Efficient Valuation of GMWB Annuities: A Variance Reduction Approach

Jennifer L. Wang
Ming-hua Hsieh
Dept. of of Risk Management and Insurance
National Chengchi University, TAIWAN

Yu-Fen Chiu
Dept. of Financial Engineering and Actuarial Mathematics,
Soochow University, TAIWAN

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Contract Specification

We consider the following contract:

- Single premium f_0
- The initial value of the sub-account F_0 equals f_0.
- Annually a certain percentage (g) of the guaranteed amount f_0 will be withdrawn from the sub-account for T years (gT is usually equals 1).
- At the beginning of year t ($t = 0, 1, \ldots, T-1$), guarantee fee (α times F_t) and a fixed management fee K are withdrawal from the sub-account by the insurer.

Contract Specification

Above GMWB contract provides the following cash-flows to the policy holder

$$\begin{aligned} c_t &= gf_0, \quad t = 1, 2, \ldots, T-1; \\ c_T &= \max(gf_0, F_T) \end{aligned}$$

GMWB Option

The cash-flow received at time T can be decomposed into

$$c_T = \max(gf_0, F_T) = gf_0 + \max(F_T - gf_0, 0)$$

These cash-flows can be decomposed into a term annuity with annual payment gf_0 and an option-like payment $\max(F_T - gf_0, 0)$.

We call the option-like payment the GMWB option.

The fair value of a VA contract with GMWB

- The fair value of a VA contract with GMWB is the sum of fair values of the term annuity and the GMWB option
- The fair value of the term annuity is easy to compute, since its value only depends on the current term structure of interest rates.
- The problem of fair valuation of a VA contract with GMWB reduced to the valuation problem of GMWB option.

Valuation of GMWB option

Based on risk-neutral valuation principle, the fair value of GMWB option can be expressed as

$$E_Q\left[\frac{\max(F_T - gf_0, 0)}{B(T)}\right]$$

where E_Q denote the expectation under risk neutral measure and $B(T)$ denotes the account value of a money market account with initial account value 1.

Dynamics of the sub-account

The value of the sub-account depends on the annual returns of the invested mutual fund.

Let $S(t)$ be NAV of the invested mutual fund at year t. Then the annual return of the invested mutual fund over the t-th year would be:

$$R_t = \frac{S(t)}{S(t-1)}$$

Let us denote F_t^- the account value at year t before withdraws and F_t^+ the account value at year t after withdraws.

Dynamics of the sub-account

The process of the account value can then be described

$$
\begin{aligned}
F_0^- &= f_0, \\
F_0^+ &= \max((1-\alpha)F_0^- - K, 0), \\
F_t^- &= R_t F_{t-1}^+, \quad t = 1, 2, \cdots, T \\
F_t^+ &= \max((1-\alpha)F_t^- - K - gf_0, 0), \quad t = 1, 2, \cdots, T-1
\end{aligned}
$$

Dynamics of the invested mutual fund and MMA

- The value of GMWB option only depends on F_T and F_T in turns only depends on the joint distribution of $(R_1, \cdots, R_T)$. Therefore, the dynamic of $S(t)$ can be very flexible.
- For simulation based method, the only restriction is that the sample of $(R_1, \cdots, R_T)$ is easy to generate.
- The dynamic of $B(t)$ can be derived from the selected interest rate model.

Monte Carlo method

Suppose that we wish to estimate $\beta = EX$, where X is the output of a complex stochastic process. In our case,

$$X = \frac{\max(F_T - gf_0, 0)}{B(T)}.$$

A naive Monte Carlo procedure would generate n independent copies of X, and produce the standard estimate

$$\beta_{\text{naive}} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

where $X_1, \ldots, X_n$ are independent copies of X

Control Variates

Let Y be a d by 1 random vector and each component of Y is correlated with X. Let (μ, Σ) denote the mean vector and covariance matrix of Y. The mean vector is known. Suppose that the covariance between X and Y_i is c_i and $c = (c_1, \ldots, c_d)^T$. Define control variates

$$C = Y - \mu$$

It is clear that the mean vector of $C = 0$, covariance matrix of $C = \Sigma$, and the covariance between X and C_i is c_i.

Control Variates

Let $\lambda \in \Re^d$ and define

$$X_C(\lambda) = X - \lambda^T C$$

It is obvious that $E[X_C(\lambda)] = \beta$ and

$$\mathrm{Var}[X_C(\lambda)] = \sigma_X^2 - 2\lambda^T c + \lambda^T \Sigma \lambda$$

The minimizer of above formula

$$\lambda^* = \Sigma^{-1} c$$

and

$$\mathrm{Var}[X_C(\lambda^*)] = \sigma_X^2 - 2(\Sigma^{-1}c)^T c + (\Sigma^{-1}c)^T \Sigma (\Sigma^{-1}c)$$

Control Variates

Hence

$$\mathrm{Var}[X_C(\lambda^*)] = \sigma_X^2 - c^T \Sigma^{-1} c < \sigma_X^2$$

Let $X_C^{(i)}(\lambda^*)$, $i = 1, \ldots, n$, be independent copies of $X_C(\lambda^*)$. Then it is obvious that

$$\beta_{\text{control}} = \frac{1}{n} \sum_{i=1}^{n} X_C^{(i)}(\lambda^*)$$

is a more efficient estimate for β.

Estimators with control variates

Assume $S(t)$ is a Levy process and then $R_1, \ldots, R_T$ are independent. We propose two estimators with control variates. Define

$$\begin{aligned} H_1 &= ((1-\alpha)f_0 - K)R_1, \\ H_t &= ((1-\alpha)H_{t-1} - K - gf_0)R_t, \quad t = 2, \cdots, T \end{aligned}$$

and set

$$C_1 = H_T - E[H_T], \quad C_2 = \prod_{t=1}^{T} R_t - E\left[\prod_{t=1}^{T} R_t\right]$$

Numerical example I

Table: Point estimates $\times 10^5$ ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.008$)

K	β_{naive}	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	2.5517	2.5585	2.5566	2.5584
2000	2.4681	2.4666	2.4679	2.4667
3000	2.3783	2.3757	2.3777	2.3758
4000	2.2856	2.2881	2.2886	2.2882

Numerical example I

Table: Standard errors ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.008$)

K	β_{naive}	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	440.7616	64.4768	146.5464	63.5819
2000	434.3924	67.3809	148.4233	66.3715
3000	426.0773	70.3953	149.9440	69.2145
4000	419.0520	73.3947	151.6640	72.0314

Numerical example I

Table: Variance Ratios ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.008$)

K	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	46.7305	9.0460	48.0552
2000	41.5615	8.5656	42.8352
3000	36.6345	8.0746	37.8950
4000	32.5992	7.6343	33.8448

Numerical example II

Table: Point estimates $\times 10^5$ ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.005$)

K	β_{naive}	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	2.8568	2.8607	2.8604	2.8608
2000	2.7634	2.7629	2.7620	2.7628
3000	2.6650	2.6652	2.6638	2.6651
4000	2.5683	2.5713	2.5708	2.5713

Numerical example II

Table: Standard errors ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.005$)

K	β_{naive}	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	472.6329	62.7985	151.2676	62.0168
2000	470.0813	65.8974	154.2422	65.0286
3000	462.5048	69.1576	156.3422	68.1162
4000	454.2942	71.8611	157.7726	70.7243

Numerical example II

Table: Variance Ratios ($f_0 = 1000000, r = 0.04, T = 20, g = 0.05, \sigma = 0.16, n = 1000000, \alpha = 0.005$)

K	β_{C_1}	β_{C_2}	β_{C_1,C_2}
1000	56.6435	9.7624	58.0804
2000	50.8874	9.2884	52.2561
3000	44.7252	8.7515	46.1033
4000	39.9656	8.2911	41.2607

Conclusions and extensions

- The selected control variates are effective from the numerical examples.
- The algorithm is easy to generalize to more complex $S(t)$ and $B(t)$ processes
- The algorithm is easy to generalize to life-long GMWB (model T driven by a specific mortality model)
- The algorithm can extend to value contracts with g is time dependent.