Deep xVA

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Abstract of Part 1

We extend the literature on BSDE and xVA along the following directions

- We solve the consistency problem between multiple discounting rules used by different traders and the discount rate of the xVA portfolio.
- We address the existence of multiple aggregation levels for contingent claims in the portfolio between the bank and the counterparty.
- We provide a mathematical treatment of pre-deal pricing.
- We include initial margin.

What is counterparty risk

Definition 1

The risk taken on by an entity entering an Over The Counter (OTC) contract with one (or more) counterparty having a relevant default probability. As such, the counterparty might not respect its payment obligations.

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- In a traditional credit risk setting the default risk is the one of the reference entity/underlying of the contract.
- In counterparty credit risk the underlying contingent claim might even be default-free (e.g. Interest Rate Swap). We are interested in the default risk of the agent with whom we are making the transaction.

Background

- The perspective is that of an investment Bank where in the front office we have:
  - A trading desk: (modulo sales department) interaction with corporate customers and hedging of market risks with interbank counterparties $\Rightarrow$ funding risk.
  - An xVA desk, having the task of calculating and hedging counterparty credit risk and funding.
- The picture is then completed by a treasury department (ALM) and a collateral desk who manages the margin calls.

Setting

- $T < \infty$.
- Two agents named the Bank (B) and the counterparty (C).
- Probability space $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ where $\mathcal{G}$ is the set of all possible events, $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is a filtration of sub-σ-algebras of of $\mathcal{G}$ that is assumed to satisfy the usual assumptions. $\mathcal{G}_0$ is assumed to be trivial.
- $\mathcal{G} = \mathcal{F} \vee \mathcal{H}$ for $\mathcal{H} = \mathcal{H}^B \vee \mathcal{H}^C$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$ where $\mathbb{H} = \mathbb{H}^B \vee \mathbb{H}^C$ with $\mathbb{H}^i = \left(\mathcal{H}_t^i\right)_{t\geq 0}$ for $\mathcal{H}_t^i = \sigma\left(H_u \middle| u \leq t\right)$, and $H_t^i := 1_{\{\tau^i \leq t\}}$ denotes the standard default indicator process of either the bank or the counterparty involving the random default times τ^i, $i \in \{C, B\}$. We set $\tau = \tau^C \wedge \tau^B$.
- Any local $(\mathbb{F}, \mathbb{P})$-martingale is a local $(\mathbb{G}, \mathbb{P})$-martingale.

In the first part of the paper, we set up a replication framework in purely diffusive setting by relying on Bielecki and Rutkowski (2015) and Bichuch et al. (2018).

Notations

Definition 2

Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{G})$. The subspace of all $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted processes X such that

$$\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \|X_t\|^2 \, dt\right] < \infty \tag{1}$$

is denoted by $\mathcal{H}^{2,d}(\mathbb{Q})$. We set $\mathcal{H}^2(\mathbb{Q}) := \mathcal{H}^{2,1}(\mathbb{Q})$.
The subspace of all $\mathbb{R}^d$-valued, $\mathbb{F}$-adapted processes X such that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{t \in [0,T]} \|X_t\|^2\right] < \infty \tag{2}$$

is denoted by $\mathcal{S}^{2,d}(\mathbb{Q})$. We set $\mathcal{S}^2(\mathbb{Q}) := \mathcal{S}^{2,1}(\mathbb{Q})$.

Basic Traded Assets

- S^i, $i = 1, \ldots, d$ the ex-dividend price (i.e. the price) of risky securities with associated cumulative dividend processes D^i.
  - For suitable coefficient functions:

$$\begin{cases} dS_t &= \mu(t, S_t)dt + \sigma(t, S_t)dW_t^{\mathbb{P}} \\ S_0 &= s_0 \in \mathbb{R}^d \end{cases}$$

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- A Bank account associated to the generic rate r^x is defined as

$$B_t^x := \exp\left\{ \int_0^t r_u^x du \right\}.$$

  - r placeholder rate/overnight rate
  - $r^{f,b}, r^{f,l}$ unsecured funding borrowing and lending rates.
  - $r^{c,b}, r^{c,l}$ collateral borrowing and lending rates.
  - $r^{I,b}, r^{I,l}$ initial margin borrowing and lending rates.
  - r^i repo rate for the i-th risky asset.

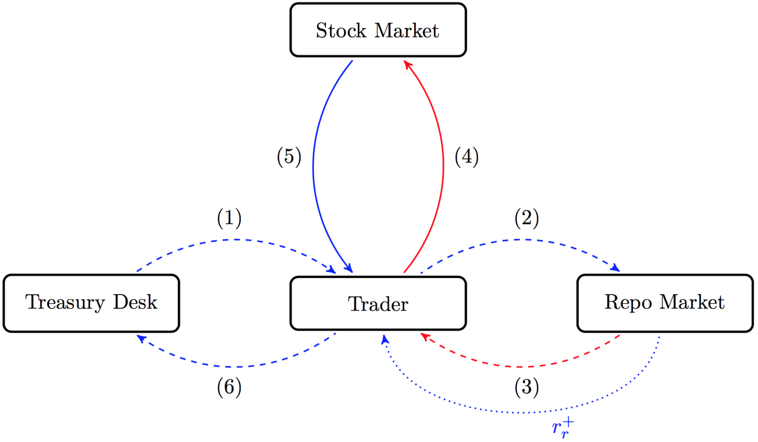
Basic Traded Assets

- We introduce two risky bonds with maturity T and rate of return $\bar{r}_t^i + \lambda_t^i$ issued by the bank and the counterparty via

$$dP_t^i = \left(\bar{r}_t^i + \lambda_t^i\right) P_t^i dt - P_{t-}^i dH_t^i,\ i \in \{B, C\}$$

- Positions in risky assets are financed via repo transactions.

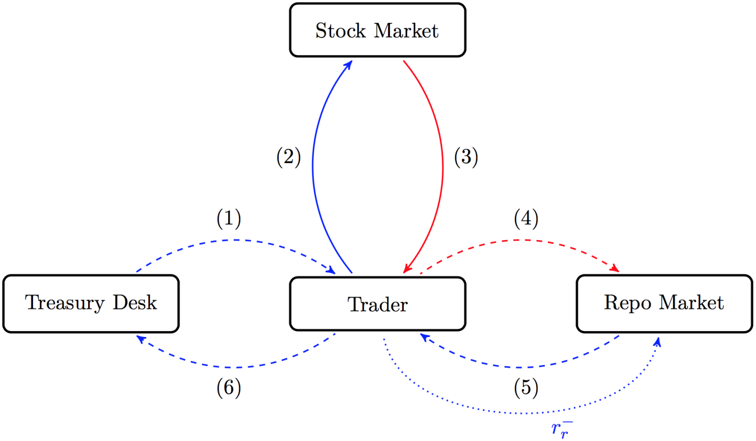
Repo trading (security driven)



Source Bichuch et al. (2018), in turn based on Equation (2.17) in Bielecki and Rutkowski (2015)

$$\psi_t^i B_t^i = -\xi_t^i S_t^i \tag{3}$$

Repo trading (cash driven)



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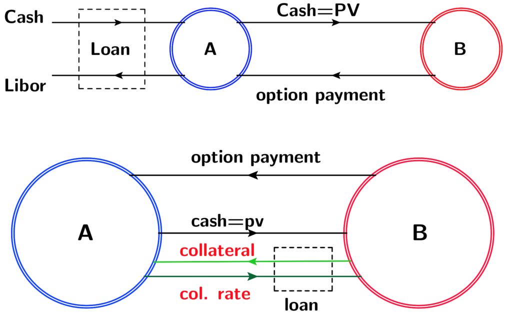
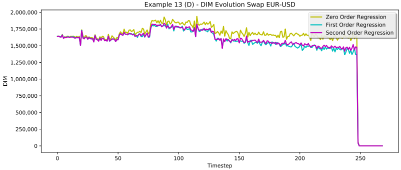
Variation Margin

- Let $C = (C_t)_{t \geq 0}$ denote the variation margin (in cash), exchanged to reduce counterparty exposure.
- The variation margin can be rehypothecated.
- C is assumed to be a Lipschitz function of a reference quantity (in our case the value of the contract in a perfect world).
- $C > 0$ the bank provides collateral. $C < 0$ the bank receives collateral.



Initial Margin

- A misnomer: Initial margin is not initial, it is continuously updated during the lifetime of the trade



- Initial margin is typically a risk measure such as VaR or ES of the future evolution of a reference quantity (e.g. the value of the contract in a perfect world). **Technical conditions - ABSDEs.**
- We distinguish between posted and received initial margin I^{TC} and I^{FC} and use the same sign convention as for variation margin.
- Extremely complex from a computational perspective (brute force approach - nested historical simulations inside a forward risk neutral simulation model).
- Expensive: received initial margin must be segregated. Remuneration

Contingent claims - Close-out condition

- Represented by means of a dividend process $A = (A_t)_{t \geq 0}$.
- A is $\mathbb{F}$-adapted.
- we define the process $\bar{A} = (\bar{A}_t)_{t \geq 0}$ by setting

$$\bar{A}_t := 1_{\{t<\tau\}} A_t + 1_{\{t \geq \tau\}} A_{\tau-} \ , \tag{4}$$

Assumption 1

Assume that $A \in \mathcal{S}^2(\mathbb{Q})$ and $A_T \in L^2(\mathcal{F}_T, \mathbb{Q})$.

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When default happens, we have a payoff structure dictated by the **close-out** condition. $\mathcal{V}$ place-holder for a generic reference value process.

$$\begin{aligned} \theta_\tau(\mathcal{V}, C, I) = \mathcal{V}_\tau &+ 1_{\{\tau^C < \tau^B\}} (1 - R^C) \left(\mathcal{V}_\tau - C_{\tau-} + I^{FC}_{\tau-} \right)^- \\ &- 1_{\{\tau^B < \tau^C\}} (1 - R^B) \left(\mathcal{V}_\tau - C_{\tau-} - I^{TC}_{\tau-} \right)^+ \end{aligned} \tag{5}$$

Close-out condition - explaination

Let us assume that $\mathcal{V}$ is the marked-to-market value of a contract in the absence of counterparty credit risk (more on this later). For simplicity we set $I_t^{TC} \equiv I_t^{FC} \equiv 0 \; d\mathbb{P} \otimes dt$-a.s.

Example 1: The bank has sold a call option to the counterparty. This means that

- $\mathcal{V}_t > 0$ for all t.
- The bank is the collateral provider: $C_t > 0$.

Example 2: The bank has purchased a call option from the counterparty. This means that

- $\mathcal{V}_t < 0$ for all t.
- The bank is the collateral receiver: $C_t < 0$.

Example 1: The bank has sold a call option

The counterparty defaults first

- The bank nets the amount owed to the counterparty (i.e. $\mathcal{V}_\tau$) with the collateral he/she already posted and pays to the liquidators of the counterparty $\mathcal{V}_\tau - C_{\tau_-}$
- The close-out inclusive of collateral is $\theta_\tau(\mathcal{V}, C, 0) = \mathcal{V}_\tau$.
- The close-out exclusive of collateral is $\mathcal{V}_\tau - C_{\tau_-}$ that coincides with the amount to be returned to (the liquidators of) the counterparty.

The bank defaults first

- The counterparty will only receive a fraction $R_B(\mathcal{V}_\tau - C_{\tau_-})$ from the bank.
- The close-out inclusive of collateral is $\theta_\tau(\mathcal{V}, C, 0) = \mathcal{V}_\tau - (1 - R_B)(\mathcal{V}_\tau - C_{\tau_-}) = C_{\tau_-} + R_B(\mathcal{V}_\tau - C_{\tau_-})$.
- The close-out exclusive of collateral is $R_B(\mathcal{V}_\tau - C_{\tau_-})$ that coincides with the amount to be returned to the counterparty.

Example 2: The bank has purchased a call option

The bank defaults first

- The counterparty is supposed to return $-\mathcal{V}_\tau$ to the bank. This amount is net with the collateral already held by the bank. In summary the counterparty return $-(\mathcal{V}_\tau - C_{\tau_-})$. The corresponding wealth of the bank at τ_B is then $\mathcal{V}_\tau - C_{\tau_-}$
- The close-out inclusive of collateral is $\theta_\tau(\mathcal{V}, C, 0) = \mathcal{V}_\tau$.
- The close-out exclusive of collateral is then $\mathcal{V}_\tau - C_{\tau_-}$ as expected.

The counterparty defaults first

- The bank only receives a fractional recovery of the value i.e. $-R_C(\mathcal{V}_\tau - C_{\tau_-})$ from the counterparty. Hence the wealth of the trader at τ_C is $R_C(\mathcal{V}_\tau - C_{\tau_-})$
- The close-out inclusive of collateral is $\theta_\tau(\mathcal{V}, C, 0) = \mathcal{V}_\tau - (1 - R_C)(\mathcal{V}_\tau - C_{\tau_-})$.
- Consequently, the close-out exclusive of collateral is $R_C(\mathcal{V}_\tau - C_{\tau_-})$ that coincides with the wealth of the trader.

Trading strategies

Definition 3

A dynamic portfolio, denoted by φ, is given by

$$\varphi = \left(\xi^1, \dots, \xi^d, \xi^B, \xi^C, \psi^1, \dots, \psi^d, \psi^B, \psi^C, \psi^{f,b}, \psi^{f,l}, \psi^{c,b}, \psi^{c,l}, \psi^{I,b}, \psi^{I,l}\right),$$

where the components are well-behaved integrands representing quantities invested in different basic traded assets.

Definition 4

A CSA between the bank and the counterparty is represented through the pair (C, I), where C is the variation margin and I is the intitial margin.

Definition 5

A *collateralized hedger's trading strategy* associated to the collateralized contract $\bar{A}$ and the CSA (C, I) is a quintuplet $\left(x, \varphi, \bar{A}, C, I\right)$, where $x \in \mathbb{R}$ is the initial endowment and φ is a dynamic portfolio.

Wealth process

Definition 6

The *wealth process* or *value process* $V(\varphi) = (V_t(\varphi))_{t\in[0,T]}$ associated to a collateralized hedger's trading strategy $\left(x, \varphi, \bar{A}, C, I\right)$ is given by

$$\begin{aligned} V_t(\varphi) := &\sum_{i=1}^{d} \left(\xi_t^i S_t^i + \psi_t^i B_t^i\right) \\ &+ \sum_{j\in\{B,C\}} \left(\xi_t^j P_t^j + \psi_t^j B_t^j\right) + \psi_t^{f,b} B_t^{f,b} + \psi_t^{f,l} B_t^{f,l} \\ &- \left(\psi_t^{c,b} B_t^{c,b} + \psi_t^{c,l} B_t^{c,l} + \psi_t^{I,l} B_t^{I,l}\right) . \end{aligned} \tag{6}$$

Self-financing condition

Definition 7

Given the initial endowment x, a collateralized hedger's trading strategy $(x, \varphi, \bar{A}, C, I)$ associated to the collateralized contract $\bar{A}$ and the CSA (C, I) is said to be *self-financing* if for any $t \in [0, T]$ the wealth process $V_t(\varphi)$ satisfies

$$
\begin{aligned}
V_t(\varphi) &= x + \sum_{i=1}^{d} \int_{(0,t]} \xi_u^i \left(\mu^i(u, S_u)du + \sum_{k=1}^{d} \sigma^{i,k}(u, S_u) dW_u^{k,\mathbb{P}} + \kappa^i(u, S_u)du \right) \\
&+ \sum_{i=1}^{d} \int_0^t \psi_u^i dB_u^i + \sum_{j \in \{B,C\}} \int_0^t \left(\xi_u^j dP_u^j + \psi_u^j dB_u^j \right) - \bar{A}_t \\
&+ \int_0^t \psi_u^{f,b} dB_u^{f,b} + \int_0^t \psi_u^{f,l} dB_u^{f,l} - \int_0^t \psi_u^{c,b} dB_u^{c,b} - \int_0^t \psi_u^{c,l} dB_u^{c,l} \\
&- \int_0^t \psi_u^{I,b} dB_u^{I,b} - \int_0^t \psi_u^{I,l} dB_u^{I,l}.
\end{aligned} \tag{7}
$$

Admissibility of strategies and Replication of claims

Definition 8

A collateralized hedger's trading strategy is *admissible* if it is self-financing and the associated value process $V(\varphi)$ is bounded from below.

Definition 9

A self-financing collateralized hedger's trading strategy $(0, \varphi, \bar{A}, C, I)$ is said to *replicate* the collateralized contract $\bar{A}$ if $V_{\hat{\tau}}(\varphi) = 0$, where $\hat{\tau} := \tau \wedge T$.

Absence of Arbitrage - Basic traded assets

Assumption 2

We assume r_t^f bounded from below and $r_t^{f,l} \leq r_t^{f,b}$, $d\mathbb{P} \otimes dt$-a.s.

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Definition 10

The cumulative dividend price associated to the i-th asset is given by

$$S_t^{i,cld} := S_t^i + B_t^i \int_{(0,t]} \frac{dD_u^i}{B_u^i}, \quad i = 1, \dots d \;,\; t \in [0, T]. \tag{8}$$

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Proposition 1

Let Assumption 2 hold. Moreover, assume that $r_t^{f,l} \geq r_t^i$, $i = 1, \dots, d$, $r_t^{f,l} \geq r_t^j$, $j \in \{B, C\}$, $\mathbb{P}$-a.s., for all $t \ \in \ [0, T]$, and that there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted asset price processes

$$\tilde{S}_t^{i,cld} := \frac{S_t^{i,cld}}{B_t^i}, \ i = 1, \dots d, \qquad \tilde{P}_t^j := \frac{P_t^j}{B_t^j}, \ j \in \{B, C\}, \tag{9}$$

are local martingales. Then, the market consisting of the basic traded assets $(0, \varphi, 0, 0, 0)$ is free of arbitrage opportunities.

Full $\mathbb{G}$-BSDE

The full contract $\mathbb{G}$-BSDE for the portfolio's dynamics has the form on $\{\tau > t\}$

$$\begin{cases} -dV_t(\varphi) = d\bar{A}_t + (f(t,V,C,I) - r_t V_t(\varphi))\,dt \\ \qquad\qquad - \sum_{k=1}^{d} Z_t^k dW_t^{k,\mathbb{Q}} - \sum_{j\in\{B,C\}} U_t^j dM_t^{j,\mathbb{Q}} \\ V_\tau(\varphi) = \theta_\tau(\mathcal{V}, C, I). \end{cases} \tag{10}$$

$$\begin{aligned} Z_t^k &:= \sum_{i=1}^{d} \xi_t^i \sigma^{i,k}(t, S_t), \\ U_t^j &:= -\xi_t^j P_{t-}^j, \\ f(t,V,C,I) &:= -\Big[(r_t^{f,l} - r_t)\left(V_t(\varphi) - C_t - I_t^{TC}\right)^+ \\ &\qquad - (r_t^{f,b} - r_t)\left(V_t(\varphi) - C_t - I_t^{TC}\right)^- \\ &\qquad + (r_t^{c,l} - r_t)C_t^+ - (r_t^{c,b} - r_t)C_t^- + (r_t^{I,l} - r_t)I_t^{TC} - r_t^{I,b} I_t^{FC}\Big]. \end{aligned}$$

Full $\mathbb{G}$-BSDE

BSDE (10) is complicated

- presence of the risky dividend process $\bar{A}$.
- presence of a risk measure (specified later) inside the driver - Anticipative BSDE (ABSDE).
- it is specified up to a random time horizon.
- it features jumps.
- if $\mathcal{V} = V$ complicated! Choose the **clean value** (aka risk-free value) instead (What is the risk-free value?).

Proving existence and uniqueness of (10) is non-trivial: we do not attack the problem directly but follow an indirect approach proposed by Crépey.

Full $\mathbb{G}$-BSDE

We prove in Theorem 13 that there exists a unique solution (V, Z, U) for the $\mathbb{G}$-BSDE (10), and the process V assumes the following form on $\{\tau > t\}$

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,\tau\wedge T]} \frac{d\bar{A}_u}{B_u^r} + \int_t^{\tau\wedge T} \frac{f(u, V, C, I)}{B_u^r} du + 1_{\{\tau \leq T\}} \frac{\theta_\tau(\mathcal{V}, C, I)}{B_\tau^r} \middle| \mathcal{G}_t \right], \tag{11}$$

where $B_t^r := \exp\left(\int_0^t r_u du\right)$, $t \in [0, T]$.
We first need to clarify who is $\mathcal{V}$.

Clean value process under $\mathbb{F}$

Assumption 3 (Clean market)

A clean market under $\mathbb{F}$ without bid-offer spreads is defined by

(i) *no bid-offer spread in the funding accounts, i.e.,* $r_t^{f,l} = r_t^{f,b} = r^f$;

(ii) *no bid-offer spread in the collateral accounts, i.e.,* $r_t^{c,l} = r_t^{c,b} = r^c$;

(iii) *the collateral rate is equal to the fictious rate, i.e.,* $r^c = r$;

(iv) *there is no default, i.e.* $\hat{\tau} = T$, *and risky bonds are excluded from the market;*

(v) *there is no exchange of initial margin;*

(vi) *the dividend process A is $\mathbb{F}$-adapted;*

(vii) *perfect collateralization, i.e.,* $\hat{V}_t \equiv C_t$, *for all* $t \in [0, T]$, *where we use* $\hat{V}$ *to denote the value process of a collateralized hedging strategy in the fictious market without default-risk.*

$\mathbb{F}$-BSDE

$\mathbb{F}$-BSDE for the clean value process.

$$\begin{cases} -d\hat{V}_t(\varphi) = dA_t - r_t \hat{V}_t(\varphi)dt - \sum_{k=1}^{d} \hat{Z}_t^k dW_t^{k,\mathbb{Q}} \\ \hat{V}_T(\varphi) = 0. \end{cases} \tag{12}$$

- Existence and uniqueness are standard. (Theorem 3.18).
- The clean value of the derivative is then (Theorem 3.19).

$$\hat{V}_t(\varphi) = \mathbb{E}^{\mathbb{Q}} \left[B_t^r \int_{(t,T]} (B_u^r)^{-1} \, dA_u \middle| \mathcal{F}_t \right]$$

From now on $\mathcal{V} = \hat{V}_t$

Ansatz

We look for a solution to the $\mathbb{G}$-BSDE of the form

$$V_t(\varphi) = \hat{V}_t(\varphi) - XVA_t \tag{13}$$

xVAs

Given the risk-free close-out we can define valuation adjustments deduced form the $\mathbb{G}$-BSDE.

Definition 11

We define the following valuation adjustments:

$$
\begin{aligned}
CVA_t &:= B_t^r \mathbb{E}^{\mathbb{Q}} \left[1_{\{\tau<T\}} 1_{\left\{\tau^C<\tau^B\right\}} (1-R^C) \frac{1}{B_\tau^r} \left(\hat{V}_\tau(\varphi) - C_{\tau-} + I_{\tau-}^{FC} \right)^+ \middle| \mathcal{G}_t \right], \\
DVA_t &:= B_t^r \mathbb{E}^{\mathbb{Q}} \left[1_{\{\tau<T\}} 1_{\left\{\tau^B<\tau^C\right\}} (1-R^B) \frac{1}{B_\tau^r} \left(\hat{V}_\tau(\varphi) - C_{\tau-} - I_{\tau-}^{TC} \right)^- \middle| \mathcal{G}_t \right], \\
FVA_t &:= B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau\wedge T} \frac{(r_u^{f,l} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^+ - (r_u^{f,b} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^-}{B_u^r} du \middle| \mathcal{G}_t \right], \\
ColVA_t &:= B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau\wedge T} \frac{(r_u^{c,l} - r_u) C_u^+ - (r_u^{c,b} - r_u) C_u^-}{B_u^r} du \middle| \mathcal{G}_t \right], \\
MVA_t &:= B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau\wedge T} \frac{(r_u^{I,l} - r_u) I_u^{TC} - r_u^{I,b} I_u^{FC}}{B_u^r} du \middle| \mathcal{G}_t \right].
\end{aligned}
$$

Finally, we set

$$
XVA_t := CVA_t - DVA_t + FVA_t + ColVA_t + MVA_t. \tag{14}
$$

Discussion

Remark 1

The recursive nature of pricing equations.

- *Upon inspection of the FVA term we notice that, in general, xVA BSDE have a recursive nature: the exposure is proportional to the full value of the transaction V and not only to the clean value $\hat{V}$. This results, in general in a high complexity of the numerical scheme.*

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- *Practitioner's papers such as Burgard Kjaer (2013) avoid the recursivity issue by means of ad-hoc assumptions on the funding strategies, see e.g. the funding strategy "semi-replication with no shortfall on default". In general however, the bank needs to fund the clean value* **and** *the value adjustments.*

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- *Our recursive FVA representation is ultimately in line with the one presented in Piterbarg (2010).*

Example: Piterbarg (2010)

Let us set $I_t^{TC} = I_t^{FC} = 0$, $r^{f,b} = r^{f,l} = r^f$, $r^{c,b} = r^{c,l} = r^c$ and $\tau^C = \tau^B = \infty$. Then the driver of the full BSDE has the form

$$f(t, V, C, 0) := -\Big((r_t^f - r_t)\,(V_t(\varphi) - C_t) + (r_t^c - r_t)C_t\Big)$$

Example: Piterbarg (2010)

Let us set $I_t^{TC} = I_t^{FC} = 0$, $r^{f,b} = r^{f,l} = r^f$, $r^{c,b} = r^{c,l} = r^c$ and $\tau^C = \tau^B = \infty$. Then the driver of the full BSDE has the form

$$f(t, V, C, 0) := -\left((r_t^f - r_t)(V_t(\varphi) - C_t) + (r_t^c - r_t)C_t\right)$$

And the integral representation is of the form

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA_u}{B_u^r} + \int_t^T \frac{f(u, V, C, 0)}{B_u^r} du \middle| \mathcal{G}_t\right]. \tag{15}$$

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If we set $r_t = r_t^f$ $d\mathbb{P} \otimes dt$-a.s. then we obtain

$$V_t(\varphi) = B_t^{r^f} \mathbb{E}^{\mathbb{Q}}\left[\int_{(t,T]} \frac{dA_u}{B_u^{r^f}} + \int_t^T (r_u^f - r_u^c)\frac{C_u}{B_u^{r^f}} du \middle| \mathcal{G}_t\right]. \tag{16}$$

This corresponds to equation (3) in Piterbarg (2010).

Example: Piterbarg (2010)

Let us set $I_t^{TC} = I_t^{FC} = 0$, $r^{f,b} = r^{f,l} = r^f$, $r^{c,b} = r^{c,l} = r^c$ and $\tau^C = \tau^B = \infty$. Then the driver of the full BSDE has the form

$$f(t, V, C, 0) := -\Big((r_t^f - r_t)(V_t(\varphi) - C_t) + (r_t^c - r_t) C_t \Big)$$

And the integral representation is of the form

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^r} + \int_t^T \frac{f(u, V, C, 0)}{B_u^r} du \,\middle|\, \mathcal{G}_t \right]. \tag{15}$$

If we set $r_t = r_t^f$ $d\mathbb{P} \otimes dt$-a.s. then we obtain

$$V_t(\varphi) = B_t^{r^f} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^{r^f}} + \int_t^T (r_u^f - r_u^c) \frac{C_u}{B_u^{r^f}} du \,\middle|\, \mathcal{G}_t \right]. \tag{16}$$

This corresponds to equation (3) in Piterbarg (2010). Let us now set $r_t = r_t^c$ $d\mathbb{P} \otimes dt$-a.s. We obtain

$$V_t(\varphi) = B_t^{r^c} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_u}{B_u^{r^c}} - \int_t^T (r_u^f - r_u^c) \frac{(V_t(\varphi) - C_u)}{B_u^{r^c}} du \,\middle|\, \mathcal{G}_t \right], \tag{17}$$

which corresponds to equation (5) in Piterbarg (2010).

Existence and uniqueness theory for the full $\mathbb{G}$-BSDE

We show that (10) admits a unique solution. We proceed along the following steps. Introduce

$$\theta_t^C := (1-R^C)\left(\hat{V}_t - C_t + I_t^{FC}\right)^-, \ \theta_t^B \quad := (1-R^B)\left(\hat{V}_t - C_t - I_t^{TC}\right)^+, \quad (18)$$

Definition 12

We call *pre-default XVA-BSDE* the following $\mathbb{F}$-BSDE on $[0, T]$ with null terminal condition in T :

$$\begin{cases} -d\overline{XVA}_t = \bar{f}(\hat{V}_t - \overline{XVA}_t)dt - \sum_{k=1}^d \bar{Z}_t^k dW_t^{k,\mathbb{Q}} \\ \overline{XVA}_T = 0, \end{cases} \quad (19)$$

where

$$\bar{f}(\hat{V}_t - \overline{XVA}_t) := -f(t, \hat{V} - \overline{XVA}, C, I) - (r_t + \lambda_t^{C,\mathbb{Q}} + \lambda_t^{B,\mathbb{Q}})\overline{XVA}_t \quad (20)$$
$$- \lambda_t^{C,\mathbb{Q}}\theta_t^C + \lambda_t^{B,\mathbb{Q}}\theta_t^B, \quad (21)$$

for θ^B, θ^C defined as in (18).

Initial margin - ABSDE

$$I_t := \rho_t(\hat{V}_{t:T})_{t\in[t,T]}, \tag{22}$$

Assumption 4

For any $X \in \mathcal{S}^2(\mathbb{Q})$, the process $(\rho_s(X_{t:T}))_{s\in[0,T]}$ is in $\mathcal{H}^2(\mathbb{Q})$. There exists a constant $C_{\bar{f}} > 0$ and a family of measures $(\nu_s)_{s\in[0,T]}$ on $\mathbb{R}$ such that $\nu_t([t;T]) = 1$, for every $t \in [0,T]$, and, for any y^1, $y^2 \in \mathcal{S}^2(\mathbb{Q})$, we have

$$\left|\rho_t(y^1_{t:T}) - \rho_t(y^2_{t:T})\right| \leq C_\rho \mathbb{E}\left[\int_t^T |y^1_s - y^2_s|\nu_t(ds)\Big|\mathcal{F}_t\right], \ dt\otimes d\mathbb{P} \text{ a.e. } . \tag{23}$$

Existence and uniqueness of the ABSDE under $\mathbb{F}$

Proposition 2

Under Assumptions 1 and 4, the $\mathbb{F}$-BSDE (19) *is well posed and has a unique solution* $(\overline{XVA}, \bar{Z}) \in \mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q})$.

Given $\overline{XVA}$ under $\mathbb{F}$, we use it to construct XVA under $\mathbb{G}$.

Proposition 3

Let $\left(\overline{XVA}, \bar{Z}\right)$ *be the unique solution of the pre-default XVA-BSDE* (19)*. Define*

$$X_t := \overline{XVA}_t J_t + 1_{\{\tau \le t\}} \vartheta_\tau, \; t \in [0, \tau \wedge T], \tag{24}$$

where $J_t := 1_{\{t<\tau\}} = 1 - H_t$*. Then, under Assumptions 1 and 4, the process* $\left(X, \tilde{Z}, \tilde{U}\right)$ *solves the* $\mathbb{G}$*-BSDE on* $\{\tau > t\}$

$$\begin{cases} -dX_t &= -\left[f(t, \hat{V} - XVA, C, I) + r_t XVA_t\right] dt - \sum_{k=1}^d \tilde{Z}_t^k dW_t^{k,\mathbb{Q}} - \sum_{j \in \{B,C\}} \tilde{U}_t^j dM_t^{j,\mathbb{Q}} \\ X_\tau &= 1_{\{\tau \le T\}} \left(\hat{V}_\tau(\varphi) - \theta_\tau(\hat{V}, C, I)\right) \end{cases} \tag{25}$$

with respect to the filtration $\mathbb{G}$*. Moreover,*

$$X_t = XVA_t, \; t \in [0, T].$$

Existence and uniqueness for the BSDE for V under $\mathbb{G}$.

Theorem 13

Let $V_t := \hat{V}_t - XVA_t$, $t \in [0, T]$, on $\{\tau > t\}$,. Then, under Assumptions 1 and 4, the triplet $(V, Z, U) \in \mathcal{S}^2(\mathbb{Q}) \times \mathcal{H}^{2,d}(\mathbb{Q}) \times \mathcal{H}^{2,2}(\mathbb{Q})$ solves the $\mathbb{G}$-BSDE (10), where Z and U are given by

$$Z_t^k = \hat{Z}_t^k - \tilde{Z}_t^k,\ k = 1, \ldots, d, \tag{26}$$

$$U_t^j = -\tilde{U}_t^j,\ j \in \{B, C\}. \tag{27}$$

Moreover, the process V satisfies

$$V_t(\varphi) = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,\tau\wedge T]} \frac{d\bar{A}_u}{B_u^r} + \int_t^{\tau\wedge T} \frac{f(u, V, C, I)}{B_u^r} du + 1_{\{\tau \le T\}} \frac{\theta_\tau(\mathcal{V}, C, I)}{B_\tau^r} \middle| \mathcal{G}_t \right] \tag{28}$$

CSA discounting

Problem

Every trading desk in the bank uses different discount curves for pricing depending on the choice of a particular collateral account.

But the $\mathbb{G}$-BSDE is based on r-discounting. $\rightarrow$ **inconsistency between the pricing rule of the front office and the xVA desk**.

Problem 14 (xVA-CSA consistency problem)

Produce a price decomposition in terms of clean value and xVA such that

(i) *the representation is coherent with the full $\mathbb{G}$-BSDE, and*

(ii) *the clean price in the representation corresponds to the one prescribed by the front-office function.*

Solving the inconsistency problem

Multiple claims!

Front-office clean value $\hat{P}_t^m$, $m = 1, \ldots, K$, obtained from the $\mathbb{F}$-BSDE

$$\begin{cases} -d\hat{P}_t^m = \sum_{k=1}^d \hat{Z}_t^{k,m} dW_t^{k,\mathbb{Q}} - dA_t^m - \hat{r}^m \hat{P}_t^m dt, \\ \hat{P}_T^m = 0. \end{cases} \tag{29}$$

xVA desk computes the clean value $\hat{V}_t^m$, $m = 1, \ldots, K$, as the solution to the $\mathbb{F}$-BSDE (12),

$$\begin{cases} -d\hat{V}_t^m = \sum_{k=1}^d \hat{Z}_t^{k,m} dW_t^{k,\mathbb{Q}} - dA_t^m - r_t \hat{V}_t^m dt, \\ \hat{V}_T^m = 0, \end{cases} \tag{30}$$

for each possible claim A^m.

Lemma 15

For any $m = 1, \dots, K$, let $(\hat{V}^m, \hat{Z}^{1,m}, \dots, \hat{Z}^{d,m})$ be the unique solution of the $\mathbb{F}$-BSDE (30). Then the value process $\hat{V}^m$ admits the two equivalent representations

i) *xVA-discounting representation*

$$\hat{V}_t^m = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_t^m}{B_u^r} \Big| \mathcal{F}_t \right], \tag{31}$$

ii) *CSA-discounting representation*

$$\hat{V}_t^m = \hat{P}_t^m - DiscVA_t^m, \tag{32}$$

where $DiscVA_t^m$ represents the discounting valuation adjustment, defined as

$$DiscVA_t^m := B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (r_u - \hat{r}^m) \frac{\hat{V}_u^m}{B_u^{\hat{r}^m}} du \Big| \mathcal{F}_t \right], \tag{33}$$

and $\hat{P}^m$ is the value process in the solution $(\hat{V}^m, \hat{Z}^{1,m}, \dots, \hat{Z}^{d,m})$ of the $\mathbb{F}$-BSDE (29)

$$\hat{P}_t^m = B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} \frac{dA_t^m}{B_u^{\hat{r}^m}} \Big| \mathcal{F}_t \right]. \tag{34}$$

Proposition 4

Under the preceeding assumptions the portfolio-wide $\mathbb{G}$-BSDE admits the following integral representation

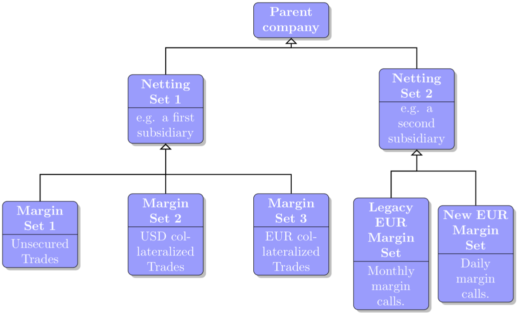
$$V_t(\varphi) = \sum_{m=1}^{K} B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_{(t,T]} (B_u^{\hat{r}^m})^{-1} dA_u^m \Big| \mathcal{G}_t \right] - XVA_t = \sum_{m=1}^{K} \hat{P}_t^m - XVA_t,$$

on the event $\{\tau > t\}$, $t \in [0, T]$, *where*

$$\begin{aligned}
XVA_t &:= FVA_t + ColVA_t + MVA_t + CVA_t - DVA_t + DiscVA_t \\
&= B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^{\tau \wedge T} (B_u^m)^{-1} \left[(r_u^{f,l} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^+ - (r_u^{f,b} - r_u) \left(V_u(\varphi) - C_u - I_u^{TC} \right)^- \right] du \Big| \mathcal{G}_t \right] \\
&+ MVA_t + ColVA_t \\
&- B_t^r \mathbb{E}^{\mathbb{Q}} \left[1_{\{\tau < T\}} 1_{\{\tau^C < \tau^B\}} (1 - R^C) \frac{1}{B_\tau^r} \left(\sum_{m=1}^{N} \hat{P}_\tau^m - C_{\tau-} + I_{\tau-}^{FC} - \sum_{m=1}^{N} DiscVA_\tau^m \right)^- \Bigg| \mathcal{G}_t \right] \\
&+ B_t^r \mathbb{E}^{\mathbb{Q}} \left[1_{\{\tau < T\}} 1_{\{\tau^B < \tau^C\}} (1 - R^B) \frac{1}{B_\tau^r} \left(\sum_{m=1}^{N} \hat{P}_\tau^m - C_{\tau-} + I_{\tau-}^{TC} - \sum_{m=1}^{N} DiscVA_\tau^m \right)^+ \Bigg| \mathcal{G}_t \right] \\
&+ \sum_{m=1}^{K} B_t^{\hat{r}^m} \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (r_u - \hat{r}_u^m) \frac{\hat{V}_u^m}{B_u^{\hat{r}^m}} du \Big| \mathcal{G}_t \right].
\end{aligned} \tag{35}$$

Sketch: multiple aggregation levels

Multiple collateral agreements, multiple subportofolios... more realism.



However

- Well posedness of the pricing BSDEs follows along the previous arguments.
- Lengthy formulas for the valuation adjustments in line with the previous ones.

Margin sets and netting sets

Definition 16

A margin (or funding) set $\mathcal{M}$ is a set of claims whose aggregated clean values (exposures) are fully or partially covered by a CSA (collateral agreement). We let $N_{\mathcal{M}}$ denote the number of margin sets in the portfolio $\mathcal{P}$.

All trades within a margin set share the same funding source.

Definition 17

A netting set $\mathcal{N}$ is a set of margin sets whose post-margin exposures can be aggregated. We let $N_{\mathcal{N}}$ denote the number of netting sets in the portfolio $\mathcal{P}$.

Example of portfolio with multiple aggregation levels.

$$
\begin{aligned}
\mathcal{P} &= \underbrace{\{A^1, \dots, A^{N_1}\}}_{\mathcal{M}_1} \cup \underbrace{\{A^{N_1+1}, \dots, A^{N_2}\}}_{\mathcal{M}_2} \cup \dots \cup \underbrace{\{A^{N_{N_\mathcal{M}-1}+1}, \dots, A^{N_{N_\mathcal{M}}}\}}_{\mathcal{M}_{N_\mathcal{M}}} \\
&= \underbrace{\{\mathcal{M}_1, \dots, \mathcal{M}_{M_1}\}}_{\mathcal{N}_1} \cup \underbrace{\{\mathcal{M}_{M_1+1}, \dots, \mathcal{M}_{M_2}\}}_{\mathcal{N}_2} \cup \dots \cup \underbrace{\{\mathcal{M}_{M_{N_\mathcal{M}-1}+1}, \dots, \mathcal{M}_{M_{N_\mathcal{M}}}\}}_{\mathcal{N}_{N_\mathcal{N}}}
\end{aligned}
\tag{37}
$$

BSDEs of XVA on multiple aggregation levels

$$
\begin{aligned}
CVA_t^K := \sum_{m_2=1}^{N_{\mathcal{N}}} B_t^r \mathbb{E}^{\mathbb{Q}} \Bigg[1_{\{\tau<T\}} 1_{\{\tau^C<\tau^B\}} \frac{(1-R^C)}{B_\tau^r} \\
\times \Bigg[\sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} \hat{P}_\tau^{m,m_1,m_2} - DiscVA_\tau^{m,m_1,m_2} - C_{\tau-}^{\mathcal{M}_{m_1},\mathcal{N}_{m_2}} - I_{\tau-}^{TC,\mathcal{M}_{m_1},\mathcal{N}_{m_2}} \Bigg]^- \Bigg| \mathcal{G}_t \Bigg],
\end{aligned} \tag{38}
$$

and similarly for DVA, FVA etc...

$$
XVA_t^K := FVA_t^K + ColVA_t^K + MVA_t^K + CVA_t^K - DVA_t^K, \tag{39}
$$

$$
\widehat{XVA}_t^K := XVA_t^K + \sum_{m_2=1}^{N_{\mathcal{N}}} \sum_{m_1=1}^{|\mathcal{N}_{m_2}|} \sum_{m=1}^{|\mathcal{M}_{m_1}|} DiscVA_t^{m,m_1,m_2} \tag{40}
$$

and finally write the whole portfolio value as

$$
\mathscr{V}_t^K(\varphi) := \sum_{m=1}^{K} \hat{P}_t^m - \widehat{XVA}_t^K \tag{41}
$$

How much xVA should the bank charge?

- **Given an existing portfolio of K claims between the bank and the counterparty, how much xVA should be charged by the bank for a candidate $K+1$-th claim?**

1. **Stand-alone scenario:** the $(K+1)$-th contingent claim and the corresponding xVA are evaluated in isolation.
2. **Incremental xVA charge:** to account for portfolio effects involving margin and netting sets, two different scenarios are compared.
    1. *Base scenario:* $\mathscr{V}_t^K(\varphi)$ as in formula (41). Value of the portfolio before the inclusion of the candidate new trade.
    2. *Full scenario:* $\mathscr{V}_t^{K+1}(\varphi)$, computed in line with formula (41). Value of the portfolio after the inclusion of the candidate $(K+1)$-th contingent claim.

Incremental xVA charge

The bank determines the price to be charged to the counterparty as the difference between the value of the portfolio under the full and the base scenario, i.e. the bank charges the *incremental value* ΔV_t^{K+1}, defined as

$$\Delta V_t^{K+1}(\varphi) := \mathscr{V}_t^{K+1}(\varphi) - \mathscr{V}_t^{K}(\varphi). \tag{42}$$

By analyzing (42) we can isolate the impact of the $(K+1)$-th trade as follows.

$$\begin{aligned}\Delta V_t^{K+1}(\varphi) &:= \mathscr{V}_t^{K+1}(\varphi) - \mathscr{V}_t^{K}(\varphi) \\ &= \sum_{m=1}^{K+1} \hat{P}_t^m - \widehat{XVA}_t^{K+1} - \sum_{m=1}^{K} \hat{P}_t^m + \widehat{XVA}_t^{K} \\ &= \hat{P}_t^{K+1} - \left(XVA_t^{K+1} - XVA_t^{K}\right) - DiscVA_t^{K+1} \\ &= \hat{P}_t^{K+1} - \Delta XVA_t - DiscVA_t^{K+1}\end{aligned} \tag{43}$$

where, in the last step, we implicitly defined the *incremental xVA charge*

$$\Delta XVA_t := XVA_t^{K+1} - XVA_t^{K} \tag{44}$$

Non-linearity

Definition 18

The *non-linearity effect* on the $(K+1)$-th contingent claim is defined as

$$NL_t(V^{K+1}) := V_t^{K+1}(\varphi) - \Delta V_t^{K+1}(\varphi), \tag{45}$$

where $V_t^{K+1}(\varphi)$ is determined by solving the stand-alone $\mathbb{G}$-BSDE and $\Delta V_t^{K+1}(\varphi)$ is the incremental charge as defined in (43).

Remark 2

Let us observe the following.

- *The clean valuation of the contingent claim is still linear.*
- *We typically have* $\Delta XVA_t - XVA_t \neq 0$. *The stand-alone xVA of the $(K+1)$-th claim is higher than ΔXVA.*
- $NL_t(V^{K+1}) = 0$ *only when there are not portfolio/netting effects.*

Abstract of Part 2

We present a novel computational framework for portfolio-wide risk management problems, where the presence of a potentially large number of risk factors makes traditional numerical techniques ineffective. The new method utilises a coupled system of BSDEs for the valuation adjustments (xVA) and solves these by a recursive application of a neural network based BSDE solver. This not only makes the computation of xVA for high-dimensional problems feasible, but also produces hedge ratios and dynamic risk measures for xVA, and allows simulations of the collateral account.

...About BSDEs...

We recall the following fundamental result (Feynman-Kac formula):

Let $u \in C^{1,2}\left([0,T] \times \mathbb{R}^d, \mathbb{R}\right)$ solution of the Cauchy problem

$$\partial_t u + \mathcal{L}(t, x, \nabla_x u, D_x^2 u) + F(t, x, u, \sigma^\top \nabla_x u) = 0, \qquad u(T, x) = g(x)$$

where $F(t, x, u, \sigma^\top \nabla_x u) := -ku + f(t, x, u, \sigma^\top \nabla_x u)$ for a Lipschitz nonlinear function f and

$$\mathcal{L}(t, x, \nabla_x u, D_x^2 u) := \mu(t, x) \nabla_x u + \frac{1}{2} \mathrm{Tr}[\sigma\sigma^\top(t, x) D_x^2 u]$$

then $u(t, x)$ admits the stochastic representation on $[0, T] \times \mathbb{R}^d$

$$u(t, x) = \mathbb{E}^{\mathbb{Q}}\left[g(X_T) e^{-k(T-t)} + \int_t^T f(s, X_s, u, \sigma^\top \nabla_x u) e^{-k(s-t)} \, \mathrm{d}s \middle| \mathcal{F}_t \right]$$

with

$$X_t = x + \int_0^t \mu(s, X_s) \, \mathrm{d}s + \int_0^t \sigma(s, X_s) \, \mathrm{d}W_s^{\mathbb{Q}}$$

...About BSDEs...

...moreover the function u satisfies a forward-backward SDE (FBSDE):

$$\begin{cases} X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s^{\mathbb{Q}}, \\ Y_t = g(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s^\top \mathrm{d}W_s^{\mathbb{Q}}. \end{cases}$$

or, equivalently,

$$\begin{cases} \mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}} & X_0 = x \\ -\mathrm{d}Y_t = F(t, X_t, Y_t, Z_t)\,\mathrm{d}t - Z_t^\top \mathrm{d}W_t^{\mathbb{Q}} & Y_T = g(X_T) \end{cases}$$

The solution of the BSDE is a couple (Y, Z) where:

- Y_t: controlled process (value process) s.t. $Y_t = u(t, X_t)$;
- Z_t: control process (hedging strategy) s.t. $Z_t = (\sigma^\top \nabla_x u)(t, X_t)$.

...BSDEs and stochastic control...

An important fact to keep in mind: FBSDEs are intrinsically linked to the following stochastic optimal control problem

$$\underset{y,\ Z=(Z_t)_{t\in[0,T]}}{\text{minimise}} \ \mathbb{E}\left[\left|g(X_T) - Y_T^{y,Z}\right|^2\right]$$

$$\text{subject to:} \begin{cases} X_t = x + \int_0^t \mu(s, X_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s)\,\mathrm{d}W_s^{\mathbb{Q}}, \\ Y_t^{y,Z} = y - \int_0^t F\left(s, X_s, Y_s^{y,Z}, Z_s\right)\mathrm{d}s + \int_0^t Z_s^\top \mathrm{d}W_s^{\mathbb{Q}}. \end{cases}$$

Indeed, if a solution of the FBSDE exists it is a minimiser for this problem. A discretized version of this optimal control problem is at the basis of the proposed solver.

The financial framework

- Two agents: the bank (B, our perspective), the counterparty (C);
- Time horizon: $T < \infty$;
- Probability space: $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ satisfying the usual assumptions;
- Default times: τ^i, for $i \in \{B, C\}$ and $\tau = \min(\tau^B, \tau^C)$. Exponentially distributed random variables with intensity

$$\Gamma_t^i = \int_0^t \lambda_s^i \mathrm{d}s, \qquad t \in [0, T],\ i \in \{B, C\};$$

- Events structure: $\mathcal{G} = \mathcal{F} \vee \mathcal{H}$ and $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$, where:
  $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is the reference space for risk factors; $(\Omega, \mathcal{H}^i, \mathbb{H}^i, \mathbb{P})$, $i \in \{B, C\}$ is the reference space for default events,
  i.e. $\mathbb{H}^i = (\mathcal{H}_t^i)_{t \geq 0}$ for $\mathcal{H}_t^i = \sigma\left(H_u^i \middle| u \leq t \right)$ and $H_t^i := \mathbf{1}_{\tau^i \leq t}$.
  $\mathbb{H} := \mathbb{H}^B \vee \mathbb{H}^C$.
- Risk neutral probability measure: $\mathbb{Q}$;

The financial framework

- Risky assets: $X_t = (X_t^1, \ldots, X_t^d)$ solution of a SDE:

$$\mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}};$$

- Cash accounts: $B_t^r = \exp\left(\int_0^t r_u \mathrm{d}u\right);$
- Defaultable bonds: $T^\star \leq T$

$$\mathrm{d}P_t^i = \left(r_t^i + \lambda_t^i\right) P_t^i \mathrm{d}t - P_{t-}^i \mathrm{d}H_t^i,$$

- Collaterals: C_t exchanged between the parties.

The portfolio

Consider a portfolio of M (European, for simplicity) contracts.
The value of the m-th contract is:

$$Y_t^m = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T_m-t)}g_m(X_{T_m})|\mathcal{F}_t\right], \qquad t \in [0, T_m].$$

and solves the following (decoupled) FBSDE:

$$\begin{cases} \mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}} \\ -\mathrm{d}Y_t^m = -rY_t^m\mathrm{d}t - \sum_{k=1}^d Z_t^{k,m}\mathrm{d}W_t^{k,\mathbb{Q}} \\ X_0 = x \\ Y_{T^m}^m = g_m(X_{T_m}). \end{cases}$$

The value of the portfolio is: $\quad Y_t := \sum_{m=1}^{M} Y_t^m \qquad t \in [0, T].$

This is called the clean value of the portfolio.

What valuation adjustments (xVAs) are?

xVAs are further terms to be added to, or subtracted from, the clean portfolio value in order to obtain the final value of the contract:

$$\text{Full portfolio value: } \overline{Y}_t = Y_t - XVA_t \qquad t \in [0, T]$$

We are interested to pre-default ($t < \tau$) values: $\overline{XVA}_t$.

Theoretical results: we refer to Crépey ('15), Biagini-Gnoatto-Oliva ('19).

Valuation adjustments

Consider the following process:

$$\overline{XVA}_t = -\overline{CVA}_t + \overline{DVA}_t + \overline{FVA}_t + \overline{ColVA}_t,$$

where, denoted $\tilde{r} = (\tilde{r}_t)_{t\in[0,T]}$ for $\tilde{r} := r + \lambda^C + \lambda^B$,

$$\overline{CVA}_t := B_t^{\tilde{r}}\,\mathbb{E}^{\mathbb{Q}}\Big[(1-R^C)\int_t^T \frac{1}{B_u^{\tilde{r}}}\,(Y_u - C_u)^-\,\lambda_u^C \mathrm{d}u \Big| \mathcal{F}_t\Big],$$

$$\overline{DVA}_t := B_t^{\tilde{r}}\,\mathbb{E}^{\mathbb{Q}}\Big[(1-R^B)\int_t^T \frac{1}{B_u^{\tilde{r}}}\,(Y_u - C_u)^+\,\lambda_u^B \mathrm{d}u \Big| \mathcal{F}_t\Big],$$

$$\begin{aligned}\overline{FVA}_t := {}& B_t^{\tilde{r}}\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \frac{(r_u^{f,l} - r_u)\left(Y_u - \overline{XVA}_u - C_u\right)^+}{B_u^{\tilde{r}}}\mathrm{d}u \Big| \mathcal{F}_t\Big] \\ & - B_t^{\tilde{r}}\,\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \frac{(r_u^{f,b} - r_u)\left(Y_u - \overline{XVA}_u - C_u\right)^-}{B_u^{\tilde{r}}}\mathrm{d}u \Big| \mathcal{F}_t\Big],\end{aligned}$$

$$\overline{ColVA}_t := B_t^{\tilde{r}}\,\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \frac{(r_u^{c,l} - r_u)C_u^+ - (r_u^{c,b} - r_u)C_u^-}{B_u^{\tilde{r}}}\mathrm{d}u \Big| \mathcal{F}_t\Big].$$

Valuation adjustments

The following BSDE representation also holds:

$$\begin{cases} -\mathrm{d}\overline{XVA}_t = f\left(Y_t, \overline{XVA}_t\right)\mathrm{d}t - \sum_{k=1}^{d} U_t^k \mathrm{d}W_t^{k,\mathbb{Q}} \\ \overline{XVA}_T = 0, \end{cases}$$

where

$$\begin{aligned} & f\left(Y_t, \overline{XVA}_t\right) := \\ & -\left(1-R^C\right)\left(Y_t - C_u\right)^- \lambda_t^C \\ & +\left(1-R^B\right)\left(Y_t - C_u\right)^+ \lambda_t^B \\ & +\left(r_t^{f,l} - r_t\right)\left(Y_t - \overline{XVA}_t - C_t\right)^+ - \left(r_t^{f,b} - r_t\right)\left(Y_t - \overline{XVA}_t - C_t\right)^-, \\ & +\left(r_t^{c,l} - r_t\right)C_t^+ - \left(r_t^{c,b} - r_t\right)C_t^- - \left(r_t + \lambda_t^C + \lambda_t^B\right)\overline{XVA}_t \end{aligned}$$

($r^{f,l}$, $r^{f,b}$: unsecured funding rate (lending/borrowing), $r^{c,l}$, $r^{c,b}$: collateral rate (lending/borrowing), r: reference rate, R^j: recovery rate)

XVA computation

Therefore, the computation of valuation adjustments involves the numerical solution of BSDEs in possibly high dimension:

- for the values Y_t^m, $m = 1, \dots, M$;
- for the $\overline{XVA}$ itself.

XVA computation

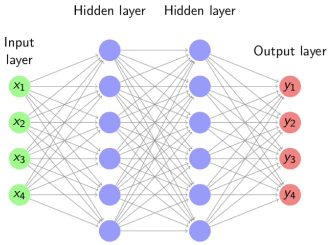
References:

- Computation of Y_t^m, $m = 1, \dots, M$:
  PDE techniques, MC simulations;
- Computation of $\overline{XVA}$:
  PDE techniques (only in some specific framework), nested MC simulations, regression based techniques of "Longstaff-Schwartz" type (coupled with Picard iteration for recursive XVAs)

  See for instance: Cesari et al. ('10), Shöftner ('08) , Abbas-Turki, Crépey, Diallo ('18), de Graaf, Kandhai, Reisinger ('18), Albanese, Crépey, Hoskinson, Saadeddine ('19) et al.

The method we propose strongly relies on the deep BSDE solver by E-Han-Jentzen ('17) .
Similar idea: She-Gercu ('17).

Artificial Neural Networks



- Consider $\mathcal{L}+1 \in \mathbb{N} \setminus \{1,2\}$ layers (input/hidden/output layers), each one consisting of ν_ℓ, $\ell = 0, \dots, \mathcal{L}$ nodes.
- Feedforward neural network $\varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$

$$x \in \mathbb{R}^d \xmapsto{\varphi} \mathcal{A}_{\mathcal{L}} \circ \varrho \circ \mathcal{A}_{\mathcal{L}-1} \circ \ldots \circ \varrho \circ \mathcal{A}_1(x) \in \mathbb{R}^d$$

where $\mathcal{A}_\ell$ $\ell = 1, \dots, \mathcal{L}$ are affine transformations of the form

$$\mathcal{A}_\ell(x) := \mathcal{W}_\ell x + \beta_\ell, \quad \ell = 1, \dots, \mathcal{L}$$

and ϱ is an activation function, for us $\varrho(x) := \max(x, 0)$.

The Deep BSDE solver by E-Han-Jentzen ('17)

Consider the FBSDE:

$$\begin{cases} \mathrm{d}X_t = \mu(t, X_t)\mathrm{d}t + \sigma(t, X_t)\mathrm{d}W_t^{\mathbb{Q}} & X_0 = x \\ -\mathrm{d}Y_t = F\left(t, X_t, Y_t, Z_t\right)\mathrm{d}t - Z_t^\top \mathrm{d}W_t^{\mathbb{Q}} & Y_T = g(X_T) \end{cases}$$

- Euler-Maruyama discretization:

$$\begin{cases} X_{n+1} = X_n - \mu(t_n, X_n)\Delta t + \sigma(t_n, X_n)\Delta W_n, & X_0 = x \\ Y_{n+1}^{y,Z} = Y_n^{y,Z} - F(t_n, X_n, Y_n^{y,Z}, Z_n)\Delta t + {Z_n}^\top \Delta W_n, & Y_0^{y,Z} = y. \end{cases}$$

- Approximation of the control Z by an ANN (parametrized by ρ):

$$\begin{cases} X_{n+1} = X_n - \mu(t_n, X_n)\Delta t + \sigma(t_n, X_n)\Delta W_n, & X_0 = x \\ Y_{n+1}^{y,\rho} = Y_n^{y,\rho} - F(t_n, X_n, Y_n^{y,\rho}, Z_n)\Delta t + \varphi_n^\rho(X_n)^\top \Delta W_n, & Y_0^{y,\rho} = y. \end{cases}$$

- Optimization:

$$\underset{y,\ \rho}{\text{minimise}}\ \mathbb{E}\left[\left|g(X_T) - Y_T^{y,\rho}\right|^2\right].$$

Deep BSDE solver

Algorithm 1: Deep algorithm for exposure simulation

Set parameters: N, L ;

▷ *time steps N , L paths for Monte Carlo loop*

Fix architecture of ANN;

▷ *intrinsically defines the dimension of parameters ρ*

Deep BSDE solver for clean values computation(N,L)

Simulate L paths $(X_n^{(\ell)})_{n=0,\dots,N}$, $\ell = 1,\dots,L$ of the forward dynamics;

Define the neural networks $(\varphi_n^{\rho})_{n=1,\dots,N}$;

for $m = 1,\dots,M$ **do**

minimize over y **and** ρ

$$\frac{1}{L}\sum_{\ell=1}^{L}\left(g_m(X_N^{(\ell)}) - \mathcal{Y}_N^{m,\rho,y,(\ell)}\right)^2 \qquad \left(\text{recall: } Y_{t_N}^m - g_m(X_{t_N}) = 0\right)$$

subject to

$$\begin{cases} \mathcal{Y}_{n+1}^{m,\rho,y,(\ell)} = \mathcal{Y}_n^{m,\rho,y,(\ell)} + r_n \mathcal{Y}_n^{m,\rho,y,(\ell)} \Delta t + (\mathcal{Z}_n^{\rho,(\ell)})^\top \Delta W_n^{(\ell)}, \\ \mathcal{Y}_0^{m,\rho,y,(\ell)} = y \\ \qquad \mathcal{Z}_n^{\rho,(\ell)} = \varphi_n^{\rho}(X_n^{(\ell)}). \end{cases}$$

Save the optimizer $(\bar{y}^m, \bar{\rho}^m)$.

end

end

Deep algorithm for non-recursive adjustments computation

Non recursive XVAs (i.e. CVA and DVA) can be written in the following general form:

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_t^T \Phi(u, Y_u)\mathrm{d}u\Big|\mathcal{F}_t\Big]$$

Algorithm 2: Deep algorithm for non-recursive adjustments

Set parameters: N, L, P ;

▷ *time steps N , paths for inner (L) and outer (P) Monte Carlo loop*

Fix architecture of ANN;

▷ *intrinsically defines the dimension of parameters* ρ

Apply Algorithm 1.

Simulate $(\mathcal{Y}_n^{m,(p)})_{n=0\dots N, p=1\dots P}$ **with** $y = \bar{y}^m$, $\rho = \bar{\rho}^m$, $m = 1, \dots, M$

Define $\mathcal{Y}_n^{(p)} = \sum_{m=1}^M \mathcal{Y}_n^{m,(p)}$ for $n = 0, \dots, N$, $p = 1, \dots, P$;

Compute the adjustment as

$$\frac{1}{P}\sum_{p=1}^{P}\left(\sum_{n=0}^{N}\eta_n\Phi(t_n, \mathcal{Y}_n^{(p)})\right)$$

where η_n are weights of the used quadrature form.

Deep algorithm for XVA computation

Algorithm 3: Deep algorithm for xVA simulation

Set parameters;

Fix architecture of ANNs;

Apply Algorithm 1

Simulate $(\mathcal{Y}_n^{m,(p)})_{n=0\dots N, p=1\dots P}$ **with** $y = \bar{y}^m$, $\rho = \bar{\rho}^m$, $m = 1, \dots, M$

Define $\mathcal{Y}_n^{(p)} = \sum_{m=1}^M \mathcal{Y}_n^{m,(p)}$ for $n = 0, \dots, N$, $p = 1, \dots, P$;

Deep BSDE solver for adjustment computation *(N,P)*:

Define the neural networks $(\psi_n^\zeta)_{n=1,\dots,N}$;

minimize over ν **and** ζ

$$\frac{1}{P}\sum_{p=1}^{P}\left(\mathcal{X}_N^{\zeta,\nu,(p)}\right)^2 \qquad \left(\text{recall: } \overline{XVA}_T = 0\right)$$

subject to

$$\begin{cases} \mathcal{X}_{n+1}^{\zeta,\nu,(p)} = \mathcal{X}_n^{\zeta,\nu,(p)} - f(\mathcal{Y}_n^{(p)}, \mathcal{X}_n^{\zeta,(p)})\Delta t + (\mathcal{U}_n^{\zeta,(p)})^\top \Delta W_n^{(p)}, \\ \mathcal{X}_0^{\zeta,\nu,(p)} = \nu \\ \mathcal{U}_n^{\zeta,(p)} = \psi_n^\zeta. \end{cases}$$

end

Numerical results

Single stock with Black-Sholes dynamics

$$\mathrm{d}X_t = rX_t\mathrm{d}t + \sigma X_t \mathrm{d}W_t^{\mathbb{Q}}, \qquad X_0 = x_0$$

and a contingent claim defined by

$$Y_t = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}g(X_T)|\mathcal{F}_t\right].$$

The discounted positive and negative discounted expected exposure of Y are:

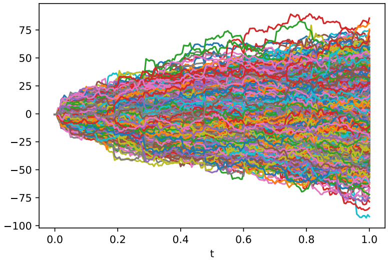
$$DEPE(s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(s-t)}\left(Y_s\right)^{+}\Big|\mathcal{F}_t\right], \tag{46}$$

$$DENE(s) = -\mathbb{E}^{\mathbb{Q}}\left[e^{-r(s-t)}\left(Y_s\right)^{-}\Big|\mathcal{F}_t\right]. \tag{47}$$

We consider:

- A forward on X: $g(X_T) = X_T - x_0$.
- A European call: $g(X_T) = \Big(X_T - x_0\Big)^{+}$

A forward on X



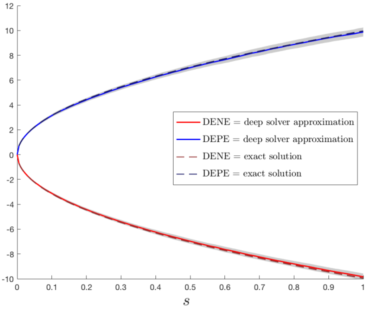


Figure: Approximated clean value (left) and EPE and ENE (right) for a forward contract. Parameters used: $\sigma = 0.25$, $r = 0$, $x_0 = 100$, $T = 1$.

A European Call option

The pathwise exposure Y at time $s \in [t, T]$ is given by the Black-Sholes formula.

It follows immediately that

$$DEPE(s) = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(s-t)} Y_s \middle| \mathcal{F}_t \right] = Y_t, \quad \text{and} \quad DENE(s) = 0.$$

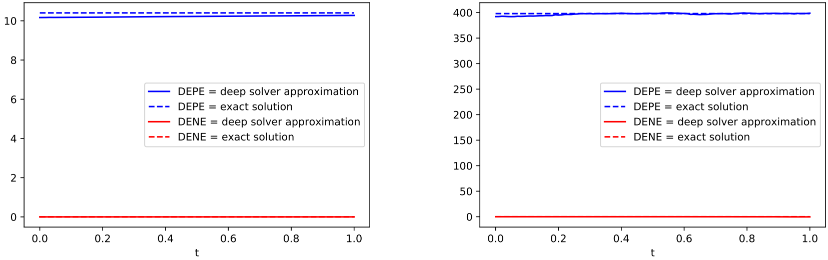
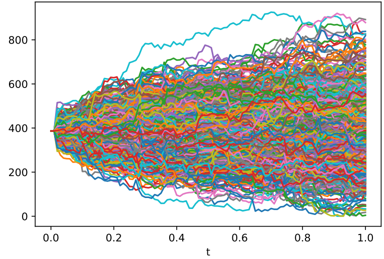


Figure: DEPE and DENE for a European call option (left) and for a European basket option with 100 underlyings (right). Parameters used: $\sigma = 0.25$, $r = 0.01$, 100, $T = 1$

A basket call: $g(S_T^1, \ldots, S_T^d) = \left(\sum_{i=1}^d S_T^i - K\right)^+$, $d = 100$.



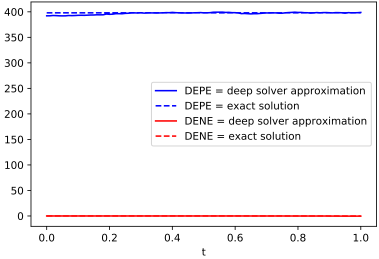


Figure: Exposure (l) and DEPE and DENE (r) for similar parameters.

Parameters used: outer MC paths $P = 1024$, inner MC paths $L = 64$, internal layers $\mathcal{L} - 1 = 2$, $\nu = d + 10 = 110$, $\mathcal{I} = 4000$, time steps $N = 100$.

The price varies over t between 393.02 and 400.82.

A time 0 MC estimate with 10^6 paths is 398.08, with C. I. [397.61, 398.56].

Forward – FVA

In this stylised test, we

- ignore CVA and DVA, i.e., set $\tau^C = \tau^B = +\infty$;
- assume fully uncollateralized, i.e. $C_t \equiv 0$;
- set $r^{c,b} = r^{c,l} = r = 0.02$, and $r^{f,b} = r^{f,l} = 0.04$.

Then $V_t = \widehat{V}_t - \overline{FVA}_t$, where

$$\overline{FVA}_t = B_t^r \mathbb{E}^{\mathbb{Q}} \left[\int_t^T (B_u^r)^{-1} (r_u^f - r_u) \left(\widehat{V}_u - \overline{FVA}_u \right) \mathrm{d}u \middle| \mathcal{F}_t \right].$$

An analytic solution is available: $\overline{FVA}_0^{\text{exact}} = 0.0392$.

The deep xVA solver gives $\overline{FVA}_0 = 0.0395$.

($N = 100$, $L = 64$ and $P = 2048$ with 2 hidden layers with $d + 20 = 21$ nodes)

Conclusions and future directions

- In the performed numerical tests the approach leads to good numerical results;
- The deep solver is appealing for products depending on many risk factors.
- The deep solver returns by construction the control of the BSDE, hence we have access to all greeks by means of automatic differentiation.
- Numerical tests on first order greeks against known solutions show a good level of precision.

Also

- theoretical estimates to validate results in absence of benchmarks: a posteriori error bounds as in Han-Long ('18).
- pathwise simulation of the collateral account is possible.
- Our approach can be readily used to compute risk measures at portfolio level (VaR, ES...)

Thank you very much for your attention!