Joint survival annuity derivative valuation in a linear-rational Wishart mortality model

José Da Fonseca[†] and **Patrick Wong**[⋆]

Longevity 19 – September 2024

[†] Auckland University of Technology

[⋆] UNSW Sydney

Outline

1. Setup
2. Model
3. Pricing framework
4. Key results
5. Numerical experiments

Setup

Requisite setup

Suppose we have two individuals x and y.

- Future lifetimes: τ_x and τ_y.
- Mortality intensities: $(\mu_x(t))_{t\geq 0}$ and $(\mu_y(t))_{t\geq 0}$.

Consider their joint survival probability:

$$\mathbb{Q}(\tau_x > T, \tau_y > T | \mathcal{G}_t) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \mu_x(s)+\mu_y(s)ds}\right],$$

where $\mathcal{G}_t$ is an appropriate filtration and $\mathbb{E}_t^{\mathbb{Q}}[\cdot] = \mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{G}_t]$

- Can we price annuities or guaranteed annuity options on their joint survival?

Instruments of Interest

1. Denote a joint survival bond purchased at t payable at T by $SB(t, T)$:
$$SB(t,T) = P(t,T)\mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \mu_x(s)+\mu_y(s)ds}\right],$$
where $P(t, T)$ is the discounting factor.
2. A joint survival annuity purchased at t with payables at $\{T_i\}_{i=1}^N$ is hence given by
$$\sum_{i=1}^{N} SB(t, T_i).$$
3. The guaranteed joint survival annuity option (GAO) is:
$$\overline{C}\,(0,T,T_N) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^T \mu_x(s)+\mu_y(s)ds}\left(\sum_{i=1}^{N} SB(T,T_i) - 1/g\right)_+\right]$$

- We will assume for this work that $r = 0$, and so $P(t, T) = 1$ for all $t, T \geq 0$.

Challenges

Previous models struggled in either one or more ways:

1. The future value of the (joint) survival annuity does not admit a known density.
    - The joint survival bond is an exponential affine function of the process.
2. Having an exact price of the GAO.
    - Impossible if we do not have the density of the future value of the joint survival annuity.
3. Constraints in the way we can correlate state variables. See Duffie et al. (2003) for vector affine processes.
4. Positivity of the mortality intensities.

Our model handles all of these problems.

Model

Wishart process

We aim to price the instruments outlined before using a Wishart process:

$$dv_t = (\omega + mv_t + v_t m^\top)dt + \sqrt{v_t}dw_t\sigma + \sigma^\top dw_t^\top \sqrt{v_t}\,,$$

- v_t and σ are a $n \times n$ positive-definite matrices.
- We use the Bru (1991) case: $\omega = \beta\sigma^2$ with $\beta \geq n + 1$.
- m is a $n \times n$ real matrix with the real component of its eigenvalues being negative.
- $\{w_t;\, t \geq 0\}$ is a $n \times n$ matrix of independent Brownian motions.

Covariation between components

In the case $n = 2$ we can find

$$d\langle v_{11,\cdot}, v_{11,\cdot}\rangle_t = 4v_{11,t}\left(\sigma_{11}^2 + \sigma_{12}^2\right)$$
$$d\langle v_{22,\cdot}, v_{22,\cdot}\rangle_t = 4v_{22,t}\left(\sigma_{22}^2 + \sigma_{12}^2\right)$$
$$d\langle v_{11,\cdot}, v_{22,\cdot}\rangle_t = 4v_{12,t}\,\sigma_{12}\left(\sigma_{11} + \sigma_{22}\right)$$

The correlation between the state variables $v_{11,t}$ and $v_{22,t}$ are primarily driven by σ_{12} and will be zero if $\sigma_{12} = 0$.

Moment generating function

Wishart process is affine. MGF is exponentially affine:

$$\Phi(t,\theta_1,\theta_2,v)=\mathbb{E}\left[\exp\left(\text{tr}[\theta_1 v_t]+\int_0^t \text{tr}[\theta_2 v_s]ds\right)\right],$$

where $\theta_1,\ \theta_2$ are real symmetric matrices, and so

$$\Phi(t,\theta_1,\theta_2,v_0)=\exp\left(\text{tr}[a(t,\theta_1,\theta_2)v_0]+b(t,\theta_1,\theta_2)\right).$$

- $a(t,\theta_1,\theta_2)$ is a deterministic matrix Riccati ODE.
- $b(t,\theta_1,\theta_2)$ is a deterministic scalar solution dependent on a.

Pricing framework

We want to obtain the price for

$$\mathrm{SB}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T \mu_x(s)+\mu_y(s)ds}\right].$$

Rogers (1997) gave a powerful recipe to obtain this price and also to recover the implied dynamics. This is done by specifying a **potential**.

1. Given a Markov process and its infinitesimal generator $\mathcal{L}$, define the following function g by

$$g = (\alpha - \mathcal{L})f,$$

where we choose a real number α and function $f > 0$ so that g is positive.

2. By choosing the probabilistic **potential** ζ_t (non-negative supermartingale and $\lim_{t\to\infty}\mathbb{E}[\zeta_t] = 0$) to be

$$\zeta_t := e^{-\alpha t} f(v_t),$$

we can use Rogers (1997) to set the joint survival bond price as:

$$\mathrm{SB}(t, T) = e^{-\alpha(T-t)}\,\mathbb{E}_t\left[\frac{\zeta_T}{\zeta_t}\right].$$

- Note, this means the bond price is under $\mathbb{P}$ and not $\mathbb{Q}$.

3. By assumption of our joint survival bond price being given by

$$\mathrm{SB}(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T (\mu_x(s)+\mu_y(s))ds}\right],$$

   then Rogers (1997) states that we can relate the mortality intensities and our model by

$$\mu_x(t) + \mu_y(t) = \frac{(\alpha - \mathcal{L})f(v_t)}{f(v_t)}.$$

Notes about our construction

Key takeaways from the potential approach of Rogers (1997):

- Our framework requires the specification of
  1. A Markov process v - In our case we use the Wishart process.
  2. The function

     $$g = (\alpha - \mathcal{L})f,$$

     where f is a positive function and α is chosen such that g is positive.
- As long as g and f is positive, then $\zeta_t := e^{-\alpha t} f(v_t)$ is a **potential**.
- The joint survival bond price depends on α and f.
- The joint survival bond price is priced under $\mathbb{P}$, not $\mathbb{Q}$.
- The choice of α and f influence the dynamics that we retrieve for $\mu_x(t) + \mu_y(t)$.

Key results

To use Rogers (1997), we require a suitable f and α.

Lemma 3.1 / 3.2

Let $u_0 \in \mathsf{M}(n)$ *then there exist a matrix function* $a_0(t) \in \mathsf{M}(n)$ *and a scalar function* $b_0(t)$ *such that*

$$\mathbb{E}\left[\mathrm{tr}[u_0 v_t]\right] = \mathrm{tr}[a_0(t) v_0] + b_0(t).$$

- **Lemma 3.2** is more practical to implement.

We set f to be

$$f(v_t) = 1 + \mathrm{tr}[u_0 v_t],$$

where $u_0 = u_1 + u_2$ and u_1, u_2 are chosen to be semi positive-definite matrices, which implies f is positive (as v_t is positive-definite).

Joint survival bond

If $f(v_t) = 1 + \text{tr}[u_0 v_t]$ and we assume (for now) there is an α such that

$$g(v_t) = (\alpha - \mathcal{L})f(v_t) > 0.$$

The price of the joint survival bond is given by **Proposition 4.1**:

$$\text{SB}(t,T) = e^{-\alpha(T-t)} \frac{1 + \mathbb{E}_t[\text{tr}[u_0 v_T]]}{1 + \text{tr}[u_0 v_t]},$$

where we use **Lemma 3.2** to find the numerator.

- Joint survival bond is a linear-rational function of the process.
- Joint survival bonds with different maturities have the same denominator, it means the annuity is also a linear-rational function of the process.

Implied mortality dynamics

If $n = 2$, m diagonal, $u_1 = e_{11}$, $u_2 = e_{22}$ and $f(v_t) = 1 + \text{tr}[u_0 v_t]$, then the implied mortality intensity dynamics are:

$$\mu_x(t) = \frac{\alpha/2 + \alpha v_{11,t} - \omega_{11} - 2m_{11} v_{11,t}}{1 + v_{11,t} + v_{22,t}},$$
$$\mu_y(t) = \frac{\alpha/2 + \alpha v_{22,t} - \omega_{22} - 2m_{22} v_{22,t}}{1 + v_{11,t} + v_{22,t}}.$$

These are positive as long as $\alpha > 2\max(\omega_{11}, \omega_{22})$ and their correlation is driven by $d\langle v_{11,.}, v_{22,.}\rangle_t$ and therefore by σ_{12}.

- Note, even if $\sigma_{12} = 0$, $\mu_x(t)$ and $\mu_y(t)$ will not be independent due to the common denominator.

Guaranteed joint survival annuity option

The guaranteed survival annuity price is given by **Proposition 5.1**

$$\overline{C}(0,T,T_N) = e^{-\alpha T} \frac{\mathbb{E}\left[\left(b_4(T,T_N) + \text{tr}[a_4(T,T_N)\, v_T]\right)_+\right]}{1+\text{tr}[u_0 v_0]}$$

where $b_4(T,T_N)$ and $a_4(T,T_N)$ admit a closed form.

If we define $Y_T = b_4(T,T_N) + \text{tr}[a_4(T,T_N)v_T]$ we can utilise the affine structure of v:

$$\begin{aligned}\Phi_Y(z) &= \mathbb{E}\left[e^{\mathrm{i}zY_T}\right]\\ &= e^{\mathrm{i}zb_4(T,T_N)}\Phi(T, \mathrm{i}za_4(T,T_N), 0_n, v_0)\end{aligned}$$

- Expectation can then be computed through a Fourier inversion.

Approximations are also provided:

- A Gaussian approximation.
- Spectral decomposition approximation.

OTHER RESULTS

We also have other results given in the paper:

- Limiting distribution of v_t.
- Distributions of the MGF under different restrictions (m, σ diagonal).
- The c.d.f and p.d.f of $\mu_x(t)$ and $\mu_y(t)$, and thus can compute its moments to potentially estimate our model.
- The density of the future value of the joint survival annuity which can be used for risk management of the annuity.
- Cumulants of $Y_T = b_4(T, T_N) + \text{tr}[a_4(T, T_N)v_T]$ and therefore can be used to develop statistical estimators.

Numerical experiments

A brief rundown of this section:

- Presentation of mortality intensity moments.
- Sensitivity analyses.
- GAO approximations.
- Dependence vs independence.

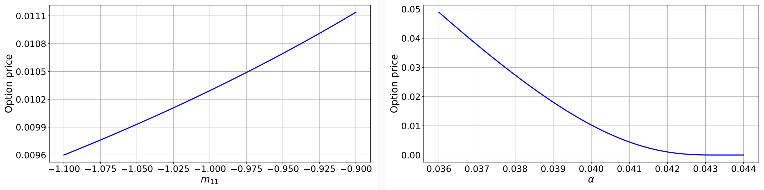
Mortality intensity moments

Table II: Moments of $\mu_x(t)$ and $\mu_y(t)$.

t	1	2	5	10
$\mathbb{E}[\mu_x(t)]$	0.0193	0.0199	0.0200	0.0200
$\mathrm{Var}(\mu_x(t))$	1.06×10^{-4}	1.20×10^{-4}	1.20×10^{-4}	1.20×10^{-4}
$\mathbb{E}[\mu_y(t)]$	0.0195	0.0198	0.0198	0.0198
$\mathrm{Var}(\mu_y(t))$	2.59×10^{-5}	2.83×10^{-5}	2.86×10^{-5}	2.86×10^{-5}

- Li et al. (2023) have excess mortality at 0.0132
- Xu et al. (2020) have expected mortality of around 0.0107.
- Correlation of $\mu_x(0)$ and $\mu_y(0)$ is 0.40.

Figure 1: Sensitivity analyses.

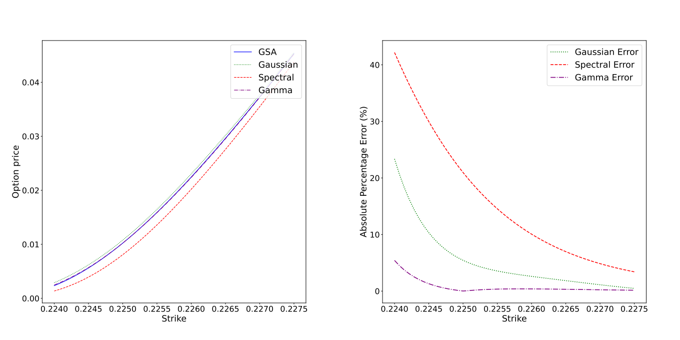


Recall

$$\mu_x(t) = \frac{\alpha/2 + \alpha v_{11,t} - \omega_{11} - 2m_{11}v_{11,t}}{1 + v_{11,t} + v_{22,t}}.$$

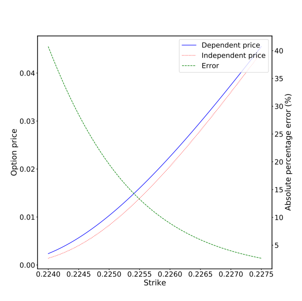
- If m_{11} decreases, then $\mu_x(t)$ and $\mu_y(t)$ both increase. More likely to die.
- If α increases, then $\mu_x(t)$ and $\mu_y(t)$ both decrease. More likely to survive.

Figure 2: Comparison of approximation methods and errors.



- All three approximations become more accurate as guaranteed rate g increases (more ITM).
- Gamma approximation is the most accurate and the Gaussian(Spectral) over(under)-estimates the GAO.

Figure 3: Pricing difference for independence and dependence assumption.



- For the dependent price we set $\sigma_{12} = 0$ since the correlation between $\mu_x(t)$ and $\mu_y(t)$ is driven by $d\langle v_{11,.}, v_{22,.}\rangle_t$.
- Logically consistent result: probability of both annuitants being alive is higher under positive dependence than independence.
- Impact of not accounting for dependency accurately can be economically significant (40%)!

CONCLUSION

The linear-rational Wishart mortality model:

- Is a highly analytical affine process in the space of positive-definite matrices.
- Allows us to compute the joint survival bond and annuity through the potential approach of Rogers (1997).
- Gives us the implied dynamics of the mortality intensities through Rogers (1997).
- Can provide an exact price of the GAO, and we further provide two approximations.

Some works in progress:

- Deriving the option Greeks and risk management measures on the GAO.
- Adapt the model to multi-population, multi-cohort or even cross-asset type problems.
- Implementation and calibration of the model.

Thank you!

p.a.wong@unsw.edu.au

References

M. F. Bru. Wishart processes. *Journal of Theoretical Probability*, 4(4): 725–751, 1991. doi: 10.1007/BF01259552.

D. Duffie, D. Filipović, and W. Schachermayer. Affine processes and applications in finance. *The Annals of Applied Probability*, 13(3): 984 – 1053, 2003. doi: 10.1214/aoap/1060202833.

H. Li, H. Liu, Q. Tang, and Z. Yuan. Pricing extreme mortality risk in the wake of the COVID-19 pandemic. *Insurance: Mathematics and Economics*, 108:84–106, 2023. doi: 10.1016/j.insmatheco.2022.11.002.

L. C. G. Rogers. The potential approach to the term structure of interest rates and foreign exchange rates. *Mathematical Finance*, 7 (2):157–176, 1997. doi: 10.1111/1467-9965.00029.

Y. Xu, M. Sherris, and J. Ziveyi. Continuous-time multi-cohort mortality modelling with affine processes. *Scandinavian Actuarial Journal*, 2020(6):526–552, 2020. doi: 10.1080/03461238.2019.1696223.