Efficient simulation of the SABR model

Jaehyuk Choi[1], Lilian Hu[2], and Yue Kuen Kwok[2]

[1]Mathematics of Finance (MAFN) Program, Columbia University
[2]Financial Technology Thrust, Hong Kong University of Science and Technology (Guangzhou)

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My research streams

SABR model

- Choi et al. (2019). Hyperbolic normal stochastic volatility model. *J. Futures Markets*. **Simulation for** $\beta = 0$.
- Choi and Wu (2021a). The equivalent CEV volatility of the SABR model. *JEDC*. **Analytic approximation**.
- Choi and Wu (2021b) A note on the option price and 'Mass at zero in the uncorrelated SABR model and implied volatility asymptotics.' *Quant. Finance*. **Numerical integral for** $\rho = 0$.
- Choi and Seo (2024). Option pricing under the normal SABR model with Gaussian quadratures. *J. Com. Finance*. **Numerical integral for** $\beta = 0$.

Simulation of stochastic volatility models

- Choi and Kwok (2024). Simulation schemes for the **Heston model** with Poisson conditioning. *EJOR*.
- Choi (2025). Exact simulation scheme for the **OUSV model** using the Karhunen-Loève Expansions. *ORL (Forthcoming)*.

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Stochastic-alpha-beta-rho (SABR) model

Volatility dynamics

$$\frac{d\sigma_t}{\sigma_t} = \nu \, dZ_t$$

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Forward price dynamics

$$\frac{dF_t}{F_t^{\beta}} = \sigma_t\, dW_t = \sigma_t\,(\rho dZ_t + \rho_* dX_t)$$

$$dW_t\, dZ_t = \rho \quad \text{or} \quad dX_t\, dZ_t = 0$$

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Useful coefficients

$\rho_* = \sqrt{1-\rho^2}$, $\beta_* := 1-\beta$, and $\hat{\nu} := \nu\sqrt{h}$ (h is time step)

- One of the most popular stochastic volatility (SV) models used in financial engineering (Hagan et al., 2002).
- A choice of *backbone* (volatility v.s. spot change dynamics): $0 \le \beta \le 1$
- Parsimonious and intuitive parameters to fit smile: σ_0, ρ, ν (and β)
- Asymptotic approximation for the equivalent BS volatility (HKLW formula): valid when $\nu\sqrt{T} \ll 1$.

HKLW formula (Hagan et al., 2002)

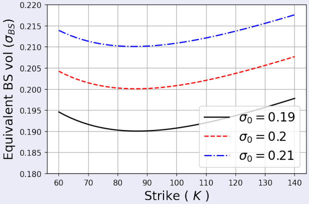
$$\sigma_{\text{BS}}(K) = \frac{\sigma_0}{F_0^{\beta_*}} \frac{H(\zeta)}{k^{\beta_*/2}} \frac{1 + \left(\frac{\beta_*^2}{24\,K^{\beta_*}}\sigma_0^2 + \frac{\rho\beta}{4\,k^{\beta_*/2}}\sigma_0\nu + \frac{2-3\rho^2}{24}\nu^2\right)T}{1 + \frac{\beta_*^2}{24}\log^2 k + \frac{\beta_*^4}{1920}\log^4 k}$$

$$\text{where} \quad k = \frac{K}{F_0}, \quad \zeta = \frac{\nu}{\sigma_0} k^{\beta_*/2} \log k, \quad \text{and} \quad H(\zeta) = \cdots.$$

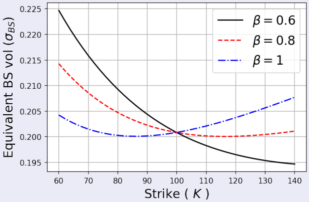
Roles of the SABR parameters

Base parameters: $\sigma_0 = 0.2,\ \nu = 0.2,\ \rho = 0.1,\ \beta = 1.$

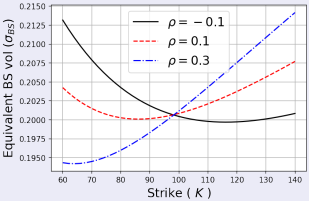
Initial vol σ_0 (level)



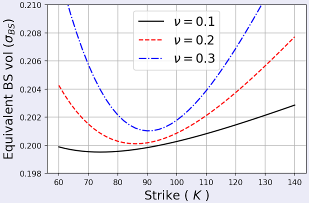
Elasticity β (skew/backbone)



Correlation ρ (skew)



Vol-of-vol ν (smile)



SABR Literature

Analytic approximations (European option)

- Aim to obtain equivalent volatility (improve HKLW formula).
- BS model: Obłój (2007); Paulot (2015); Lorig et al. (2017).
- CEV model: Yang et al. (2017), Choi and Wu (2021a).
- Inaccurate and arbitrageable when $\hat{\nu} := \nu\sqrt{T} \gg 1$ or $K \ll F_0$.
- More advanced approaches: (Hyb) ZC Map (Antonov and Spector, 2012).

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Monte-Carlo simulation

- Flexible method for pricing path-dependent derivatives and other purposes.
- Small-time approximation: Chen et al. (2012), Leitao et al. (2017a,b, QF), Cui et al. (2021, EJOR)
- Piecewise Semi-Exact MC: Cai, Song and Chen (2017, OR), Kyriakou et al. (2023, OR)
- Closed-form MC for $\beta = 0$: Choi et al. (2019)

Challenges in the SABR model simulation

- When $0 < \beta < 1$, an absorbing boundary should be imposed to preserve martingality of F_t.
- Euler/Milstein schemes with naive $F_0 \geq 0$ suffer from non-negligible bias.

Euler/Milstein schemes

$$F_{t+h} \approx F_t + \sigma_t F_T^{\beta} \sqrt{h} W + \frac{\beta \sigma_t^2 F_t^{2\beta-1}}{2} h(W^2 - 1) \quad \text{for} \quad W \sim \mathcal{N}(0,1).$$

- Time step (h) should be very small, ending up high computation cost.
- We need efficient simulation schemes for a reasonably big h.
- Chen et al. (2012), Leitao et al. (2017a,b), Cai et al. (2017), Cui et al. (2018), Kyriakou et al. (2023).

Stochastic integral of the SABR SDE

- The volatility process follows a geometric BM:

$$\frac{d\sigma_t}{\sigma_t} = \nu\, dZ_t, \quad \sigma_t = \sigma_0 \exp\left(\nu Z_t - \frac{\nu^2 t}{2}\right) \quad \Rightarrow \quad \int_0^T \sigma_t dZ_t = \frac{1}{\nu}(\sigma_T - \sigma_0).$$

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- As in the other SV models, the conditional average (integrated) variance plays a key role as stochastic time clock:

$$I_t^h(\sigma_t, \sigma_{t+h}) := \frac{1}{\sigma_t^2 h} \int_t^{t+h} \sigma_s^2 \, \mathrm{d}s \qquad \left(\lim_{\nu \downarrow 0} I_t^h \to 1\right).$$

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- The price process is integrated to

$$\int_0^T \frac{dF_t}{F_t^\beta} = \int_0^T \sigma_t\,(\rho dZ_t + \rho_* dX_t) = \frac{\rho}{\nu}(\sigma_T - \sigma_0) + \rho_* \int_0^T \sigma_t dX_t$$

$$\sim \underbrace{\frac{\rho}{\nu}(\sigma_T - \sigma_0)}_{\text{conditional expectation}} + \underbrace{\rho_* \sigma_0 \sqrt{T I_0^T}}_{\text{conditional volatility}}\ X, \quad (X \sim \mathcal{N}(0,1))$$

- The joint sampling of (σ_T, I_0^T) is critical for the SABR simulation schemes.

Simulation procedures

All SABR simulation algorithms need the steps below from time t to $t+h$:

- Given σ_t, simulate volatility σ_{t+h}:

$$\sigma_{t+h} = \sigma_t \exp\left(\nu Z_h - \frac{\nu^2 h}{2}\right) = \sigma_t \exp\left(\hat{\nu}\hat{Z}\right), \quad \hat{Z} \sim \mathcal{N}(-\hat{\nu}/2, 1).$$

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- **Step 1**: Given σ_t and σ_{t+h} (or $\hat{Z}$), simulate average variance I_t^h.
- **Step 2**: Given F_t, σ_{t+h} (or $\hat{Z}$), and I_t^h, simulate asset price F_{t+h}.
  - Distribution is not analytically tractable for $0 < \beta < 1$.
  - **Step 2a**: How to approximate the distribution of F_{t+h}?
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Contribution of our study

- We propose new algorithms for **Step 1, 2a, and 2b**.
- In particular, we innovate the method for **Step 2a** used in all existing studies.
- Our algorithms are highly efficient and accurate.
- The algorithms can be applied selectively to the existing methods.

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Normal SABR model ($\beta = 0$)

- Conditional price distribution:

$$F_T \,|\, \sigma_T, I_0^T \;\sim\; \mathcal{N}\Big(\underbrace{\bar{F}_0^T}_{\text{mean}},\; \underbrace{\rho_*^2 \sigma_0^2 T I_0^T}_{\text{var}} \Big)$$

where the conditional expectation $\bar{F}_0^T$ is given by

$$\bar{F}_0^T = F_0 + \frac{\rho}{\nu}\big(\sigma_T - \sigma_0\big). \tag{1}$$

- **Step 2** is trivial as long as I_0^T is sampled.
- A closed-form exact simulation is available in Choi et al. (2019).
- This study no longer considers $\beta = 0$ case.

Lognormal SABR model ($\beta = 1$)

- Conditional price distribution:

$$F_T \,|\, \sigma_T, I_0^T \;\sim\; \mathcal{LN}\Big(\underbrace{\bar{F}_0^T}_{\text{mean}},\; \underbrace{\rho_*^2 \sigma_0^2 T I_0^T}_{\text{logvar}} \Big)$$

$$\sim\; \bar{F}_0^T \;\exp\left(\rho_* \sigma_0 \sqrt{T I_0^T}\, X - \frac{\rho_*^2 \sigma_0^2 T I_0^T}{2} \right),$$

where the conditional expectation $\bar{F}_0^T$ is given by

$$\bar{F}_0^T = F_0 \exp\left(\frac{\rho}{\nu}(\sigma_T - \sigma_0) - \frac{\rho^2 \sigma_0^2 T I_0^T}{2} \right).$$

- **Step 2** is trivial as long as I_0^T is sampled.
- The preservation of martingality for any σ_0:

$$E\left(\bar{F}_0^T\right) = 1 \quad (\text{a key intuition}).$$

Zero vol-of-vol ($\nu = 0$) case: CEV model

- Constant-elasticity-of-variance (CEV) model and distribution:

$$\frac{dF_t}{F_t^\beta} = \sigma_0\, dW_t \quad \text{and} \quad F_T :\sim \mathcal{CEV}_\beta\Big(\underbrace{F_0}_{\text{mean}}, \underbrace{\sigma_0^2 T}_{\text{var}}\Big).$$

- When $0 < \beta < 1$ and $\nu \downarrow 0$, SABR $\rightarrow \mathcal{CEV}_\beta$. (a key intuition)
- The $\mathcal{CEV}_\beta$ distribution is closely connected to, but not equal to, the noncentral chi-squared (χ^2) distribution.
- **Step 2a** is resolved.
- **Step 2b**: Sampling CEV distribution is not trivial. But see below.

Uncorrelated ($\rho = 0$) SABR model $0 < \beta < 1$

- F_t follows the CEV process with a stochastic time τ:

$$\frac{dF_\tau}{F_\tau^\beta} = dW_\tau \quad \text{where} \quad \tau = \int_0^t \sigma_s^2 ds = \sigma_0^2 t\, I_0^t.$$

- Conditional price distribution:

$$F_T \,|\, I_0^T \sim \mathcal{CEV}_\beta\Big(\underbrace{F_0}_{\text{mean}}, \underbrace{\rho_*^2 \sigma_0^2 T I_0^T}_{\text{var}} \Big). \quad (\rho_* = 1)$$

- Approximate option price is available in Choi and Wu (2021b).
- How to sample F_T from the CEV distribution?
- No analytic distribution of F_T if $\rho \neq 0$.

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[Step 1] Simulation of I_t^h in other studies

- Chen et al. (2012) used the lognormal variable matched to the small-time mean and variance:

$$\mu = E(I_t^h) \approx 1 + \hat\nu\, Z + \cdots \quad \text{and} \quad \mu_2' = \mathrm{Var}\left(I_t^h\right) \approx \frac{\hat\nu^2}{3},$$

  where Z is the normal variate used in σ_{t+h}.

- Cai et al. (2017) used the Laplace transform of $1/I_t^h$ (Matsumoto and Yor, 2005):

$$E\left(e^{-s/I_t^h}\right) = \exp\left(-\frac{\phi_x^2(\hat\nu^2 s) - x^2}{2\hat\nu^2}\right)$$

$$\text{for} \quad x = \log(\sigma_{t+h}/\sigma_t) \quad \text{and} \quad \phi_x(\lambda) = \mathrm{acosh}(s e^{-x} + \cosh(x)).$$

  Then, sample $1/I_t^h$ from the numerically inverted CDF.

- Leitao et al. (2017a) use Fourier transform of $1/I_t^h$.
- Kyriakou et al. (2023) uses moment-matched Pearson-family distribution for quick sampling.

[Step 1] Sampling I_t^h from an SLN distribution

- Shifted lognormal (SLN) random variable, $Y \sim \mathcal{SLN}(\mu, \sigma^2, \lambda)$:

$$Y \sim \mu \left[(1-\lambda) + \lambda \exp\left(\sigma X - \frac{\sigma^2}{2} \right) \right] \quad \text{with} \quad \mu > 0,\ \sigma > 0,\ 0 < \lambda \le 1,$$

  where $X \sim \mathcal{N}(0,1)$, μ is the mean of Y, λ and σ^2 are the weight and variance of the LN component.

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- When $\lambda = 1$, $\mathcal{SLN}$ is reduced to the $\mathcal{LN}(\mu, \sigma^2)$.
- The mean, coef. of variation (std/mean), skewness, and ex-kurtosis are

$$E(Y) = \mu,\ v = \lambda\sqrt{w},\ s = \sqrt{w}(w+3),\ \text{and } \kappa = w(w^3 + 6w^2 + 15w + 16),$$

  where $w = \exp\left(\sigma^2\right) - 1$ $(w > 0)$.

- The parameters can be analytically fitted to μ, v, and s:

$$\sigma = \sqrt{\ln(1+w)} \quad \text{for} \quad w = 4\sinh^2\left(\frac{1}{6} \operatorname{acosh}\left(1 + \frac{s^2}{2} \right) \right)$$

$$\text{and} \quad \lambda = \frac{v}{2\sinh\left(\frac{1}{6}\operatorname{acosh}\left(1 + \frac{s^2}{2}\right)\right)}.$$

[Step 1] Four conditional moments of I_t^h

We derive the first four raw moments of I_t^h (i.e., $\mu = E(I_t^h)$ and $\mu_k' = E\left((I_t^h)^k\right)$):

$$\mu = \left(\frac{\sigma_{t+h}}{\sigma_t}\right) m_1,$$

$$\mu_2' = \left(\frac{\sigma_{t+h}}{\sigma_t}\right)^2 \frac{1}{\hat{\nu}^2} \left[m_2 - c\, m_1\right],$$

$$\mu_3' = \left(\frac{\sigma_{t+h}}{\sigma_t}\right)^3 \frac{1}{8\hat{\nu}^4} \left[3m_3 - 8cm_2 + \left(4c^2 + 1\right) m_1\right],$$

$$\mu_4' = \left(\frac{\sigma_{t+h}}{\sigma_t}\right)^4 \frac{1}{24\,\hat{\nu}^6} \left[2m_4 - 9cm_3 + \left(12c^2 + 2\right) m_2 - c\left(4c^2 + 3\right) m_1\right],$$

where the conditioning variable, σ_{t+h} and $\hat{Z}$, are exchangeable by $\hat{Z} = \frac{1}{\hat{\nu}} \ln(\sigma_{t+h}/\sigma_t)$, and the coefficients, c and m_k $(k = 1, 2, 3, 4)$, are the functions of them:

$$c(\hat{Z}) := \cosh(\hat{\nu}\hat{Z}) = \frac{1}{2}\left(\frac{\sigma_{t+h}}{\sigma_t} + \frac{\sigma_t}{\sigma_{t+h}}\right) \quad \text{and} \quad m_k(\hat{Z}) := \frac{N(\hat{Z} + k\hat{\nu}) - N(\hat{Z} - k\hat{\nu})}{2k\hat{\nu}\; n\left(\sqrt{\hat{Z}^2 + (k\hat{\nu})^2}\right)},$$

and $n(z)$ and $N(z)$ are the normal PDF and CDF.

- $I_0^T(\hat{Z})$ is closely related to the exponential functional of BM, defined by

$$A_t := \int_0^t e^{2Z_s}\,\mathrm{d}s, \quad I_0^T = \frac{1}{\hat{\nu}^2} A_{\hat{\nu}^2} \quad \left(\hat{\nu} := \nu\sqrt{T}\right).$$

- The derivation of the moments is based on the analytic conditional moments of the exponential functional of BM (Matsumoto and Yor, 2005, (5.4)):

$$E\left((A_t)^n \,|Z_t = x\right) = \frac{e^{nx}}{n!t}\int_x^\infty b e^{(x^2-b^2)/2t}\left[\cosh(b) - \cosh(x)\right]^n db.$$

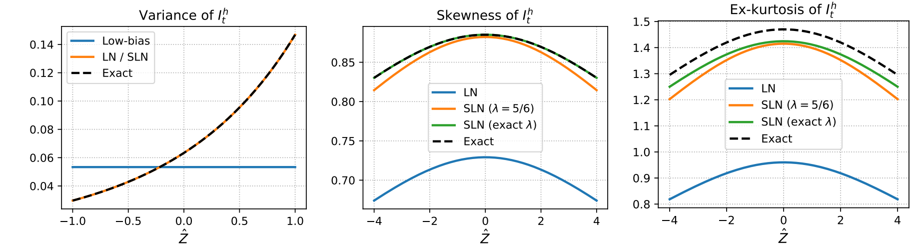
- Kennedy et al. (2012) derived the mean and variance without resorting to Matsumoto and Yor (2005, (5.4)).
- Kyriakou et al. (2023) numerically obtained the four moments from the Laplace transform.

[Step 1] A simple SLN approach with λ fixed

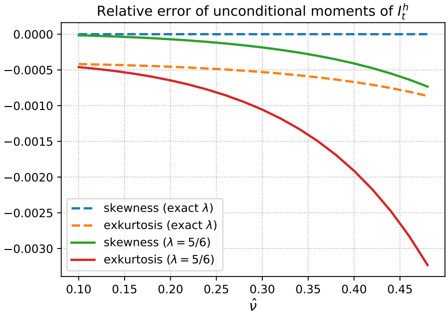
- In the small-time limit ($\hat{\nu} \downarrow 0$), the weight of SLN converges $\lambda \to 5/6$.
- Match σ to the mean μ and coef. of variance v of I_t^h:

$$I_t^h \sim \frac{\mu}{6}\left[1 + 5\exp\left(\sigma X - \frac{\sigma^2}{2}\right)\right] \quad \text{with} \quad \sigma = \sqrt{\ln\left(1 + \frac{36}{25}v^2\right)},$$

- Effectively same performance compared to the calibrated λ (exact skewness). (Majority of error is from **Step 2a**.)
- Computationally simpler.



Variance (left), skewness (middle), and Ex-kurtosis (right) of I_t^h as functions of $\hat{Z}$ for $\hat{\nu} = 0.4$. We compare our SLN approximation with $\lambda = 5/6$ (yellow), the SLN approximation fitted up to skewness (green), and the LN approximation fitted up to variance (blue) to the true values (dashed).



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[Step 2a] Approximation of F_{t+h} distribution

- Conditional on F_t, σ_t, σ_{t+h}, and I_t^h, assume F_{t+h} follows

$$F_{t+h} \sim \text{Distribution} \ (\ \text{Mean}, \quad \text{Variance} \ ).$$

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- Conditional on F_t, σ_t, σ_{t+h}, and I_t^h, assume F_{t+h} follows

$$F_{t+h} \sim \underbrace{\mathcal{CEV}_\beta}_{\text{from } \nu \downarrow 0 \text{ case}} \left(\text{Mean}, \quad \text{Variance} \right).$$

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- Approximate dynamics with GBM with $\sigma_t := \sigma_t / F_t^{\beta_*}$:

$$\frac{dF_s}{F_s} = \frac{\sigma_s}{F_s^{\beta_*}}\,\mathrm{d}W_s \approx \frac{\sigma_s}{F_t^{\beta_*}}\,\mathrm{d}W_s \quad \text{for} \quad t \le s \le t+h,$$

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- $\bar{F}_t^h$ from the lognormal ($\beta = 1$) case:

$$\bar{F}_t^h \approx F_t \exp\left(\frac{\rho}{\nu} \frac{(\sigma_{t+h} - \sigma_t)}{F_t^{\beta_*}} - \frac{\rho^2 \sigma_t^2 h\, I_t^h}{2\, F_t^{2\beta_*}} \right)$$

- Preserves martingality: $E(F_{t+h}) = E(\bar{F}_t^h) = F_t$.
- Exact when $\rho = 0$ since $\bar{F}_t^h = F_t$.

[Step 2b] Random variables to be used

- $\chi^2(\delta, r)$ be the NCX2 random variable with degree of freedom δ and noncentrality r. The PDF and CDF of $\chi^2(\delta, r)$ are respectively expressed by

$$f_{\chi^2}(x\,;\,\delta, r) = \frac{1}{2}\left(\frac{x}{r}\right)^{\alpha/2} I_\alpha(\sqrt{x\,r})\; e^{-(x+r)/2} \quad \text{and} \quad P_{\chi^2}(x;\delta, r) = \int_0^x f_{\chi^2}(t\,;\,\delta, r)\,\mathrm{d}t,$$

  where $\alpha = \delta/2 - 1$ and $I_\alpha(z)$ is the modified Bessel function of the first kind,

$$I_\alpha(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{\alpha+2k}}{k!\,\Gamma(k+\alpha+1)}.$$

- Let $\mathcal{G}(\alpha)$ be a gamma random variable with shape parameter α and unit scale parameter. The PDF and CDF of $\mathcal{G}(\alpha)$ are:

$$f_{\mathcal{G}}(x;\alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} \quad \text{and} \quad P_{\mathcal{G}}(x;\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^x t^{\alpha-1} e^{-t}\,\mathrm{d}t.$$

- The central chi-squared (i.e., $r = 0$) distribution is a gamma distribution:

$$\chi^2(\delta, 0) \sim 2\,\mathcal{G}(\delta/2).$$

- The shifted Poisson (SP) variate with intensity λ and shift α, $N \sim \mathcal{SP}(\lambda, \alpha)$, takes non-negative integer values with mass probability function:

$$P_{\mathcal{SP}}(n; \lambda, \alpha) = \mathrm{Prob}(N = n) = \frac{1}{P_{\mathcal{G}}(\lambda; \alpha)} \frac{\lambda^{\alpha+n}\, e^{-\lambda}}{\Gamma(n + \alpha + 1)}.$$

- The SP variate arises from the series expansion of $P_{\mathcal{G}}(x; \alpha)$:

$$1 = \frac{P_{\mathcal{G}}(\lambda; \alpha)}{P_{\mathcal{G}}(\lambda; \alpha)} = \frac{1}{P_{\mathcal{G}}(\lambda; \alpha)} \sum_{n=0}^{\infty} \frac{\lambda^{\alpha+n}\, e^{-\lambda}}{\Gamma(n + \alpha + 1)} = \sum_{n=0}^{\infty} P_{\mathcal{SP}}(n; \lambda, \alpha).$$

- The Poisson distribution with intensity λ, $\mathcal{POIS}(\lambda)$, is a special case $\mathcal{SP}(\lambda, \alpha = 0)$:

$$P_{\mathcal{POIS}}(n; \lambda) = \frac{\lambda^n\, e^{-\lambda}}{n!}.$$

[Step 2b] Sampling the CEV distribution

The $\mathcal{CEV}_\beta$ and χ^2 distributions

$$f(z; z_0)\,\mathrm{d}z = \mathrm{Prob}(z_T \in [z, z+\mathrm{d}z]) = f_{\chi^2}\left(z_0; \frac{1}{\beta_*} + 2, z\right)\mathrm{d}z \quad (z_T > 0)$$

for the transformation $\quad z(y) := \dfrac{y^{2\beta_*}}{\beta_*^2 \sigma_0^2 T} \quad$ and $\quad z_t = z(F_t) \quad (0 \le t \le T)$.

[Step 2b] Sampling the CEV distribution

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$$f(z; z_0)\,\mathrm{d}z = \mathrm{Prob}(z_T \in [z, z+\mathrm{d}z]) = f_{\chi^2}\left(z_0; \frac{1}{\beta_*} + 2, z\right)\mathrm{d}z \quad (z_T > 0)$$

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Makarov and Glew (2010, §3.3)

The CEV distribution can be sampled by an SP-mixture gamma variable:

$$z_T \sim 2\,\mathcal{G}(N+1) \quad \text{where} \quad N \sim \mathcal{SP}\left(\frac{z_0}{2}, \frac{1}{2\beta_*}\right)$$

[Step 2b] Sampling the CEV distribution

The $\mathcal{CEV}_\beta$ and χ^2 distributions

$$f(z; z_0)\,\mathrm{d}z = \mathrm{Prob}(z_T \in [z, z + \mathrm{d}z]) = f_{\chi^2}\left(z_0; \frac{1}{\beta_*} + 2, z\right)\mathrm{d}z \quad (z_T > 0)$$

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Kang (2014, Algorithm 3)

$N \sim \mathcal{SP}(\lambda, \alpha)$ can be sampled by a gamma-mixture Poisson variable:

$$N \sim \mathcal{POIS}(\lambda - X) \quad \text{where} \quad X \sim \mathcal{G}(\alpha) \quad \text{conditional on} \quad X < \lambda.$$

[Step 2b] Sampling F_T in the CEV model

Simulation algorithm for $F_T \sim \mathcal{CEV}_\beta(F_0, \sigma_0^2 T)$ from Makarov-Glew-Kang:

1. Sample $X \sim 2\mathcal{G}\left(\frac{1}{2\beta_*}\right)$.
2. If $X \geq z_0 = \frac{F_0^{2\beta_*}}{\beta_*^2 \sigma_0^2 T}$, $F_T = 0$ and terminate.
3. Else, sample $z_T \sim 2\mathcal{G}\left(\mathcal{POIS}\left(\frac{z_0 - X}{2}\right) + 1\right)$.
4. Obtain F_T from $z_T = \frac{F_T^{2\beta_*}}{\beta_*^2 \sigma_0^2 T}$.

[Step 2b] Sampling F_{t+h} in the SABR model

Simulation algorithm for $F_{t+h} \sim \mathcal{CEV}_\beta(F_t, \rho_*^2\sigma_t^2 h I_t^h)$ from Makarov-Glew-Kang:

1. Sample $X \sim 2\mathcal{G}\left(\frac{1}{2\beta_*}\right)$.
2. If $X \geq z_t = \frac{F_t^{2\beta_*}}{\beta_*^2\rho_*^2\sigma_t^2 h I_t^h}$, $F_{t+h} = 0$ and terminate.
3. Else, sample $z_{t+h} \sim 2\mathcal{G}\left(\mathcal{POIS}\left(\frac{z_t - X}{2}\right) + 1\right)$.
4. Obtain F_{t+h} from $z_{t+h} = \frac{F_{t+h}^{2\beta_*}}{\beta_*^2\rho_*^2\sigma_t^2 h I_t^h}$.
5. Repeat above for $t = 0,\ h,\ 2h,\ \cdots, T-h$.

[Step 2a] Approximation of F_{t+h} in other studies

- Islah (2009)'s approximation is used in almost all SABR MC literature (Chen et al., 2012; Leitao et al., 2017a,b; Cai et al., 2017; Cui et al., 2021; Kyriakou et al., 2023).

Islah (2009)'s approximation of F_{t+h}

$q_t = F_t^{\beta_*}/\beta_*$ follows a process close to the Bessel process:

$$q_{t+h} = q_t + \frac{\rho}{\nu}(\sigma_{t+h} - \sigma_t) + \int_t^{t+h} \rho_* \sigma_s \, dW_s - \frac{\beta}{2\rho_*^2 \beta_*} \int_t^{t+h} \frac{(\rho_* \sigma_s)^2}{q_s} dt$$

$$\mathrm{Prob}(F_{t+h} > y) \approx P_{\chi^2}\left(z_t' \, \frac{1 - \beta_* \rho^2}{\beta_* \rho_*^2}, z(y) \right),$$

$$\text{where} \quad z_t' = \frac{1}{\rho_*^2 \sigma_t^2 h I_t^h} \left[\frac{F_t^{\beta_*}}{\beta_*} + \frac{\rho}{\nu}(\sigma_{t+h} - \sigma_t) \right]^2 \quad \text{and} \quad z(y) = \frac{y^{2\beta_*}}{\beta_*^2 \rho_*^2 \sigma_t^2 h I_t^h}.$$

- Only exact when $\rho = 0$.
- As $\nu \downarrow 0$, distribution does not converge to $\mathcal{CEV}_\beta$
- Martingality is broken:

$$E(F_{t+h} \,|\, \mathcal{F}_t) \neq F_t$$

[Step 2b] Sampling F_{t+h} in other studies

Various methods are used to sample F_T from Islah's CEV distribution.

- Cai et al. (2017): For a uniform random variate U, solve x with numerical root-finding:

$$\text{Prob}(F_{t+h} \le x) = U.$$

- Chen et al. (2012); Leitao et al. (2017a,b): moment-matched quadratic Gaussian (Andersen, 2008):

$$F_{t+h} \sim d(e+Z)^2 \quad \text{for} \quad Z \sim N(0,1).$$

- Grzelak et al. (2019) used stochastic collocation: efficient interpolation using orthogonal polynomial.
- In fact, Makarov and Glew (2010) and Kang (2014)' algorithm can be modified to sample Islah's CEV exactly (see numerical result).

Summary of the SABR simulation steps

- Simulate σ_{t+h} from σ_t:

$$\sigma_{t+h} = \sigma_t \exp\left(\hat{\nu}\hat{Z}\right), \quad \hat{Z} \sim \mathcal{N}(-\hat{\nu}/2, 1)$$

- **Step 1**: Simulate I_t^h given σ_t and σ_{t+h}.
  - Sample from an SLN variable matched to the conditional mean, variance, and skewness* of I_t^h.
- **Step 2**: Simulate F_{t+h} given F_t, σ_{t+h}, and I_0^T.
  - Approximate F_{t+h} with a martingale-preserving $\mathcal{CEV}_\beta$ distribution.
  - Sample F_{t+h} based on Makarov and Glew (2010) and Kang (2014)'s algorithm.

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Parameter sets

- Parameter sets to test:

Cases	F_0	σ_0	ν	ρ	β	T	K
Case 0	1	0.2	1	-0.75	0.25∼1	0.5∼2	1
Case I	1	0.25	0.3	-0.8	0.3	10	0.2∼2.0
Case II	1	0.25	0.3	-0.5	0.6	10	0.2∼2.0
Case III	0.05	0.4	0.6	0.0	0.3	1	0.02∼0.10
Case IV	1.1	0.4	0.8	-0.3	0.3	4	1.1
Case V	1.1	0.3	0.5	-0.8	0.4	1∼10	1.1

- Use $N = 10^5$ paths for one price.
- Repeat $m = 50$ times to obtain the average bias and stdev.
- Measure biases in forward and European option price.
- Use FDM for the benchmark price with the code from Yingda Song (Cai et al., 2017).
- MATLAB R2023b on a laptop with an i7-1360P 2.20 GHz processor.

Case 0: contribution of error

- Error source: I_t^h (SLN approx.) v.s. F_{t+h} (CEV approx.)
- At $\beta = 1$, no CEV approx. error. Then, decrease β.
- Price European option with $T = h$, $\sigma_0 = 20\%$, $\rho = -0.75$, and $\nu = 1$.

	FDM Price			Rel. error of our method		
β	$h = 0.5$	1	2	$h = 0.5$	1	2
1.00	0.05561	0.07701	0.10331	0.11%	0.11%	0.21%
0.75	0.05581	0.07781	0.10531	-0.31%	-0.61%	-1.01%
0.50	0.05611	0.07871	0.10741	-0.71%	-1.41%	-2.31%
0.25	0.05651	0.07971	0.10981	-1.11%	-2.11%	-3.61%

- Error from F_{t+h} (CEV approx.) is much bigger.

$$\text{Error} \quad \propto \quad \beta_* \nu^2 h$$

- Approximately $\beta_* \nu^2 h \leq 0.5$ ensures 1% error.

Case I

- Our methods accurately price challenging cases ($\nu = 0.3$ and $T = 10$.)

K/F_0	0.2	0.4	0.8	1.0	1.2	1.6	2.0
Exact option price from FDM							
FDM	0.84255	0.68906	0.40646	0.28502	0.18304	0.05343	0.01096
Bias of our simulation method $(\times 10^{-3})$							
$h = 1$	-1.22	-1.49	-0.37	0.49	1.28	1.72	1.32
$h = 1/4$	-0.46	-0.24	0.22	0.42	0.56	0.56	0.48
$h = 1/16$	-0.34	-0.20	0.00	0.05	0.11	0.10	0.10
Standard deviation of our simulation method $(\times 10^{-3})$							
$h = 1$	1.97	1.83	1.50	1.31	1.08	0.63	0.38
$h = 1/4$	1.96	1.73	1.29	1.08	0.91	0.61	0.41
$h = 1/16$	1.89	1.75	1.44	1.28	1.06	0.53	0.22
Bias of various analytic approximation methods $(\times 10^{-3})$							
HKLW	22.35	23.64	17.94	13.81	9.38	2.54	0.82
ZC Map	0.37	0.51	1.39	2.29	3.20	4.02	2.66
Hyb ZC Map	4.64	5.81	4.07	2.29	0.43	-1.26	-0.30

Case II

- Our methods accurately price challenging cases ($\nu = 0.3$ and $T = 10$.)

K/F_0	0.2	0.4	0.8	1.0	1.2	1.6	2
Exact option price from FDM							
FDM	0.82886	0.66959	0.39772	0.29118	0.20690	0.10018	0.05014
Bias of our simulation method $(\times 10^{-3})$							
$h = 1$	-0.14	-0.30	-0.42	-0.43	-0.43	-0.40	-0.30
$h = 1/4$	0.45	0.37	0.27	0.20	0.10	-0.02	0.00
$h = 1/16$	0.01	-0.01	0.02	0.04	0.03	0.00	-0.03
Stdev of our simulation method $(\times 10^{-3})$							
$h = 1$	2.23	2.09	1.78	1.65	1.51	1.20	0.93
$h = 1/4$	2.21	2.10	1.85	1.70	1.51	1.14	0.88
$h = 1/16$	2.46	2.32	2.01	1.79	1.58	1.22	0.97
Bias of various analytic approximation methods $(\times 10^{-3})$							
HKLW	11.65	15.94	16.69	14.67	12.18	8.56	6.88
ZC Map	-1.56	-1.57	0.56	2.37	4.02	5.77	5.33
Hyb ZC Map	3.07	4.53	3.86	2.37	1.00	-0.10	0.52

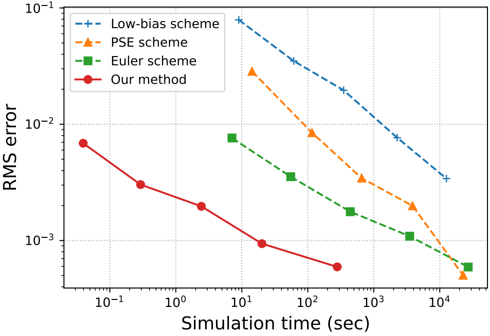
Case III (accuracy of I_t^h)

- Compare with other MC methods: Euler, low-bias (Chen et al., 2012), and PSE (Cai et al., 2017) schemes.
- Validates the SLN approximation of I_t^h (CEV distribution is exact if $\rho = 0$).
- With one jump ($T = h = 1$), our method is more accurate than existing methods.

K/F_0	0.4	0.8	1	1.2	1.6	2	Time (s)
Exact option price from FDM							
FDM	0.04559	0.04141	0.03942	0.03750	0.03390	0.03061	
Bias ($\times 10^{-3}$) and CPU time of our simulation method							
$h = 1$	0.00	0.00	0.00	0.00	-0.01	-0.01	0.03
Bias ($\times 10^{-3}$) and CPU time of the Euler scheme reported in Cai et al. (2017)							
$h = 1/400$	1.6	1.5	1.5	1.4	1.3	1.2	49.4
$h = 1/800$	0.7	0.6	0.5	0.5	0.4	0.3	99.1
$h = 1/1600$	-0.3	-0.3	-0.3	-0.3	-0.3	-0.3	194.0
Bias ($\times 10^{-3}$) and CPU time of the low-bias scheme reported in Cai et al. (2017)							
$h = 1/4$	0.5	0.5	0.5	0.4	0.4	0.4	78.4
$h = 1/8$	0.4	0.4	0.4	0.3	0.3	0.2	175.8
Bias ($\times 10^{-3}$) and CPU time of the PSE scheme reported in Cai et al. (2017)							
$h = 1$	0.1	0.2	0.2	0.2	0.2	0.2	98.3

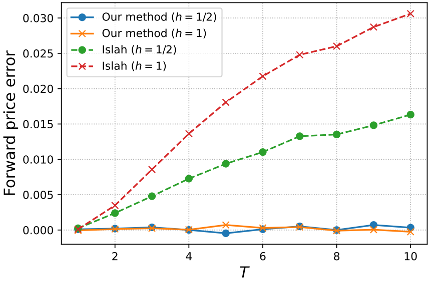
Case IV. (RMS error v.s. CPU time)

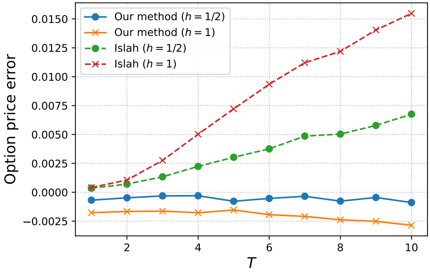
- RMS error $= \sqrt{\text{Stdev}^2 + \text{Bias}^2}$.
- For the same level of error, our scheme is at least 100 times faster than existing methods.
- The decay rate is slower than the PSE scheme (Cai et al., 2017). Big part of RMS error is from bias.



Case V (martingality preservation)

- Measure the biases of $E(F_T)$ and $E((F_T - K)^+)$ as T grows.
- Use Makarov and Glew (2010) and Kang (2014) to Islah's CEV as well as ours.
- Our CEV approximation for F_{t+h} preserves martingality: $E(F_T) = F_0$.
- In Islah's, the option price bias is largely contributed from the price bias.





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Summary and conclusion

- The SABR model is an SV model widely used in industry and academia.
- There is a growing interest in the efficient simulation scheme of the SABR model.
- We propose an efficient simulation algorithms:
  - **Step 1**: Sample average variacne I_t^h from a moment-matched $\mathcal{SLN}$.
  - **Step 2a**: Assume price follows a martingale preserving $\mathcal{CEV}_\beta$ distribution.
  - **Step 2b**: Adopt Makarov-Glew-Kang's CEV simulation algorithm in the context of the SABR model.
- Our new algorithm significantly outperforms the existing algorithms in terms of error-vs-speed tradeoff.

Python implementation: PyFENG

PyFENG (Python Financial ENGineering)

- Implements standard quant finance models: Black–Scholes, Bachelier (normal), CEV, SABR, Heston models and etc.
- PyPI package: `pip install pyfeng`
- Source: https://github.com/PyFE/PyFENG/
- Implemented in pure Python (no `C/C++` extensions).

PyFengForPapers

- Inspired by PapersWithCode project.
- https://github.com/PyFE/PyfengForPapers/
- A collection of Jupyter Notebooks reproducing quant finance papers (mostly in derivative pricing and stochastic volatility) using the methods implemented in PyFENG.
- The code for this talk will soon be available in `PyFengForPapers`.


THANK YOU,
I SAY

References I

Andersen, L., 2008. Simple and efficient simulation of the Heston stochastic volatility model. Journal of Computational Finance 11, 1–42. doi:10.21314/JCF.2008.189.

Antonov, A., Spector, M., 2012. Advanced analytics for the SABR model. SSRN Electronic Journal URL: https://ssrn.com/abstract=2026350.

Cai, N., Song, Y., Chen, N., 2017. Exact simulation of the SABR model. Operations Research 65, 931–951. doi:10.1287/opre.2017.1617.

Chen, B., Oosterlee, C.W., Van Der Weide, H., 2012. A low-bias simulation scheme for the SABR stochastic volatility model. International Journal of Theoretical and Applied Finance 15, 1250016. doi:10.1142/S0219024912500161.

Choi, J., 2025. Exact simulation scheme for the Ornstein-Uhlenbeck driven stochastic volatility model with the Karhunen-Loève expansions. doi:10.48550/arXiv.2402.09243, arXiv:2402.09243.

Choi, J., Kwok, Y.K., 2024. Simulation schemes for the Heston model with Poisson conditioning. European Journal of Operational Research 314, 363–376. doi:10.1016/j.ejor.2023.10.048.

Choi, J., Liu, C., Seo, B.K., 2019. Hyperbolic normal stochastic volatility model. Journal of Futures Markets 39, 186–204. doi:10.1002/fut.21967.

Choi, J., Seo, B.K., 2024. Option pricing under the normal stochastic alpha-beta-rho model with Gaussian quadratures. Journal of Computational Finance 28, 1–20. doi:10.21314/JCF.2024.007, arXiv:2301.02797.

Choi, J., Wu, L., 2021a. The equivalent constant-elasticity-of-variance (CEV) volatility of the stochastic-alpha-beta-rho (SABR) model. Journal of Economic Dynamics and Control 128, 104143. doi:10.1016/j.jedc.2021.104143.

Choi, J., Wu, L., 2021b. A note on the option price and 'Mass at zero in the uncorrelated SABR model and implied volatility asymptotics'. Quantitative Finance 21, 1083–1086. doi:10.1080/14697688.2021.1876908.

Cui, Z., Kirkby, J., Nguyen, D., 2018. A general valuation framework for SABR and stochastic local volatility models. SIAM Journal on Financial Mathematics 9, 520–563. doi:10.1137/16M1106572.

Cui, Z., Kirkby, J.L., Nguyen, D., 2021. Efficient simulation of generalized SABR and stochastic local volatility models based on Markov chain approximations. European Journal of Operational Research 290, 1046–1062. doi:10.1016/j.ejor.2020.09.008.

Grzelak, L.A., Witteveen, J.A.S., Suárez-Taboada, M., Oosterlee, C.W., 2019. The stochastic collocation Monte Carlo sampler: Highly efficient sampling from 'expensive' distributions. Quantitative Finance 19, 339–356. doi:10.1080/14697688.2018.1459807.

Hagan, P.S., Kumar, D., Lesniewski, A.S., Woodward, D.E., 2002. Managing smile risk. Wilmott September, 84–108.

Islah, O., 2009. Solving SABR in Exact Form and Unifying it with LIBOR Market Model. SSRN Electronic Journal doi:10.2139/ssrn.1489428.

Kang, C., 2014. Simulation of the shifted Poisson distribution with an application to the CEV model. Management Science and Financial Engineering 20, 27–32. doi:10.7737/MSFE.2014.20.1.027.

Kennedy, J.E., Mitra, S., Pham, D., 2012. On the Approximation of the SABR Model: A Probabilistic Approach. Applied Mathematical Finance 19, 553–586. doi:10.1080/1350486X.2011.646523.

References II

Kyriakou, I., Brignone, R., Fusai, G., 2023. Unified Moment-Based Modeling of Integrated Stochastic Processes. Operations Research doi:10.1287/opre.2022.2422.

Leitao, Á., Grzelak, L.A., Oosterlee, C.W., 2017a. On a one time-step Monte Carlo simulation approach of the SABR model: Application to European options. Applied Mathematics and Computation 293, 461–479. doi:10.1016/j.amc.2016.08.030.

Leitao, Á., Grzelak, L.A., Oosterlee, C.W., 2017b. On an efficient multiple time step Monte Carlo simulation of the SABR model. Quantitative Finance 17, 1549–1565. doi:10.1080/14697688.2017.1301676.

Lorig, M., Pagliarani, S., Pascucci, A., 2017. Explicit implied volatilities for multifactor local-stochastic volatility models. Mathematical Finance 27, 926–960. doi:10.1111/mafi.12105.

Makarov, R.N., Glew, D., 2010. Exact simulation of Bessel diffusions. Monte Carlo Methods and Applications 16, 283–306. doi:10.1515/mcma.2010.010, arXiv:0910.4177.

Matsumoto, H., Yor, M., 2005. Exponential functionals of Brownian motion, I: Probability laws at fixed time. Probability Surveys 2, 312–347. doi:10.1214/154957805100000159.

Obłój, J., 2007. Fine-tune your smile: Correction to Hagan et al. arXiv:0708.0998 [math, q-fin] URL: https://arxiv.org/abs/0708.0998, arXiv:0708.0998.

Paulot, L., 2015. Asymptotic implied volatility at the second order with application to the SABR model, in: Friz, P., Gatheral, J., Gulisashvili, A., Jacquier, A., Teichmann, J. (Eds.), Large Deviations and Asymptotic Methods in Finance, pp. 37–69. doi:10.1007/978-3-319-11605-1_2.

Yang, N., Chen, N., Liu, Y., Wan, X., 2017. Approximate arbitrage-free option pricing under the SABR model. Journal of Economic Dynamics and Control 83, 198–214. doi:10.1016/j.jedc.2017.08.004.