FX Open Forward

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Introduction and plan

FX Open Forward allows the long party of the contract to freely convert (i.e. exercise) any arbitrary fraction of the notional at the locked exchange rate at any time during the contract lifetime, subject to the obligation that the full contractual notional needs to be exercised by the expiry date;

$\rightarrow$ Extension of the traditional FX Forward;

$\rightarrow$ The additional flexibility with the size and the timing of the exercise allows the hedger to perform currency conversion "on demand", eliminating the uncertainty of cash flows timing mismatch.

Despite the popularity of FX Open Forward, there appear to be very few studies investigating its pricing behaviours:

- Based on heuristic arguments, Kwok and Lau (2003) point out that the flexibility of exercising any arbitrary fraction of the notional during the lifetime of a FX Open Forward is redundant;

Introduction and plan

- Giribone and Ligato (2016) study numerical pricing of FX Open Forward by tree-based methods.
- Holbl and Lovric (2019) propose a least-square Monte Carlo algorithm to evaluate FX Open Forward by relating the product to a swing option.

Target: It is to characterize the optimal exercise strategy and analysis the FX open forward in the Black-Scholes model with the possibility of negative interest rates;

Plan:

- Optimal exercise strategy
- Analysis of the FX Open Forward (American forward) in a classical Black-Scholes model
- Numerical analysis
- Summary and concluding remarks

American derivative with partial early exercise

To develop intuition, let's focus on a discrete time model:

- Filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0,\dots,M}, \mathbb{Q})$
- Trading times $n \in \{0, 1, ..., M\}$, risky asset price process $S = (S_n)_{n=0,1,\dots,M}$ and domestic riskfree money market process $R = (R_n)_{n=0,1,\dots,M}$
- Derivative product with payoff $g(\cdot)$, maturity date M and notional N
- The derivative holder can partially exercise a fraction $\theta_n \in [0, N]$ of the contract and receives an immediate cash flow of $\theta_n g(S_n)$ with the constraint $\sum_{n=0}^{M} \theta_n = N$
- The time-zero fair price of the contract

$$\sup_{\theta \in \mathscr{A}} \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} \theta_n g(S_n)\right] \tag{1}$$

$$\mathscr{A} := \left\{ \theta = (\theta_n)_{n=0,\dots,M} : \theta \text{ is } \{\mathcal{F}_n\}\text{-adapted}, \ \theta_n \in [0, N] \text{ for all } n, \ \sum_{n=0}^{M} \theta_n = N \right\}.$$

American derivative with partial early exercise

Proposition

Let $\mathscr{T}$ be the set of $\{\mathcal{F}_n\}$-stopping times valued on $\{0, 1, ..., M\}$. Then

$$\sup_{\theta \in \mathscr{A}} \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} \theta_n g(S_n)\right] = N \sup_{\tau \in \mathscr{T}} \mathbb{E}\left[R_\tau^{-1} g(S_\tau)\right]. \tag{2}$$

Moreover, if τ^* is the optimiser to $\sup_{\tau \in \mathscr{T}} \mathbb{E}\left[R_\tau^{-1} g(S_\tau)\right]$, then

$$\sup_{\theta \in \mathscr{A}} \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} \theta_n g(S_n)\right] = \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} \theta_n^* g(S_n)\right] \tag{3}$$

under the choice of $\theta_n^* := N 1_{(n=\tau^*)}$.

$\rightarrow$ FX open price $=$ American option price on S with payoff g and notional N.

American derivative with partial early exercise

Proof.

1. Define $J(\theta) := \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1}\theta_n g(S_n)\right]$ and $H(\tau) := \mathbb{E}\left[R_\tau^{-1} g(S_\tau)\right]$. For $\tau \in \mathscr{T}$, consider $\hat{\theta}$ defined via $\hat{\theta}_n = N 1_{(n=\tau)}$. By construction, $\hat{\theta} \in \mathscr{A}$ and hence

$$\sup_{\theta\in\mathscr{A}} J(\theta) \geq J(\hat{\theta}) = N\mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} 1_{(n=\tau)} g(S_n)\right] = N\mathbb{E}\left[R_\tau^{-1} g(S_\tau)\right] = NH(\tau).$$

As τ is arbitrary, it leads to $\sup_{\theta\in\mathscr{A}} J(\theta) \geq N \sup_{\tau\in\mathscr{T}} H(\tau)$

2. For $\theta \in \mathscr{A}$, define a stochastic process indexed by $x \in [0, N]$:

$$\hat{\tau}(x) := \inf\left\{k \geq 0 : \sum_{i=0}^{k} \theta_i \geq x\right\} \tag{4}$$

$\hat{\tau}(x)$ represents the time when x out of N units of the notional has been exercised and is left continuous, increasing pure jump process with state space $\{0, 1, ..., M\}$ with $\hat{\tau}(0) = 0$ and $\hat{\tau}(N) \leq M$. $\square$

American derivative with partial early exercise

Proof.

Importantly, we have $\hat{\tau}(x) \in \mathscr{T}$ for all x and $\theta_n = \int_0^N 1_{(\hat{\tau}(x)=n)}\, dx$.

$$J(\theta) = \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1}\theta_n g(S_n)\right] = \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1}\left(\int_0^N 1_{(\hat{\tau}(x)=n)}\, dx\right) g(S_n)\right] \quad (5)$$

$$= \int_0^N \mathbb{E}\left[\sum_{n=0}^{M} R_n^{-1} 1_{(\hat{\tau}(x)=n)} g(S_n)\right] dx \quad (6)$$

$$= \int_0^N \mathbb{E}\left[R_{\hat{\tau}(x)}^{-1} g(S_{\hat{\tau}(x)})\right] dx \quad (7)$$

$$= \int_0^N H(\hat{\tau}(x))\, dx \le \int_0^N \sup_{\tau\in\mathscr{T}} H(\tau)\, dx = N \sup_{\tau\in\mathscr{T}} H(\tau). \quad (8)$$

Taking supremum over $\theta \in \mathscr{A}$ now yields $\sup_{\theta\in\mathscr{A}} J(\theta) \le N \sup_{\tau\in\mathscr{T}} H(\tau)$. □

Valuation of FX Open Forward under Black-Scholes model in finite maturity

$$\frac{dS_t}{S_t} = (r_d - r_f)\,dt + \sigma\,dB_t, \quad S_0 = s \tag{9}$$

with solution

$$S_u^{t,s} = s \exp\left(\left(r_d - r_f - \frac{\sigma^2}{2}\right)(u-t) + \sigma(B_u - B_t)\right), \qquad u \geq t, \quad s > 0. \tag{10}$$

where $\sigma > 0$ is the FX volatility, $r_d \in \mathbb{R}$ and $r_f \in \mathbb{R}$ are respectively the domestic and foreign interest rates.

$$V(t,s) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\left[e^{-r_d(\tau - t)} g(S_\tau) | S_t = s\right], \tag{11}$$

where $\mathcal{T}_{t,T}$ is the set of all $\{\mathcal{F}_t\}$-stopping times valued in $[t, T]$, payoff

$$g(s) = \begin{cases} s - K, & \text{purchase Open Forward} \\ K - s, & \text{sale Open Forward} \end{cases} \tag{12}$$

K the strike price and T the option expiry.

Valuation of FX Open Forward under Black-Scholes model in finite maturity

Some degenerate cases and properties for $V(t,s)$:

1. If $r_d \geq 0 \geq r_f$, it is optimal to defer the exercise of the contract until the maturity date ($\tau^* = T$);
2. If $r_f \geq 0 \geq r_d$, it is optimal to early exercise the contract immediately ($\tau^* = t$);
3. $V(t,s) \geq g(s) = s - K$ for all $(t,s) \in [0,T] \times (0,\infty)$;
4. For any fixed $s > 0$, $t \to V(t,s)$ is decreasing with $V(T,s) = g(s)$;
5. For any fixed $t \in [0,T]$, $s \to V(t,s)$ is a continuous, convex and increasing function.

Valuation of FX Open Forward under Black-Scholes model in finite maturity

Proof of (1) and (2):

Proof.

Write $J(t,s;\tau) := \mathbb{E}\left[e^{-r_d(\tau-t)}g(S_\tau)|S_t = s\right]$. Then under $r_d \geq 0 \geq r_f$, for any $\tau \in \mathcal{T}_{t,T}$ we have

$$J(t,s;\tau) = s\mathbb{E}[e^{-r_f(\tau-t)}e^{\sigma(B_\tau - B_t)-\frac{\sigma^2}{2}(\tau-t)}] - K\mathbb{E}[e^{-r_d(\tau-t)}] \tag{13}$$

$$\leq se^{-r_f(T-t)}\mathbb{E}[e^{\sigma(B_\tau - B_t)-\frac{\sigma^2}{2}(\tau-t)}] - Ke^{-r_d(T-t)} \tag{14}$$

$$= se^{-r_f(T-t)} - Ke^{-r_d(T-t)}. \tag{15}$$

$V(t,s) \leq se^{-r_f(T-t)} - Ke^{-r_d(T-t)}$, equality is attained for $\tau = T$ hence $\tau^* = T$.

Similarly, if $r_f \geq 0 \geq r_d$, then

$$J(t,s;\tau) = s\mathbb{E}[e^{-r_f(\tau-t)}e^{\sigma(B_\tau - B_t)-\frac{\sigma^2}{2}(\tau-t)}] - K\mathbb{E}[e^{-r_d(\tau-t)}] \leq s\mathbb{E}[e^{\sigma(B_\tau - B_t)-\frac{\sigma^2}{2}(\tau-t)}] - K = s - K \tag{16}$$

$\sup_{\tau\in\mathcal{T}_{t,T}} J(t,s;\tau) \leq s - K$ and equality is attained under $\tau = t$. Hence $V(t,s) = J(t,s;\tau = t) = s - K$ and $\tau^* = t$ is optimal. □

Valuation of FX Open Forward under Black-Scholes model in finite maturity

Theoretical characterisation of the optimal exercise strategy

From the standard theories of optimal stopping of Markovian process, the optimal stopping time associated with value function (11) is given by

$$\tau^* = \tau^*(t,s) = \inf\{u \geq t : V(u, S_u^{t,s}) = g(S_u^{t,s})\}. \tag{17}$$

Define the exercise (stopping) set as

$$\mathcal{D} := \{(t,s) \in [0,T] \times (0,\infty) : V(t,s) = g(s)\} \tag{18}$$

and the continuation set as

$$\mathcal{C} := \{(t,s) \in [0,T] \times (0,\infty) : V(t,s) > g(s)\}. \tag{19}$$

Then τ^* can be rewritten as $\tau^* = \inf\{u \geq t : (u, S_u^{t,s}) \in \mathcal{D}\}$

Valuation of FX Open Forward under Black-Scholes model in finite maturity

Proposition

1. *If* $r_d > 0$ *and* $r_f > 0$*, there exists a strictly positive function* $b : [0, T) \to (0, \infty)$ *such that*

$$\mathcal{C} := \{(t, s) \in [0, T) \times (0, \infty) : s < b(t)\} \tag{20}$$

$$\mathcal{D} := \{(t, s) \in [0, T) \times (0, \infty) : s \geq b(t)\} \cup \{\{T\} \times (0, \infty)\}. \tag{21}$$

2. *If* $r_d < 0$ *and* $r_f < 0$*, there exists a non-negative function* $b : [0, T) \to [0, \infty)$ *such that*

$$\mathcal{C} := \{(t, s) \in [0, T) \times (0, \infty) : s > b(t)\} \tag{22}$$

$$\mathcal{D} := \{(t, s) \in [0, T) \times (0, \infty) : s \leq b(t)\} \cup \{\{T\} \times (0, \infty)\}. \tag{23}$$

Valuation of FX Open Forward under Black-Scholes model in finite maturity

- The continuation set $\mathcal{C}$ is non-empty;
- When the rates are positive, it is optimal to early exercise the contract when the spot price is sufficiently high, similar to the American call option;
- When the rates are negative, it is optimal to early exercise the contract when the spot price is sufficiently low, very different to the American call option;
- For FX open forward, there exists a single time-dependent threshold $b(t)$ separating the waiting and continuation region, unlike the American call option which have disconnected waiting region when rates are negative.

Valuation of FX Open Forward under Black-Scholes model in finite maturity

Proposition

The early exercise boundary function $b : [0, T) \to [0, \infty)$ *has the following properties:*

1. b *is decreasing (resp. increasing) if* $r_d > 0$ *and* $r_f > 0$ *(resp.* $r_d < 0$ *and* $r_f < 0$*).*
2. $b(t) \geq \frac{r_d}{r_f} K$ *(resp.* $b(t) \leq \frac{r_d}{r_f} K$*) if* $r_d > 0$ *and* $r_f > 0$ *(resp.* $r_d < 0$ *and* $r_f < 0$*) for all* $t \in [0, T)$*.*
3. b *is continuous and* $b(T) := \lim_{t \to T} b(t) = \frac{r_d}{r_f} K$*.*

Valuation of FX Open Forward under Black-Scholes model in finite maturity

For (3), when time to expiry is very small, the exercise boundary is approximately given by $\frac{r_d}{r_f}K$:

$\rightarrow$ Suppose the current spot is $S_{T-\Delta} = s$ where the contract will expire after Δ unit of time;

- The immediate payoff is $s - K$ if the contract is exercised now;
- Otherwise, the discounted risk-neutral expected payoff at the expiry date is $se^{-r_f\Delta} - Ke^{-r_d\Delta} \approx s(1 - r_f\Delta) - K(1 - r_d\Delta)$;

Early exercise is therefore beneficial if and only if

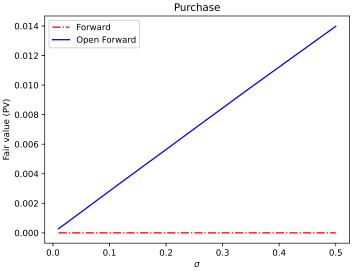
$$s - K > s(1 - r_f\Delta) - K(1 - r_d\Delta)$$
$$\Leftrightarrow$$
$$s > \frac{r_d}{r_f}K \ \textit{if} \ r_d > 0 \text{ and } r_f > 0, \text{ or } \ s < \frac{r_d}{r_f}K \ \textit{if} \ r_d < 0 \text{ and } r_f < 0$$

Forwards and Open Forwards

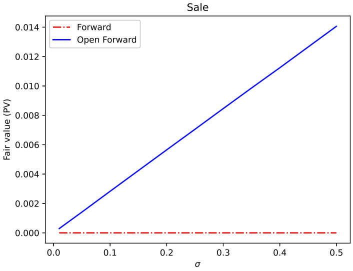
$$f_T = \begin{cases} S_T - K, & \text{for a long position;} \\ K - S_T, & \text{for a short position,} \end{cases}$$

and the fair or present values (pv) of such contracts are

$$pv = \begin{cases} e^{-r_f T} S_0 - e^{-r_d T} K, & \text{for a long position;} \\ e^{-r_d T} K - e^{-r_f T} S_0, & \text{for a short position.} \end{cases}$$

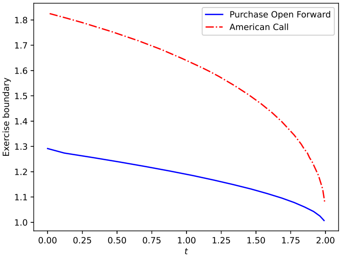


(a) Currency purchase forward and Open Forward

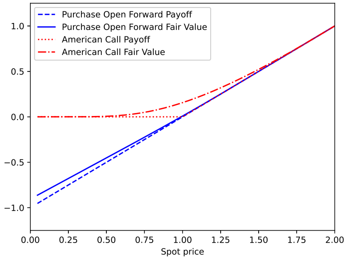


(b) Currency sale forward and Open Forward

Figure: PV comparison between forwards and Open Forwards as functions of volatility σ: $S_0 = 1,\ K = 1,\ T = 2,\ r_d = 5\%,\ r_f = 5\%$.

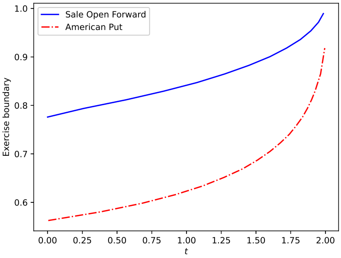


(a) Exercise boundaries for American call and purchase Open Forward, above which early exercise is optimal.

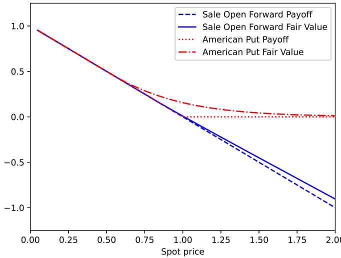


(b) Fair values of American call and purchase Open Forward as functions of spot price.

Figure: Comparison between American call and purchase Open Forward. Model and market parameters: $S_0 = 1$, $K = 1$, $T = 2$, $r_d = 5\%$, $r_f = 5\%$, $\sigma = 30\%$.



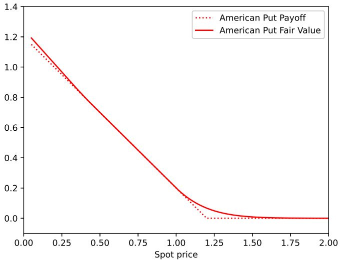
(a) Exercise boundaries for American put and sale Open Forward, below which early exercise is optimal.



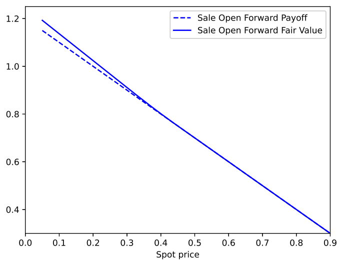
(b) Fair values of American put and sale Open Forward as functions of spot price.

Figure: Comparison between American put and sale Open Forward. Model and market parameters: $S_0 = 1$, $K = 1$, $T = 2$, $r_d = 5\%$, $r_f = 5\%$, $\sigma = 30\%$.

Numerical analysis

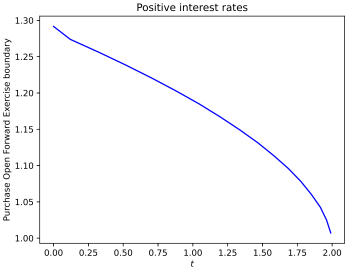


(a) American put payoff and fair value as functions of spot

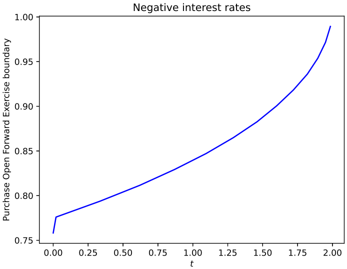


(b) Sale Open Forward payoff and fair value as functions of spot

Figure: Comparison between American put and Sale Open Forward with negative rates. Model and market parameters:
$S_0 = 1,\ K = 1.2,\ T = 1,\ r_d = -4\%,\ r_f = -12\%,\ \sigma = 20\%.$



(a) Exercise boundary of purchase Open Forward under positive rates



(b) Exercise boundary of purchase Open Forward under negative rates

Figure: Exercise boundaries comparison for purchase Open Forward with positive rates ($r_d = r_f = 5\%$) and negative rates ($r_d = r_f = -2\%$). Others model and market parameters: $S_0 = 1,\ K = 1,\ T = 2,\ \sigma = 30\%$.

Summary and concluding remarks

- We first showed the optimal strategy has a "all-or-nothing"/"bang-bang" form where the contract holder always settles the full notional at once.

  $\rightarrow$ This result simplifies significantly the problem and reduces the analysis to that of a standard American derivative;

- Then we focus on a Black-Scholes model to theoretically analyse the pricing and the optimal exercise strategy:

  $\rightarrow$ The form of the exercise strategy crucially depends on the signs of the interest rates, and is significantly different from that of American call/put option when interest rates are negative because the lack of optionality preludes the continuation region over the out-of-money regime of spot price;

- Future research:

  $\rightarrow$ One natural model extension is to better capture the stochastic dynamic of interest rates and study how the state of interest rates interacts with the early exercise decision;