785	$W_{IV}(3,4,\tau)$	0.049983	0.025846	1.1625
	0.135	*0.036*	*0.036*		*0.133*	*0.375*	*0.374*
$W_{IV}(2,5,\tau)$	0.052661	0.074805	1.1702	$W_{IV}(3,5,\tau)$	0.038806	0.029134	1.163
	0.252	*0.129*	*0.130*		*0.368*	*0.363*	*0.363*
$W_{IV}(2,6,\tau)$	0.021918	0.028842	1.163	$W_{IV}(3,6,\tau)$	0.042201	0.045078	1.1655
	0.742	*0.553*	*0.553*		*0.264*	*0.231*	*0.231*
$W_{IV}(2,7,\tau)$	0.011033	0.004199	1.1591	$W_{IV}(3,7,\tau)$	0.043421	0.070141	1.1694
	0.842	*0.947*	*0.947*		*0.160*	*0.065*	*0.065*
$W_{IV}(2,8,\tau)$	0.008558	0.002216	1.1588	$W_{IV}(3,8,\tau)$	0.015648	0.014422	1.1607
	0.878	*0.991*	*0.991*		*0.528*	*0.430*	*0.430*

Table 6: Testing Results for Fama Bliss Monthly Bond Yields. The table reports the *Sup*, *Avg*, and *Exp* values for the test statistics $W_I(\tau), W_{II}(i,\tau), W_{III}(i,\tau), W_{IV}(i,\tau), W_V(i,\tau)$, and $W_{VI}(i,\tau)$ associated with the six hypotheses formulated in equations 6 - 11. The p-values are reported in italics.

Testing overall system

	Sup	*Avg*	*Exp*
$W_I(\tau)$	0.47223	0.70067	1.0435
	0.000	*0.000*	*0.000*

Testing the level factor

	Sup	*Avg*	*Exp*
$W_{II}(1,\tau)$	0.18327	0.26953	0.95232
	0.000	*0.000*	*0.000*
$W_{III}(1,\tau)$	0.70166	1.0604	1.1158
	0.007	*0.002*	*0.002*
$W_V(1,\tau)$	0.62397	1.0427	1.1123
	0.000	*0.000*	*0.000*

Testing the slope factor

	Sup	*Avg*	*Exp*
$W_{II}(2,\tau)$	0.115	0.15889	0.92948
	0.000	*0.000*	*0.000*
$W_{III}(2,\tau)$	0.53077	0.74771	1.0503
	0.013	*0.004*	*0.005*
$W_V(2,\tau)$	0.41481	0.73693	1.0481
	0.005	*0.000*	*0.000*

Testing the curvature factor

	Sup	*Avg*	*Exp*
$W_{II}(3,\tau)$	0.19333	0.27225	0.95306
	0.000	*0.000*	*0.000*
$W_{III}(3,\tau)$	0.42918	0.64173	1.0282
	0.338	*0.146*	*0.159*
$W_V(3,\tau)$	0.48997	0.82029	1.066
	0.064	*0.003*	*0.004*

Testing the common factors

	Sup	*Avg*	*Exp*
$W_{VI}(1,2,\tau)$	0.92379	1.6091	1.23
	0.003	*0.000*	*0.000*
$W_{VI}(1,3,\tau)$	1.0419	1.7968	1.2706
	0.031	*0.000*	*0.000*
$W_{VI}(2,3,\tau)$	1.164	2.0277	1.3225
	0.021	*0.000*	*0.000*

Testing the IRs governing the level factor

	Sup	*Avg*	*Exp*
$W_{IV}(1,1,\tau)$	0.28486	0.38651	0.97683
	0.000	*0.000*	*0.000*
$W_{IV}(1,2,\tau)$	0.34409	0.45176	0.99064
	0.000	*0.000*	*0.000*
$W_{IV}(1,3,\tau)$	0.43058	0.43963	0.98835
	0.000	*0.000*	*0.000*
$W_{IV}(1,4,\tau)$	0.27042	0.3326	0.96543
	0.000	*0.000*	*0.000*
$W_{IV}(1,5,\tau)$	0.18172	0.20256	0.93849
	0.000	*0.000*	*0.000*
$W_{IV}(1,6,\tau)$	0.11036	0.11305	0.92002
	0.025	*0.016*	*0.015*
$W_{IV}(1,7,\tau)$	0.079312	0.05133	0.90737
	0.012	*0.051*	*0.050*
$W_{IV}(1,8,\tau)$	0.027267	0.010379	0.89899
	0.289	*0.529*	*0.529*
$W_{IV}(1,9,\tau)$	0.015298	0.012722	0.89946
	0.732	*0.537*	*0.537*
$W_{IV}(1,10,\tau)$	0.16416	0.18611	0.93522
	0.000	*0.001*	*0.001*
$W_{IV}(1,11,\tau)$	0.28215	0.37595	0.97472
	0.000	*0.000*	*0.000*
$W_{IV}(1,12,\tau)$	0.39883	0.59148	1.0199
	0.000	*0.000*	*0.000*
$W_{IV}(1,13,\tau)$	0.34209	0.49583	0.9992
	0.000	*0.000*	*0.000*
$W_{IV}(1,14,\tau)$	0.32182	0.4332	0.98599
	0.000	*0.000*	*0.000*
$W_{IV}(1,15,\tau)$	0.30738	0.42135	0.98373
	0.000	*0.000*	*0.000*
$W_{IV}(1,16,\tau)$	0.28336	0.33451	0.96568
	0.000	*0.000*	*0.000*
$W_{IV}(1,17,\tau)$	0.31334	0.28299	0.95532
	0.000	*0.000*	*0.000*
$W_{IV}(1,18,\tau)$	0.17546	0.14729	0.92717
	0.000	*0.001*	*0.001*
$W_{IV}(1,19,\tau)$	0.23686	0.10196	0.9179
	0.000	*0.018*	*0.018*
$W_{IV}(1,20,\tau)$	0.19447	0.080092	0.91334
	0.000	*0.036*	*0.033*
$W_{IV}(1,21,\tau)$	0.14803	0.063431	0.90988
	0.001	*0.041*	*0.040*

Testing the IRs governing the slope factor			
	Sup	*Avg*	*Exp*
$W_{IV}(2,1,\tau)$	0.015325	0.019611	0.90087
	0.565	*0.221*	*0.221*
$W_{IV}(2,2,\tau)$	0.012791	0.007502	0.8984
	0.621	*0.590*	*0.590*
$W_{IV}(2,3,\tau)$	0.083969	0.12711	0.92286
	0.014	*0.001*	*0.001*
$W_{IV}(2,4,\tau)$	0.10422	0.11245	0.9199
	0.002	*0.000*	*0.000*
$W_{IV}(2,5,\tau)$	0.16548	0.22846	0.94382
	0.001	*0.000*	*0.000*
$W_{IV}(2,6,\tau)$	0.13564	0.1945	0.93679
	0.001	*0.000*	*0.000*
$W_{IV}(2,7,\tau)$	0.12321	0.19624	0.93706
	0.005	*0.001*	*0.001*
$W_{IV}(2,8,\tau)$	0.18675	0.22926	0.94386
	0.000	*0.000*	*0.000*
$W_{IV}(2,9,\tau)$	0.12094	0.15915	0.92936
	0.004	*0.001*	*0.001*
$W_{IV}(2,10,\tau)$	0.10936	0.16118	0.92984
	0.003	*0.003*	*0.003*
$W_{IV}(2,11,\tau)$	0.08732	0.12848	0.92316
	0.016	*0.004*	*0.004*
$W_{IV}(2,12,\tau)$	0.068445	0.083145	0.91388
	0.027	*0.008*	*0.008*
$W_{IV}(2,13,\tau)$	0.04162	0.039435	0.90492
	0.101	*0.064*	*0.064*
$W_{IV}(2,14,\tau)$	0.058328	0.060814	0.9093
	0.049	*0.036*	*0.036*
$W_{IV}(2,15,\tau)$	0.028223	0.018894	0.90073
	0.170	*0.192*	*0.192*
$W_{IV}(2,16,\tau)$	0.051298	0.037002	0.90443
	0.012	*0.045*	*0.045*
$W_{IV}(2,17,\tau)$	0.008739	0.002974	0.89747
	0.770	*0.894*	*0.894*
$W_{IV}(2,18,\tau)$	0.006821	0.00286	0.89745
	0.889	*0.911*	*0.911*
$W_{IV}(2,19,\tau)$	0.016391	0.008169	0.89853
	0.451	*0.497*	*0.497*
$W_{IV}(2,20,\tau)$	0.011784	0.012301	0.89937
	0.629	*0.324*	*0.324*
$W_{IV}(2,21,\tau)$	0.011639	0.009328	0.89877
	0.605	*0.416*	*0.416*

Testing the IRs governing the curvature factor			
	Sup	*Avg*	*Exp*
$W_{IV}(3,1,\tau)$	0.004254	0.001733	0.89722
	0.963	*0.969*	*0.969*
$W_{IV}(3,2,\tau)$	0.01785	0.019643	0.90087
	0.309	*0.108*	*0.110*
$W_{IV}(3,3,\tau)$	0.047549	0.051524	0.90739
	0.064	*0.014*	*0.014*
$W_{IV}(3,4,\tau)$	0.029769	0.038349	0.9047
	0.305	*0.084*	*0.084*
$W_{IV}(3,5,\tau)$	0.062465	0.048166	0.90671
	0.107	*0.092*	*0.092*
$W_{IV}(3,6,\tau)$	0.028814	0.027685	0.90251
	0.393	*0.209*	*0.209*
$W_{IV}(3,7,\tau)$	0.024056	0.02107	0.90117
	0.566	*0.374*	*0.374*
$W_{IV}(3,8,\tau)$	0.04849	0.023871	0.90176
	0.101	*0.203*	*0.202*
$W_{IV}(3,9,\tau)$	0.011174	0.007633	0.89842
	0.716	*0.602*	*0.603*
$W_{IV}(3,10,\tau)$	0.022095	0.021407	0.90123
	0.682	*0.456*	*0.461*
$W_{IV}(3,11,\tau)$	0.008244	0.004866	0.89786
	0.929	*0.873*	*0.873*
$W_{IV}(3,12,\tau)$	0.023356	0.008579	0.89862
	0.397	*0.608*	*0.608*
$W_{IV}(3,13,\tau)$	0.057081	0.016472	0.90024
	0.341	*0.665*	*0.663*
$W_{IV}(3,14,\tau)$	0.030502	0.01019	0.89895
	0.534	*0.743*	*0.742*
$W_{IV}(3,15,\tau)$	0.039536	0.015681	0.90008
	0.325	*0.495*	*0.494*
$W_{IV}(3,16,\tau)$	0.046383	0.030354	0.90307
	0.235	*0.226*	*0.226*
$W_{IV}(3,17,\tau)$	0.030402	0.021695	0.9013
	0.330	*0.258*	*0.258*
$W_{IV}(3,18,\tau)$	0.12516	0.031969	0.90344
	0.027	*0.409*	*0.408*
$W_{IV}(3,19,\tau)$	0.021461	0.020342	0.90102
	0.352	*0.231*	*0.231*
$W_{IV}(3,20,\tau)$	0.029471	0.026966	0.90237
	0.308	*0.210*	*0.211*
$W_{IV}(3,21,\tau)$	0.016953	0.008696	0.89864
	0.441	*0.520*	*0.520*

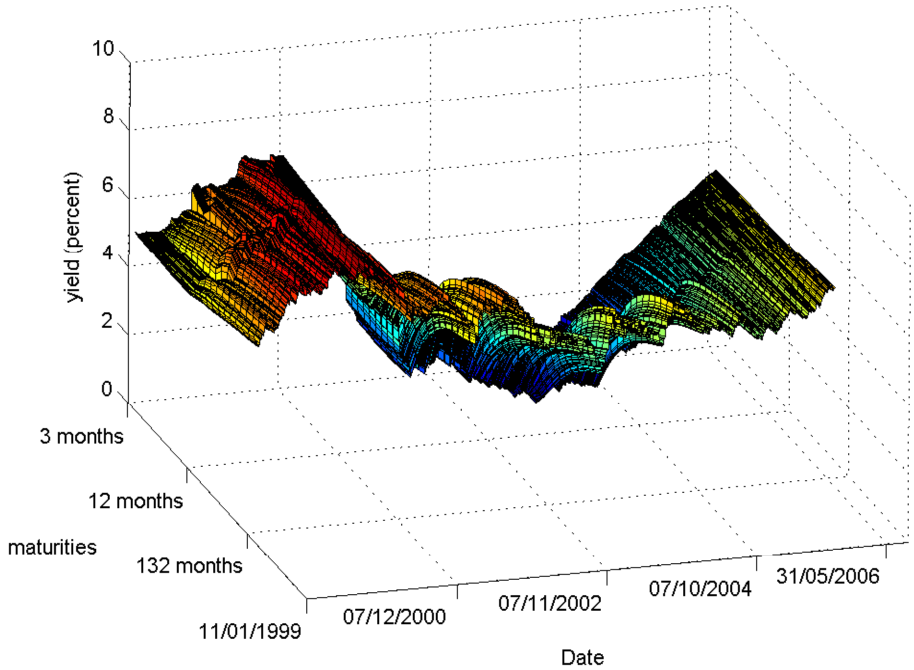

Figure 1: Zero Coupon Bond Yield Curves from January 1999 to May 2006

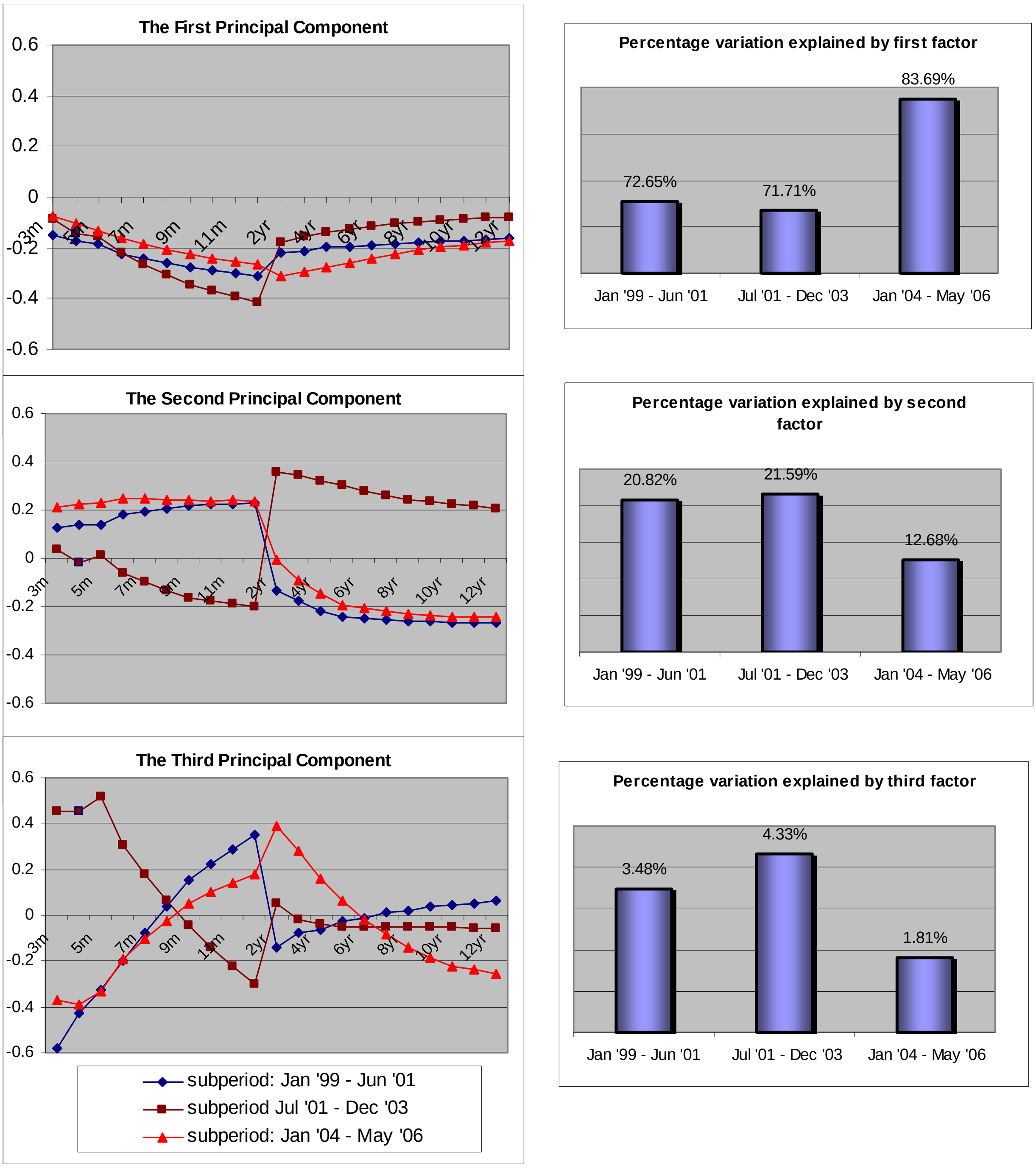

Figure 2: Three Subsample Periods. The line charts plot the first three principal components and the column charts show the percentage variations explained by those factors for the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.

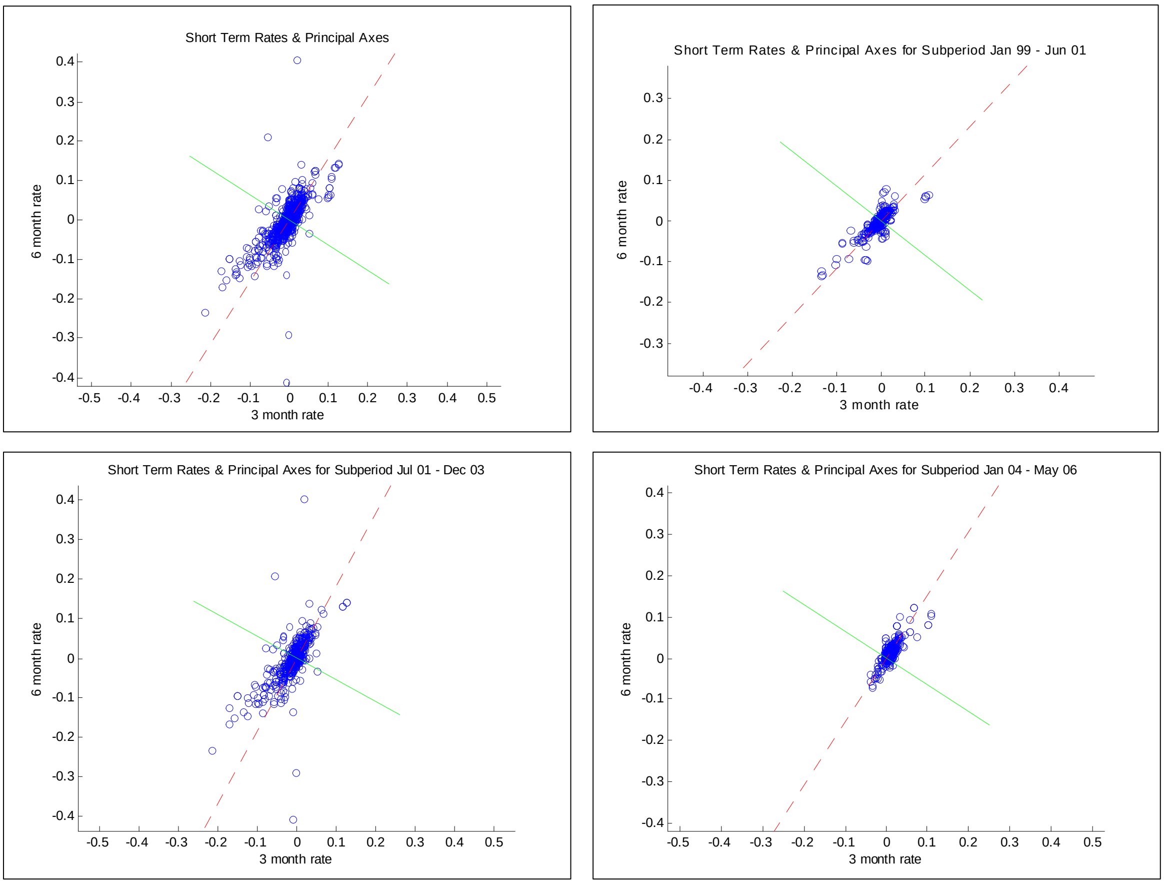

Figure 3: Short Term Rates (Yield Changes) and Principal Axes. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the three month and six month yield changes data that proxies the short end of the yield curve.

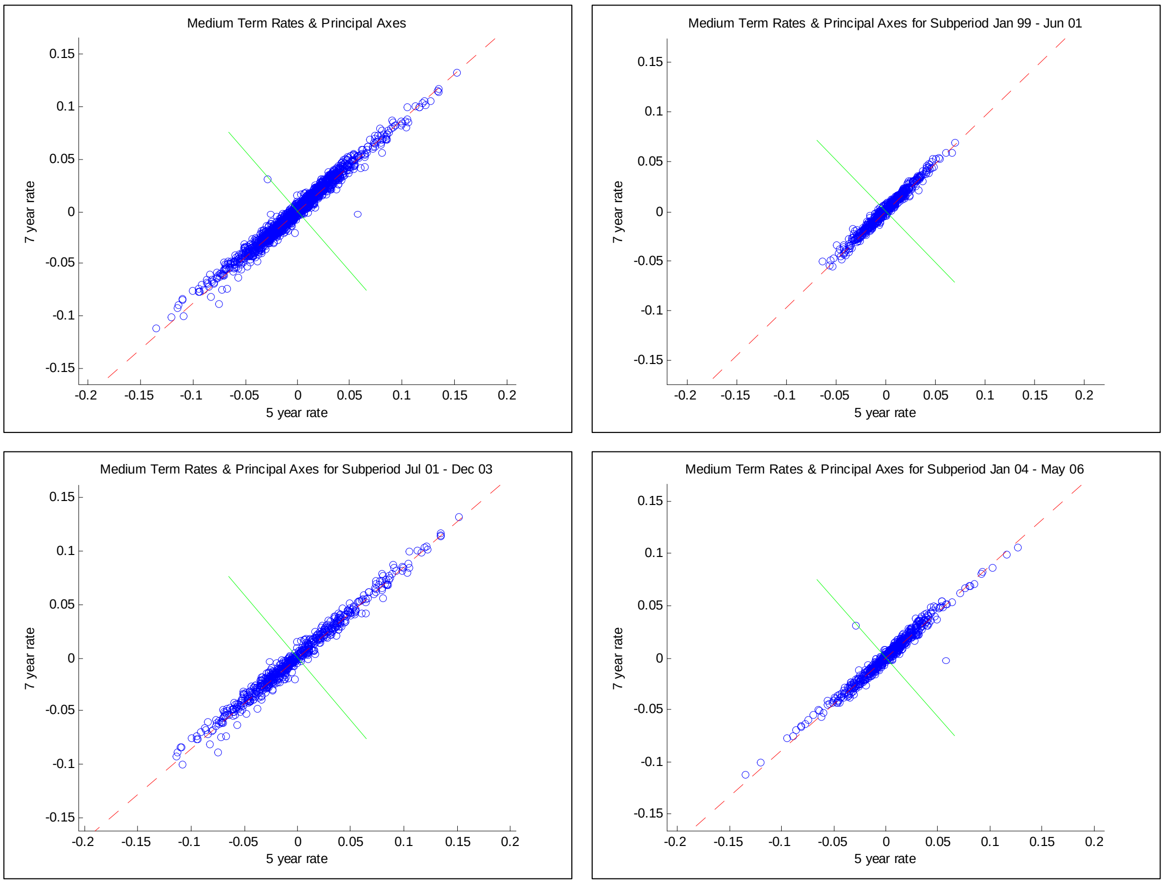

Figure 4: Medium Term Rates (Yield Changes) and Principal Axes. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the five year and seven year yield changes data that proxies the medium end of the yield curve.

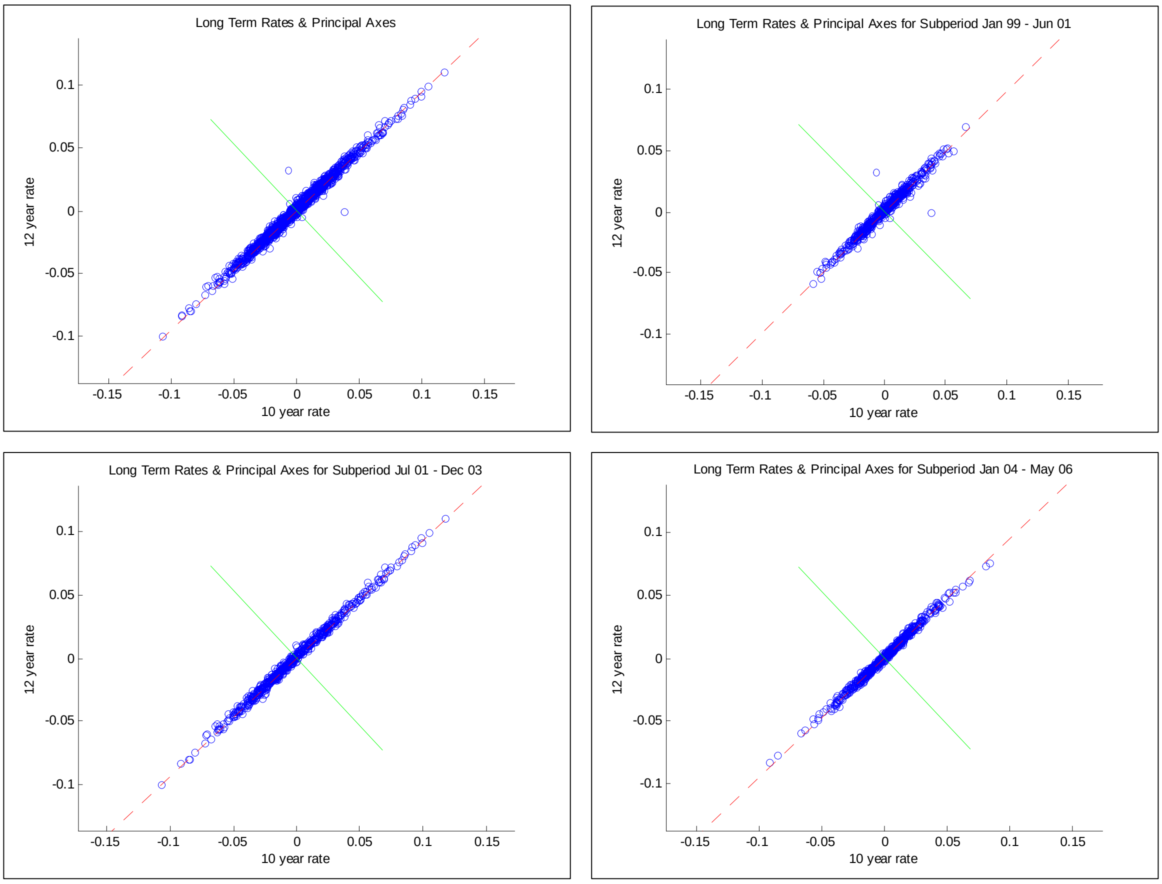

Figure 5: Long Term Rates (Yield Changes) and Principal Axes. The dashed line depicts the first principal axis and the continuous line depicts the second principal axis of the short rates considered for the whole sample period and the three subsample periods: Jan 1999 - Jun 2001, Jul 2001 - Dec 2003, and Jan 2004 - May 2006.. The two orthogonal axes are fitted onto the scatter plot of the ten year and twelve year yield changes data that proxies the long end of the yield curve.

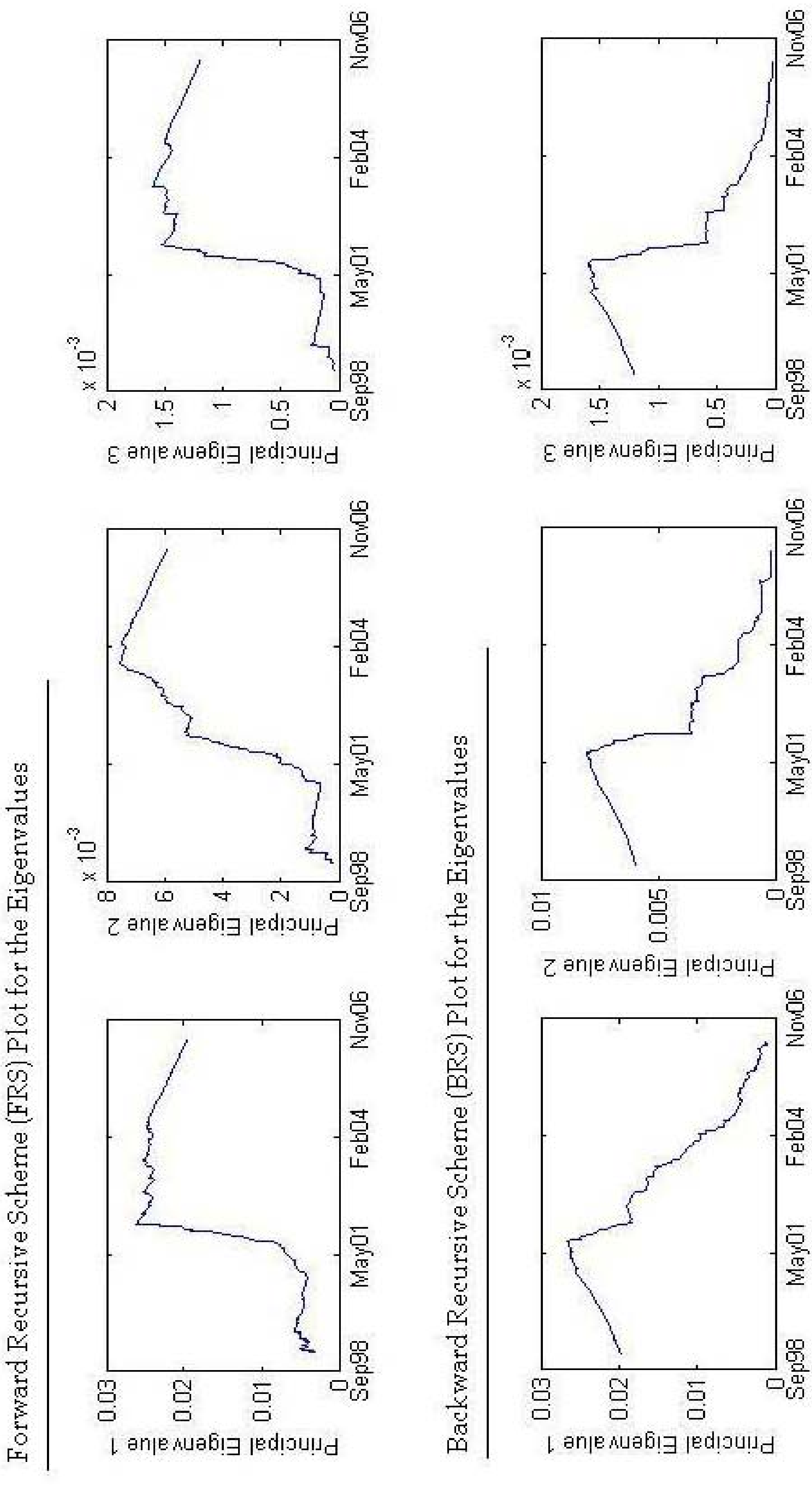

Figure 6: Forward Recursive Scheme and Backward Recursive Scheme of the Eigenvalues.

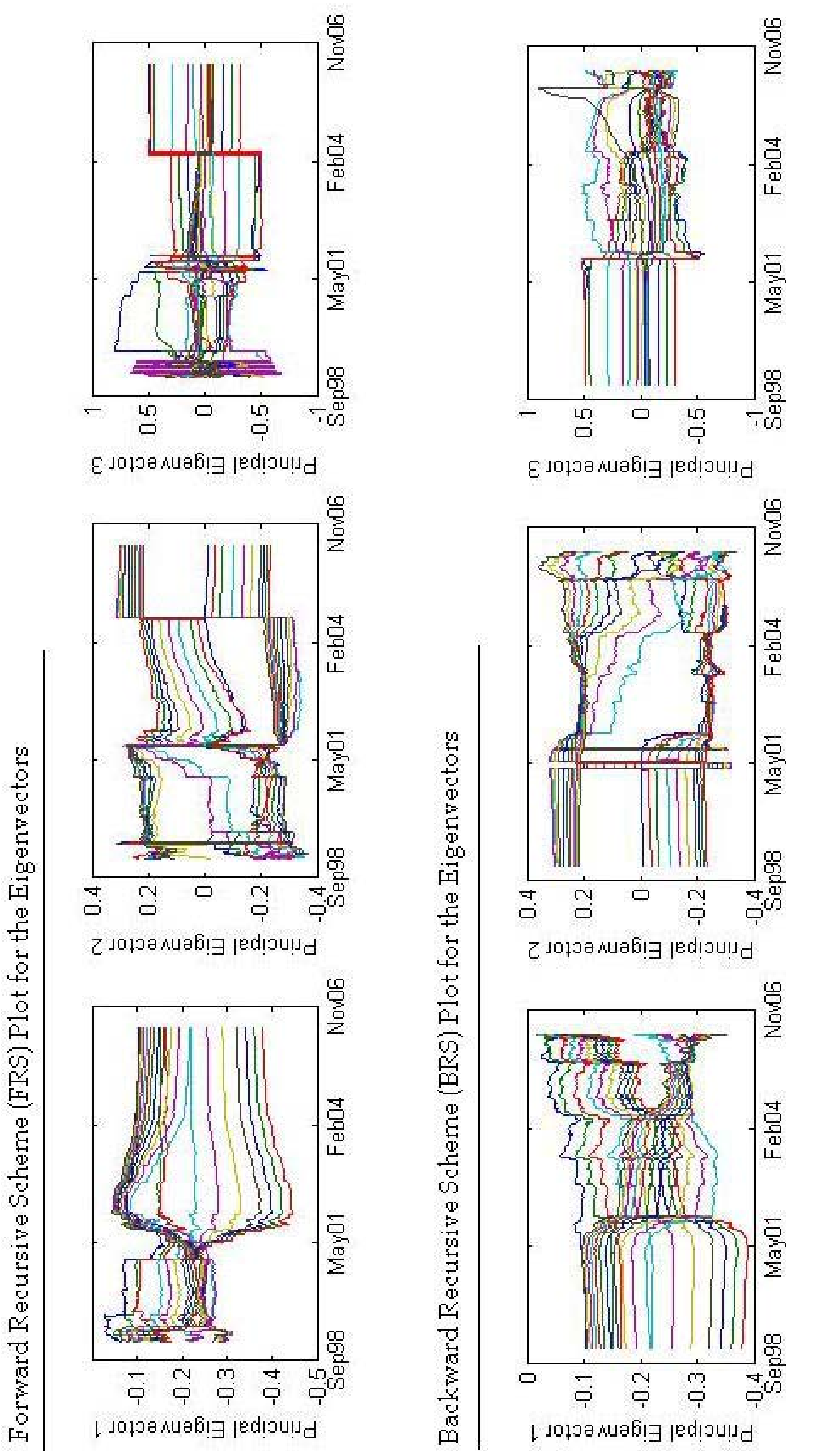

Figure 7: Forward Recursive Scheme and Backward Recursive Scheme of the Eigenvectors.

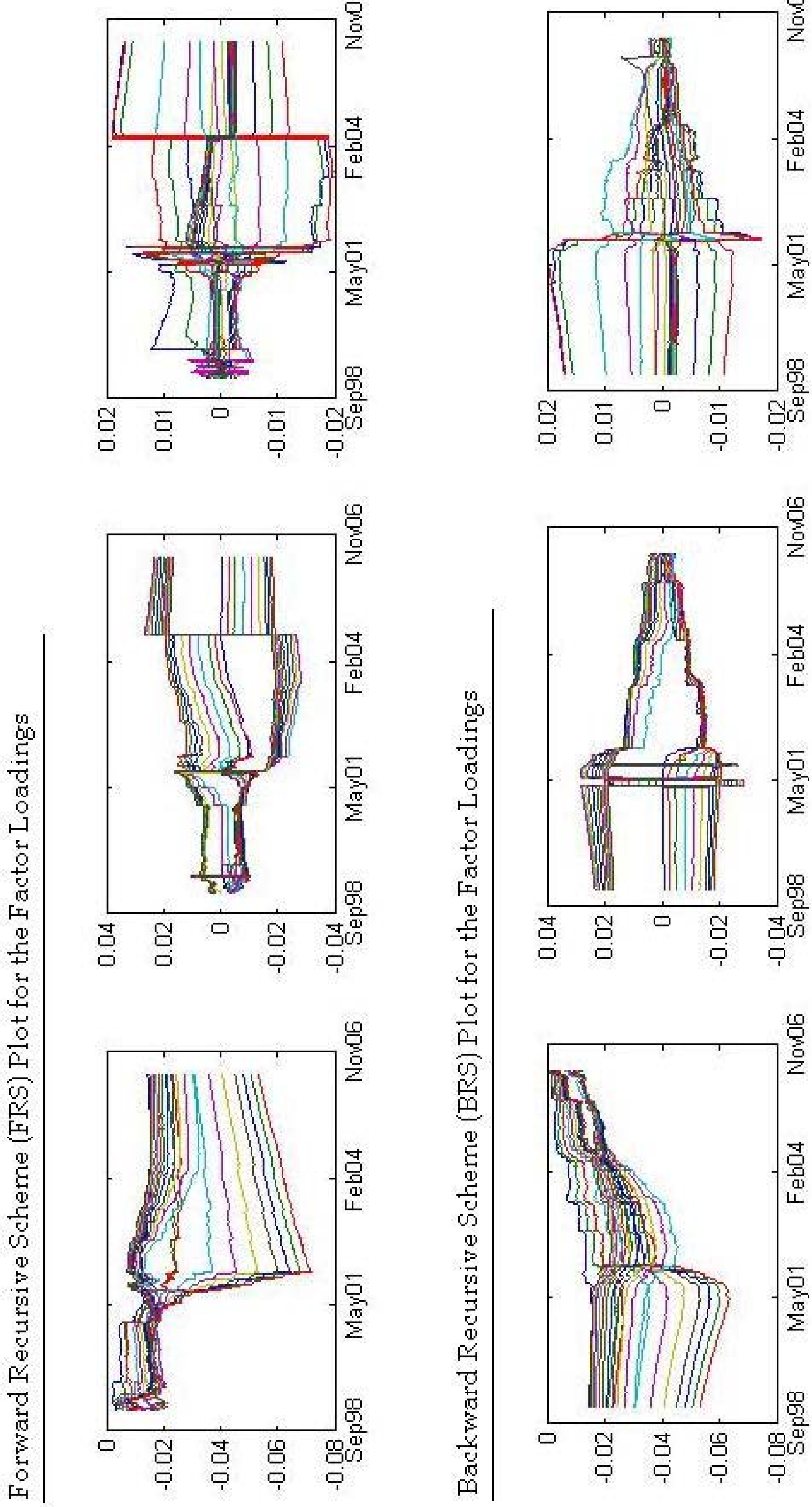

Figure 8: Forward Recursive Scheme and Backward Recursive Scheme of the Factor Loadings.