A GLMM approach for two-population mortality

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Two-population mortality modelling

- Modeling insured (small) population using information (mortality) from a reference
  - Adding credibility
  - Enhance predictions performance
- The roles of (multi)-population mortality models in securitization:
  - Basis-risk modeling and quantification
  - Pricing and managing securities, for e.g. index based transfers

Recent frameworks/models: Cairns et al. (2011), Dowd et al. (2011) Cairns et al. (2011), Li and Hardy (2011), Salhi and Loisel (2012), Li (2012, 2013), Yang and Wang (2013), Chan et al. (2014) among others.

Brass relational model

logit-regression

- The (initial) Brass (1971) model:

$$\mathrm{logit}(q_x^i) = \beta_0^i + \beta_1^i \mathrm{logit} q_x^r + \epsilon^i$$

where q_x^i and q_x^r are one-year death probabilities for population "i" and a reference "r", with $\epsilon \sim \mathcal{N}(0, \sigma)$.

- A (dynamic) Brass model would be of the form

$$\mathrm{logit}(q_{x,t}^i) = \beta_{0,t}^i + \beta_{1,t}^i \mathrm{logit} q_{x,t}^r + \epsilon_t,$$

where $\beta_{0,t}$ and $\beta_{1,t}$ have to be projected.

  - Hard to estimate parameters for small populations
  - For some dataset it is hard to capture the dynamics for $\beta_{0,t}$ and $\beta_{1,t}$

A GLMM approach (1/4)

The model assumptions

- Assume that the age-specific (random) deaths $D^i_{x,t}$ of the i^{th} insured population has a Binomial distribution $\sim \text{Bin}(L^i_{x,t}, q^i_{x,t})$, where
  - $L^i_{x,t}$ is the exposure to risk
  - $q^i_{x,t}$ is the one-year death probability
- The $D^i_{x,t}$'s density can be written in the following form (exponential family)

$$f(y;\theta^i_{x,t}) = \exp\Big(\big(y\theta^i_{x,t} - g(\theta^i_{x,t})\big) + c(y)\Big),$$

where

$$\theta^i_{x,t} = \log(q^i_{x,t}/(1-q^i_{x,t})) \quad \text{and} \quad g(\theta^i_{x,t}) = -L^i_{x,t}\log(1-q^i_{x,t})$$

A GLMM approach (2/4)

The model assumptions

- The canonical link for a *Binomial* distribution is the logit function, so the linear model is of the following form

$$\mathrm{logit}(q^i_{x,t}) = (\beta^i)' Y_{x,t} + \epsilon^i_x$$

- The time varying covariates term is of the following form

$$(\beta^i)' Y_{x,t} \equiv \beta^i_0 + \beta^i_1 \mathrm{logit}(q^r_{x,t}),$$

with $q^r_{x,t}$ is a reference mortality (e.g. national). So the full model is given by

$$D^i_{x,t} \sim \mathrm{Bin}(L^i_{x,t}, q^i_{x,t}) \quad \text{s.t.} \quad \mathrm{logit}(q^i_{x,t}) = \beta^i_0 + \beta^i_1 \mathrm{logit}(q^r_{x,t}) + \epsilon^i_x,$$

with $\epsilon^i_x \sim \mathcal{N}(0, \sigma^i_x)$

A GLMM approach (3/4)

The model assumptions

- The model is a *generalization* of the Brass (1971)'s *relational* model
- The parameters β_0^i and β_1^i have to be estimated using a MLE.
  - Need specifying an underlying model for the reference mortality $q_{x,t}^r$
  - A candidate model is as follows (CBD2):

$$\mathrm{logit} q_{x,t}^r = \kappa_{1,t} + \kappa_{2,t} x,$$

  where $(\kappa_{1,t}, \kappa_{2,t})$ are correlated random walks with joint-distribution Φ and independent of ϵ_x^i.

- The full model is as follows:

$$\begin{cases} D_{x,t}^i \sim \mathrm{Bin}(L_{x,t}^i, q_{x,t}^i), \\ \mathrm{logit}(q_{x,t}^i) = \beta_0^i + \beta_1^i(\kappa_{1,t} + \kappa_{2,t} x) + \epsilon_x^i, \\ \kappa_{1,t} = \kappa_{1,t-1} + \mu_1 + \varepsilon_{1,t}, \\ \kappa_{2,t} = \kappa_{2,t-1} + \mu_2 + \varepsilon_{2,t}, \end{cases} \tag{M0}$$

A GLMM approach (4/4)

Model Enhancement

- There are obvious difficulties to carry the model out, as some of ages experience may lack of sufficient information/deaths/exposures
- It seems that combining single (at the portfolio level) and collective (other portfolios) histories among ages may lead to better predictions, see Lai, Su and Sun (2013).
- We have at our disposal n insured portfolios (populations).
- We propose the following adjustment to the (M0):

$$\begin{cases} D^i_{x,t} \sim \mathrm{Bin}(L^i_{x,t}, q^i_{x,t}), \\ \mathrm{logit}(q^i_{x,t}) = \beta^i_0 + \beta^i_1(\kappa_{1,t} + \kappa_{2,t}x) + \alpha_x \mathrm{logit}(q^{\bullet}_{x,t-1}) + \epsilon^i_x, \\ \kappa_{1,t} = \kappa_{1,t-1} + \mu_1 + \varepsilon_{1,t}, \\ \kappa_{2,t} = \kappa_{2,t-1} + \mu_2 + \varepsilon_{2,t}, \end{cases} \tag{M1}$$

where $\mathrm{logit}(q^{\bullet}_{x,t})$ is a common factor over for a single age x over the n porfolios given as

$$\mathrm{logit}(q^{\bullet}_{x,t-1}) = \mathrm{logit}(n^{-1}\sum_{i=1}^{n} q^i_{x,t-1})$$

Comments

Estimations

- The factor $\mathrm{logit}(q^{\bullet}_{x,t-1})$ can be connected to credibility or frailty models
- The estimation of paramaters β_0^i, β_1^i and α_x can be done using the max-likelihood estimator based on

$$L(\beta_0, \beta_1, \alpha, \sigma) = \prod_i \left(\int \prod_x \prod_t f(D^i_{x,t}, \theta^i_{x,t}) \Phi(x, y) \phi(z) dxdydz \right),$$

using an adaptive gaussian Quadrature algorithms

Numerical Analysis

Datasets

- 3 French portfolios (relatively small) (years 2007 – 2012, ages 55 – 89)

	2007	2008	2009	2010	2011	2012
P1	2928	2987	3023	3074	3089	3142
P2	1592	6793	6901	7004	6637	8875
P3	19172	19468	19638	19634	19537	19758

Numerical Analysis

Goodness-of-fit/Parsimony - AIC

- 3 French portfolios (relatively small) (years $2007 - 2012$, ages $55 - 89$)

	P1	P2	P3
M0	740.63	312.55	788.50
M1	632.54	301.93	761.16

Numerical Analysis

Backtest - MQE

- 3 French portfolios (relatively small) (years 2007 – 2012, ages 55 – 89)
- Estimation period 2007 – 2010,
- Projection period 2011 – 2012

	P1	P2	P3
M0	1.36e-2	1.18e-2	2.26e-3
M1	1.07e-3	6.02e-3	1.86e-3

Conclusion

- More work should be done on the effectiveness/performance of the model
- Robustness with regard to parameters estimation
- Taking into account censored datasets
- The common factor can be considered as non-observable (use of Kalman-Filter for estimation)
- Backtest on other datasets