It's All in the Hidden States: A Hedging Method with an Explicit Measure of Population Basis Risk

Yanxin Liu and Johnny Siu-Hang Li

Presenter: Yanxin Liu

September 9, 2015

Outline

Introduction

- Population Basis Risk
  - ⇒ A risk associated with the difference in mortality experience between the population of the hedging instruments and the population of the liability being hedged
- Multi-Population Mortality Model
  - Augmented Common Factor Model (Li and Lee, 2005)
  - Co-integrated Lee-Carter Model (Li and Hardy, 2011)
  - Gravity Model (Dowd et al., 2011)
  - Two-population CBD Model (Cairns et al., 2011)
  - M-CBD Model (Zhou and Li, 2015)
- Managing Basis Risk
  - Coughlan et al.(2011) develop a framework for analysing longevity basis risk;
  - Cairns et al.(2013) use stochastic simulation to analyse several key risk factors, including the population basis risk, that influences the hedge effectiveness of a longevity hedge.

Why An Explicit Measure of Population Basis Risk is Needed?

- As the market in longevity and mortality-related risk becomes more liquid, the reference population of a standardized longevity contract can be linked to a larger range of populations,
  - e.g., the Life and Longevity Markets Association provides longevity indices for four different populations:
    - United States;
    - England and Wales;
    - Netherlands;
    - Germany.
  - Those four populations could further be used as the reference population of longevity contracts such as q-forwards.
- Without an explicit measure of basis risk,
  - the hedgers are not able to evaluate the basis risk profile for their hedge portfolios;
  - no guideline for hedgers to select the most appropriate standardized contracts in a longevity hedge.

Research Objectives

1. To introduce a new hedging method called the generalized state space hedging method;
   - allows us to decompose the underlying longevity risk into a component arising solely from the hidden states that are shared by all populations and components stemming exclusively from the hidden states that are population-specific.
   - can be used as long as the mortality model can be written in state space form;

2. To develop a quantity called standardized basis risk profile, which is an efficient measure for hedgers to select the most appropriate population among all candidate populations.

Outline

State Space Model: A General State Space Structure

Observation Equation

$$\vec{y}_t = \vec{D} + B\vec{\alpha}_t + \vec{\epsilon}_t$$

Transition Equation

$$\vec{\alpha}_t = \vec{U} + A\vec{\alpha}_{t-1} + \vec{\eta}_t$$

where

- $\vec{y}_t$ is the vector of observations at time t;
- $\vec{\alpha}_t$ is the vector of hidden states at time t;
- $\vec{D}$ and $\vec{U}$ are the vectors of constants;
- B is the design matrix (linear transformation between $\vec{y}_t$ and $\vec{\alpha}_t$);
- A is a squared matrix (first-order Markov relation of $\vec{\alpha}_t$);
- $\vec{\epsilon}_t \overset{\text{i.i.d.}}{\sim} \text{MVN}(0, R)$ and $\vec{\eta}_t \overset{\text{i.i.d.}}{\sim} \text{MVN}(0, Q)$.

State Space Model and Multi-Population Mortality Model

- Suppose that we consider n_p populations.
- We partition $\vec{\alpha}_t$ into $\vec{\alpha}_t = ((\vec{\alpha}_t^c)', (\vec{\alpha}_t^{(1)})', \ldots, (\vec{\alpha}_t^{(n_p)})')'$, where
  - $\vec{\alpha}_t^c$ represents the states that are shared by all populations being modeled;
  - $\vec{\alpha}_t^{(p)}$ represents the states that are exclusive to population p.
  - accordingly, $\vec{\eta}_t = ((\vec{\eta}_t^c)', (\vec{\eta}_t^{(1)})', \ldots, (\vec{\eta}_t^{(n_p)})')'$;
  - Q would be a block diagonal matrix with blocks $Q^c, Q^{(1)}, \ldots, Q^{(n_p)}$ being the covariance matrix of $\vec{\eta}_t^c, \vec{\eta}_t^{(1)}, \ldots, \vec{\eta}_t^{(n_p)}$, respectively.

Reformulating the current mortality models into state space form:

Illustration I: ACF Model (Li and Lee, 2005)
Illustration II: M-CBD Model (Zhou and Li, 2015)

Augmented Common Factor Model (ACF Model, Li and Lee, 2005)

- The ACF model assumes that

$$\begin{array}{rcll} \ln(m_{x,t}^{(p)}) & = & a_x^{(p)} + b_x^c k_t^c + b_x^{(p)} k_t^{(p)} + \epsilon_{x,t}^{(p)}, & \\ k_t^c & = & \mu^c + k_{t-1}^c + \eta_t^c, & \\ k_t^{(p)} & = & \mu^{(p)} + \phi^{(p)} k_t^{(p)} + \eta_t^{(p)}, & p = 1, \ldots, n_p, \end{array}$$

  where
  - $m_{x,t}^{(p)}$ is population p's central death rate at age x and in year t;
  - $a_x^{(p)}$, b_x^c, $b_x^{(p)}$, μ^c, $\mu^{(p)}$ and $\phi^{(p)}$ are constants;
  - $\sum_x b_x^c = 1$, $\sum_x b_x^{(p)} = 1$ and $|\phi^{(p)}| < 1$;
  - $\epsilon_{x,t}^{(p)} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_\epsilon^2)$, $\eta_t^c \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, Q^c)$ and $\eta_t^{(p)} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, Q^{(p)})$.

ACF Model in State Space Form

In ACF model, the observation equation and transition equation can be specified as

$$\vec{y}_t = \vec{D} + B\vec{\alpha}_t + \vec{\epsilon}_t$$

and

$$\vec{\alpha}_t = \vec{U} + A\vec{\kappa}_{t-1} + \vec{\eta}_t,$$

where

$$\begin{aligned}
\vec{y}_t &= \left(\ln(m^{(1)}_{x_a,t}),\ \ldots,\ \ln(m^{(1)}_{x_b,t}),\ \ldots, \ln(m^{(n_p)}_{x_a,t}),\ \ldots,\ \ln(m^{(n_p)}_{x_b,t})\right)',\\
\vec{D} &= \left(a^{(1)}_{x_a}, \ldots, a^{(1)}_{x_b}, \ldots, a^{(n_p)}_{x_a}, \ldots, a^{(n_p)}_{x_b}\right)',\\
\vec{\alpha}_t &= \left(k^c_t,\ k^{(1)}_t,\ \ldots,\ k^{(n_p)}_t\right)',\\
\vec{\epsilon}_t &= \left(\epsilon^{(1)}_{x_a,t},\ \ldots,\ \epsilon^{(1)}_{x_b,t},\ \ldots, \epsilon^{(n_p)}_{x_a,t},\ \ldots,\ \epsilon^{(n_p)}_{x_b,t}\right)',\\
\vec{U} &= \left(\mu^c, \mu^{(1)},\ \ldots,\ \mu^{(n_p)}\right)',
\end{aligned}$$

ACF Model in State Space Form

and

$$A = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \phi^{(1)} & 0 & \cdots & 0 \\ 0 & 0 & \phi^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi^{(n_p)} \end{pmatrix}, B = \begin{pmatrix} \vec{b}^c & \vec{b}^{(1)} & 0 & \cdots & 0 \\ \vec{b}^c & 0 & \vec{b}^{(2)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vec{b}^c & 0 & 0 & \cdots & \vec{b}^{(n_p)} \end{pmatrix}$$

with

$$\vec{b}^c = (b^c_{x_a}, \ldots, b^c_{x_b})'$$

and

$$\vec{b}^{(p)} = (b^{(p)}_{x_a}, \ldots, b^{(p)}_{x_b})', \quad p = 1, \ldots, n_p.$$

- In ACF model, the common states vector $\vec{\alpha}^c_t$ reduces to a scalar and we have $\vec{\alpha}^c_t = k^c_t$. Similarly for the population-specific states vector, we have $\vec{\alpha}^{(p)}_t = k^{(p)}_t$, for $p = 1, \ldots, n_p$.

M-CBD Model (Zhou and Li, 2015)

- The M-CBD model assumes that

$$\begin{aligned}
\mathrm{logit}(q_{x,t}^{(p)}) &= \kappa_{1,t}^{c} + \kappa_{2,t}^{c}(x-\bar{x}) + \kappa_{1,t}^{(p)} + \kappa_{2,t}^{(p)}(x-\bar{x}) + \epsilon_{x,t}^{(p)}, \\
\kappa_{i,t}^{c} &= \mu_i^{c} + \kappa_{i,t-1}^{c} + \eta_{i,t}^{c}, \\
\kappa_{i,t}^{(p)} &= \mu_i^{(p)} + \phi_i^{(p)}\kappa_{i,t-1}^{(p)} + \eta_{i,t}^{(p)},
\end{aligned}$$

for $i = 1, 2$ and $p = 1, \ldots, n_p$, where

- $q_{x,t}^{(p)}$ is population p's death probability at age x and in year t;;
- μ_i^c, $\mu_i^{(p)}$ and $\phi_i^{(p)}$ are constants;
- $\bar{x}$ is the average of the sample age range;
- $|\phi_i^{(p)}| < 1$;
- let $\vec{\eta}_t^c = (\eta_1^c(t), \eta_2^c(t))'$ and $\vec{\eta}_t^{(p)} = (\eta_1^{(p)}(t), \eta_2^{(p)}(t))'$;
- $\epsilon_{x,t}^{(p)} \overset{\text{i.i.d.}}{\sim} \mathrm{N}(0, \sigma_\epsilon^2)$, $\vec{\eta}_t^c \overset{\text{i.i.d.}}{\sim} \mathrm{MVN}(0, Q^c)$ and $\vec{\eta}_t^{(p)} \overset{\text{i.i.d.}}{\sim} \mathrm{MVN}(0, Q^{(p)})$.

M-CBD Model in State Space Form

In M-CBD model, the observation equation and transition equation can be specified as

$$\vec{y}_t = \vec{D} + B\vec{\alpha}_t + \vec{\epsilon}_t$$

and

$$\vec{\alpha}_t = \vec{U} + A\vec{\kappa}_{t-1} + \vec{\eta}_t,$$

where

$$\begin{aligned}
\vec{y}_t &= \left(\operatorname{logit}(q^{(1)}_{x_a,t}),\ \ldots,\ \operatorname{logit}(q^{(1)}_{x_b,t}),\ \ldots, \operatorname{logit}(q^{(n_p)}_{x_a,t}),\ \ldots,\ \operatorname{logit}(q^{(n_p)}_{x_b,t})\right)', \\
\vec{D} &= (0, \ldots, 0)', \\
\vec{\alpha}_t &= \left(\kappa^c_{1,t}, \kappa^c_{2,t},\ \kappa^{(1)}_{1,t}, \kappa^{(1)}_{2,t},\ \ldots,\ \kappa^{(n_p)}_{1,t}, \kappa^{(n_p)}_{2,t}\right)', \\
\vec{\epsilon}_t &= \left(\epsilon^{(1)}_{x_a,t},\ \ldots,\ \epsilon^{(1)}_{x_b,t},\ \ldots, \epsilon^{(n_p)}_{x_a,t},\ \ldots,\ \epsilon^{(n_p)}_{x_b,t}\right)', \\
\vec{U} &= \left(\mu^c_1,\ \mu^c_2,\ \mu^{(1)}_1,\ \mu^{(1)}_2\ \ldots,\ \mu^{(n_p)}_1,\ \mu^{(n_p)}_2\right)',
\end{aligned}$$

M-CBD Model in State Space Form

and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \phi_1^{(1)} & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \phi_2^{(1)} & \cdots & 0 & 0 \\ \vdots & \vdots & & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \phi_1^{(n_p)} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \phi_2^{(n_p)} \end{pmatrix},$$

$$B = \begin{pmatrix} B^* & B^* & 0 & \cdots & 0 \\ B^* & 0 & B^* & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B^* & 0 & 0 & \cdots & B^* \end{pmatrix} \quad \text{with} \quad B^* = \begin{pmatrix} 1 & x_a - \bar{x} \\ 1 & x_a + 1 - \bar{x} \\ \vdots & \vdots \\ 1 & x_b - \bar{x} \end{pmatrix}.$$

- In M-CBD model, we have $\vec{\alpha}_t^c = (\kappa_{1,t}^c, \kappa_{2,t}^c)'$ and $\vec{\alpha}_t^{(p)} = (\kappa_{1,t}^{(p)}, \kappa_{2,t}^{(p)})'$, for $p = 1, \ldots, n_p$.

Outline

Basic Set Up

- Suppose it is time t_0.
- The liability being hedged
  - A T-year temporary life annuity immediate sold to population P_L who are currently age x_0
  - The time-t value of the liability:

$$L(t) \quad = \quad \sum_{u=1}^{T-(t-t_0)} e^{-ru} \left(\prod_{s=t+1}^{t+u} (1 - q^{(P_L)}_{x_0+s-t_0-1,s|t}) \right),$$

- The hedge portfolio
  - m hedging instruments
    - q-forwards, with payoffs linked to the realized death probabilities of population P_H
  - The time-t value of the jth q-forward sold at time t:

$$H_j^{(P_H)}(t) \quad = \quad e^{-rT_j} \left(\mathrm{E}(q^{(P_H)}_{x_j,t+T_j|t}) - q^{(P_H)}_{x_j,t+T_j|t} \right)$$

The hedging goal:

- if the liability is hedged statically,

$$\min_{N_1^{(P_H)}(t_0),\ldots,N_m^{(P_H)}(t_0)} \left(\mathrm{Var} \left(L(t_0) - \mathrm{E}(L(t_0)) - \sum_{j=1}^{m} N_j^{(P_H)}(t_0) H_j^{(P_H)}(t_0) \right) \right)$$

- if the liability is hedged dynamically,

$$\min_{N_1^{(P_H)}(t),\ldots,N_m^{(P_H)}(t)} \left(\mathrm{Var} \left(L(t) - \mathrm{E}(L(t)) - \sum_{j=1}^{m} N_j^{(P_H)}(t) H_j^{(P_H)}(t) \right) \right)$$

for $t = t_0, t_0 + 1, \ldots, t_0 + T - 1$.

Evaluation of hedge effectiveness (HE):

- if the liability is hedged statically,

$$HE = 1 - \frac{\mathrm{Var}\left(L(t_0) - \mathrm{E}(L(t_0)) - \sum_{j=1}^{m} N_j(t_0) H_j^{(P_H)}(t_0) | \mathcal{F}_{t_0}\right)}{\mathrm{Var}(L(t_0) - \mathrm{E}(L(t_0)) | \mathcal{F}_{t_0})},$$

- if the liability is hedged dynamically,

$$HE = 1 - \frac{\mathrm{Var}\left(L(t_0) - \mathrm{E}(L(t_0)) - \sum_{t=t_0+1}^{t_0+T} PCF(t) | \mathcal{F}_{t_0}\right)}{\mathrm{Var}(L(t_0) - \mathrm{E}(L(t_0)) | \mathcal{F}_{t_0})},$$

where PCF_t is the present value of the unexpected cash flow occurring at time t arising from the hedge portfolio.

The Generalized State Space Hedging Method

The main procedures:

1. Variance approximation (first-order Taylor expansion about all relevant states).
2. Variance decomposition.
3. Compute the partial derivatives and obtain the optimal hedging strategy.

Step 1: Variance Approximation

The derivation of the GSS hedging strategy involves the first-order Taylor approximations of $L(t)$ and $H_j^{(P_H)}(t)$ about all relevant states vectors.

- For $L(t)$, the first-order approximation $l(t)$ is given by

$$l(t) = \hat{L}(t) + \sum_{s=t+1}^{t_0+T} (\frac{\partial L(t)}{\partial \vec{\alpha}^c_{s|t}})'(\vec{\alpha}^c_{s|t} - \hat{\vec{\alpha}}^c_{s|t}) + \sum_{s=t+1}^{t_0+T} (\frac{\partial L(t)}{\partial \vec{\alpha}^{(P_L)}_{s|t}})'(\vec{\alpha}^{(P_L)}_{s|t} - \hat{\vec{\alpha}}^{(P_L)}_{s|t}),$$

- For $H_j^{(P_H)}(t)$, the first order approximation $h_j^{(P_H)}(t)$ is given by

$$h_j^{(P_H)}(t) = \hat{H}_j^{(P_H)}(t) + (\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}^c_{t+T_j|t}})'(\vec{\alpha}^c_{t+T_j|t} - \hat{\vec{\alpha}}^c_{t+T_j|t}) + (\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}^{(P_H)}_{t+T_j|t}})'(\vec{\alpha}^{(P_H)}_{t+T_j|t} - \hat{\vec{\alpha}}^{(P_H)}_{t+T_j|t}).$$

- The target function becomes

$$\min_{N_1^{(P_H)}(t),\dots,N_m^{(P_H)}(t)} \left(\mathrm{Var}\left(l(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t) h_j^{(P_H)}(t) \right) \right).$$

Step 2.1: Variance Decomposition

$$\mathrm{Var}\left(l(t)-\sum_{j=1}^{m}N_j^{(P_H)}(t)h_j^{(P_H)}(t)\right)=V_1(t)+V_2(t)+V_3(t)+V_4(t)+V_5(t), \tag{1}$$

where

$$\begin{aligned}
V_1(t) &= \sum_{s,u=t+1}^{t_0+T}\left(\frac{\partial L(t)}{\partial\vec{\alpha}^c_{s|t}}\right)'\mathrm{Cov}(\vec{\alpha}^c_{s|t},\vec{\alpha}^c_{u|t})\left(\frac{\partial L(t)}{\partial\vec{\alpha}^c_{u|t}}\right)\\
V_2(t) &= \sum_{i=1}^{m}\sum_{j=1}^{m}\left(-N_i^{(P_H)}(t)\frac{\partial H_i^{(P_H)}(t)}{\partial\vec{\alpha}^c_{t+T_i|t}}\right)'\mathrm{Cov}(\vec{\alpha}^c_{t+T_i|t},\vec{\alpha}^c_{t+T_j|t})\left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial\vec{\alpha}^c_{t+T_j|t}}\right)\\
V_3(t) &= 2\sum_{s=t+1}^{t_0+T}\sum_{j=1}^{m}\left(\frac{\partial L(t)}{\partial\vec{\alpha}^c_{s|t}}\right)'\mathrm{Cov}(\vec{\alpha}^c_{s|t},\vec{\alpha}^c_{t+T_j|t})\left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial\vec{\alpha}^c_{t+T_j|t}}\right)\\
V_4(t) &= \sum_{s,u=t+1}^{t_0+T}\left(\frac{\partial L(t)}{\partial\vec{\alpha}^{(P_L)}_{s|t}}\right)'\mathrm{Cov}(\vec{\alpha}^{(P_L)}_{s|t},\vec{\alpha}^{(P_L)}_{u|t})\left(\frac{\partial L(t)}{\partial\vec{\alpha}^{(P_L)}_{u|t}}\right)\\
V_5(t) &= \sum_{i=1}^{m}\sum_{j=1}^{m}\left(-N_i^{(P_H)}(t)\frac{\partial H_i^{(P_H)}(t)}{\partial\vec{\alpha}^{(P_H)}_{t+T_i|t}}\right)'\mathrm{Cov}(\vec{\alpha}^{(P_H)}_{t+T_i|t},\vec{\alpha}^{(P_H)}_{t+T_j|t})\left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial\vec{\alpha}^{(P_H)}_{t+T_j|t}}\right)
\end{aligned}$$

Summary of the decomposed variance components $V_1(t), \ldots, V_5(t)$

	Relation with $\vec{N}_t^{(P_H)}$	Associated hidden states
$V_1(t)$	constant	$\vec{\alpha}_t^c$
$V_2(t)$	quadratic	$\vec{\alpha}_t^c$
$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

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$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

$$V_1(t) = \sum_{s,u=t+1}^{t_0+T} \left(\frac{\partial L(t)}{\partial \vec{\alpha}_{s|t}^c}\right)' \mathrm{Cov}(\vec{\alpha}_{s|t}^c, \vec{\alpha}_{u|t}^c) \left(\frac{\partial L(t)}{\partial \vec{\alpha}_{u|t}^c}\right)$$

Summary of the decomposed variance components $V_1(t), \ldots, V_5(t)$

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$V_1(t)$	constant	$\vec{\alpha}_t^c$
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$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

$$V_2(t) = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(-N_i^{(P_H)}(t)\frac{\partial H_i^{(P_H)}(t)}{\partial \vec{\alpha}^c_{t+T_i|t}}\right)' \mathrm{Cov}(\vec{\alpha}^c_{t+T_i|t}, \vec{\alpha}^c_{t+T_j|t})\left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}^c_{t+T_j|t}}\right)$$

Summary of the decomposed variance components $V_1(t), \dots, V_5(t)$

	Relation with $\vec{N}_t^{(P_H)}$	Associated hidden states
$V_1(t)$	constant	$\vec{\alpha}_t^c$
$V_2(t)$	quadratic	$\vec{\alpha}_t^c$
$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

$$V_3(t) \quad = \quad 2\sum_{s=t+1}^{t_0+T}\sum_{j=1}^{m}\left(\frac{\partial L(t)}{\partial \vec{\alpha}_{s|t}^c}\right)' \mathrm{Cov}(\vec{\alpha}_{s|t}^c, \vec{\alpha}_{t+T_j|t}^c)\left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_j|t}^c}\right)$$

Summary of the decomposed variance components $V_1(t), \ldots, V_5(t)$

	Relation with $\vec{N}_t^{(P_H)}$	Associated hidden states
$V_1(t)$	constant	$\vec{\alpha}_t^c$
$V_2(t)$	quadratic	$\vec{\alpha}_t^c$
$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

$$V_4(t) = \sum_{s,u=t+1}^{t_0+T} \left(\frac{\partial L(t)}{\partial \vec{\alpha}_{s|t}^{(P_L)}} \right)' \mathrm{Cov}(\vec{\alpha}_{s|t}^{(P_L)}, \vec{\alpha}_{u|t}^{(P_L)}) \left(\frac{\partial L(t)}{\partial \vec{\alpha}_{u|t}^{(P_L)}} \right)$$

Summary of the decomposed variance components $V_1(t), \ldots, V_5(t)$

	Relation with $\vec{N}_t^{(P_H)}$	Associated hidden states
$V_1(t)$	constant	$\vec{\alpha}_t^c$
$V_2(t)$	quadratic	$\vec{\alpha}_t^c$
$V_3(t)$	linear	$\vec{\alpha}_t^c$
$V_4(t)$	constant	$\vec{\alpha}_t^{(P_L)}$
$V_5(t)$	quadratic	$\vec{\alpha}_t^{(P_H)}$

$$V_5(t) = \sum_{i=1}^{m}\sum_{j=1}^{m}\left(-N_i^{(P_H)}(t)\frac{\partial H_i^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_i|t}^{(P_H)}}\right)' \mathrm{Cov}(\vec{\alpha}_{t+T_i|t}^{(P_H)}, \vec{\alpha}_{t+T_j|t}^{(P_H)}) \left(-N_j^{(P_H)}(t)\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_j|t}^{(P_H)}}\right)$$

Step 2.2: Reorganizing the order of $V_1(t), \ldots, V_5(t)$

- $V_2(t)$ and $V_5(t)$ are quadratic functions of $\vec{N}_t^{(P_H)}$, which can be expressed in matrix form as

$$V_2(t) = (\vec{N}_t^{(P_H)})' \Psi_t^{(P_H)} \vec{N}_t^{(P_H)},$$

$$V_5(t) = (\vec{N}_t^{(P_H)})' \Gamma_t^{(P_H)} \vec{N}_t^{(P_H)},$$

where both $\Psi_t^{(P_H)}$ and $\Gamma_t^{(P_H)}$ are m-by-m square matrices, with the (i,j)th element being

$$\Psi_{i,j|t}^{(P_H)} = \left(\frac{\partial H_i^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_i|t}^{c}} \right)' \mathrm{Cov}(\vec{\alpha}_{t+T_i|t}^{c}, \vec{\alpha}_{t+T_j|t}^{c}) \frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_j|t}^{c}}$$

and

$$\Gamma_{i,j|t}^{(P_H)} = \left(\frac{\partial H_i^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_i|t}^{(P_H)}} \right)' \mathrm{Cov}(\vec{\alpha}_{t+T_i|t}^{(P_H)}, \vec{\alpha}_{t+T_j|t}^{(P_H)}) \frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}_{t+T_j|t}^{(P_H)}},$$

respectively.

Step 2.2: Reorganizing the order of $V_1(t), \ldots, V_5(t)$

- $V_3(t)$ is linear function of $\vec{N}_t^{(P_H)}$, which can be expressed in matrix form as

$$V_3(t) = -2(\vec{G}_t^{(P_H)})' \vec{N}_t^{(P_H)}$$

where $\vec{G}_t^{(P_H)}$ is a m-by-1 vector with the jth element $G_j^{(P_H)}(t)$ being

$$G_j^{(P_H)}(t) = \sum_{s=t+1}^{T-(t-t_0)} \left(\frac{\partial H_j^{(P_H)}(t)}{\partial \vec{\alpha}^c_{t+T_j|t}} \right)' \mathrm{Cov}(\vec{\alpha}^c_{t+T_j|t}, \vec{\alpha}^c_{s|t}) \frac{\partial L(t)}{\partial \vec{\alpha}^c_{s|t}},$$

- $V_1(t)$ and $V_4(t)$ are free of notional amounts. We let $C(t) = V_1(t) + V_4(t)$ and do not bother to write $C(t)$ into matrix form.

Therefore, we can reorganize the order of $V_1, \ldots, V_5$, which gives

$$\mathrm{Var}(I(t) - \textstyle\sum_{j=1}^{m} N_j^{(P_H)}(t) h_j^{(P_H)}(t))$$

$$= (V_2(t) + V_5(t)) + V_3(t) + (V_1(t) + V_4(t))$$

$$= (\vec{N}_t^{(P_H)})'(\Psi_t^{(P_H)} + \Gamma_t^{(P_H)})\vec{N}_t^{(P_H)} - 2(\vec{G}_t^{(P_H)})'\vec{N}_t^{(P_H)} + C(t)$$

Step 3: Obtain the Optimal Hedging Strategy

- We first take partial derivative of $\mathrm{Var}(I(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t) h_j^{(P_H)}(t))$ with respect to $\vec{N}_t^{(P_H)}$. Then the optimal hedging strategy at different time t given $\mathcal{F}_t$ is obtained by setting the partial derivatives to zeros.
- The optimal hedging strategy $\hat{\vec{N}}_t^{(P_H)}$:

$$\hat{\vec{N}}_t^{(P_H)} = \left(\Psi_t^{(P_H)} + \Gamma_t^{(P_H)}\right)^{-1} \vec{G}_t^{(P_H)}.$$

- The minimized value of $\mathrm{Var}(I(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t) h_j^{(P_H)}(t))$:

$$\min_{\vec{N}_t^{(P_H)}} \left(\mathrm{Var}(I(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t) h_j^{(P_H)}(t))\right) = C(t) - (\vec{G}_t^{(P_H)})' \left(\Psi_t^{(P_H)} + \Gamma_t^{(P_H)}\right)^{-1} \vec{G}_t^{(P_H)}.$$

Outline

A Hypothetical Situation: A World with No Basis Risk

- When population basis risk is absent,
  - $V_4(t) = V_5(t) = 0$;
  - the choice of reference population should not affect the hedge effectiveness.

- It can be shown that,

$$V_2(t) = (\vec{N}_t^{(P_H)})' \Psi_t^{(P_H)} \vec{N}_t^{(P_H)} = (\vec{N}_t^{(P_H)})' \Lambda_t^{(P_H)} Z_t^c \Lambda_t^{(P_H)} \vec{N}_t^{(P_H)}$$

and

$$V_3(t) = \vec{G}_t^{(P_H)} \vec{N}_t^{(P_H)} = \Lambda_t^{(P_H)} \vec{G}_t^c \vec{N}_t^{(P_H)}$$

where

  - $\Lambda_t^{(P_H)}$ is a m-by-m diagonal matrix;
  - Z_t^c is a m-by-m symmetric matrix that does not involve population-specific factors;
  - $\vec{G}_t^c$ is a m-by-1 vector that does not involve population-specific factors.

A Hypothetical Situation: A World with No Basis Risk

- The optimal hedging strategy can be expressed as

$$\vec{\tilde{N}}_t^{(P_H)} = (\Lambda_t^{(P_H)})^{-1}(Z_t^c)^{-1}\vec{G}_t^c.$$

- We can treat $\Lambda_t^{(P_H)}\vec{N}_t^{(P_H)}$ as the standardized notional amounts for the reference population P_H.

- The minimized value of $\text{Var}(I(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t)h_j^{(P_H)}(t))$:

$$\min_{\Lambda_t^{(P_H)}\vec{N}_t^{(P_H)}}(\mathrm{Var}(I(t) - \sum_{j=1}^{m} N_j^{(P_H)}(t)h_j^{(P_H)}(t))) = C(t) - (\vec{G}_t^c)'(Z_t^c)^{-1}\vec{G}_t^c,$$

which is free of P_H and is the same no matter what the reference population is.

A Hypothetical Situation: A World with No Basis Risk

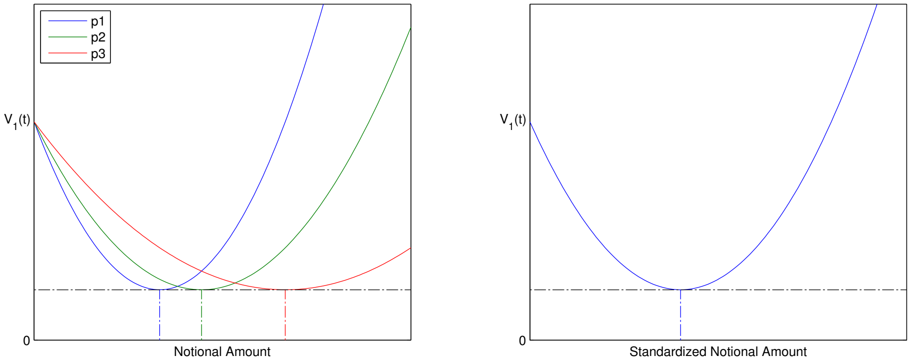


Figure: The hypothetical curves of $V_1(t) + V_2(t) + V_3(t)$ as functions of non-standardized notional amount and standardized notional amount when population basis risk is absent and when $m = 1$ instrument is being used.

A Hypothetical Situation: When Basis Risk is Present

- When population basis risk is present,
  - $V_4(t) > 0$, $V_5(t) > 0$;
  - the choice of reference population should affect the hedge effectiveness.
- We have

$$\begin{aligned} V_2(t) + V_5(t) &= (\vec{N}_t^{(P_H)})'(\Psi_t^{(P_H)} + \Gamma_t^{(P_H)})\vec{N}_t^{(P_H)} \\ &= (\vec{N}_t^{(P_H)})'\Lambda_t^{(P_H)}(Z_t^c + Z_t^{(P_H)})\Lambda_t^{(P_H)}\vec{N}_t^{(P_H)}, \end{aligned}$$

where the additional term $Z_t^{(P_H)}$ is a m-by-m symmetric matrix that is solely arising from population P_H.

A Hypothetical Situation: When Basis Risk is Present

- The optimal standardized hedging strategy would then be computed as

$$\Lambda_t^{(P_H)}\hat{\vec{N}}_t^{(P_H)} = (Z_t^c + Z_t^{(P_H)})^{-1}\vec{G}_t^c.$$

- The minimized value of $\mathrm{Var}(I(t) - \sum_{j=1}^m N_j^{(P_H)}(t)h_j^{(P_H)}(t))$ would be

$$\min_{\Lambda_t^{(P_H)}\vec{N}_t^{(P_H)}}(\mathrm{Var}(I(t) - \sum_{j=1}^m N_j^{(P_H)}(t)h_j^{(P_H)}(t))) = C(t) - (\vec{G}_t^c)'(Z_t^c + Z_t^{(P_H)})^{-1}\vec{G}_t^c.$$

- We define the standardized basis risk profile as

$$BRP(x_1, T_1, P_H) = Z_t^{(P_H)}$$

when $m = 1$.

- A smaller value of $BRP(x_1, T_1, P_H)$ implies a better hedging performance.

A Hypothetical Situation: When Basis Risk is Present

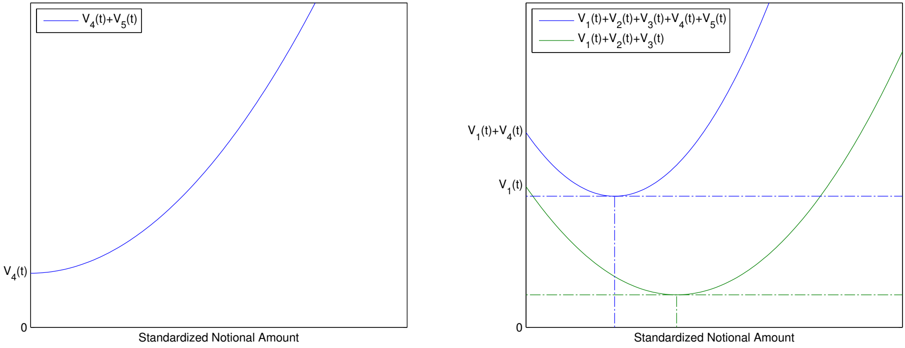


Figure: The hypothetical curves of $V_4(t) + V_5(t), V_1(t) + V_2(t) + V_3(t)$ and $V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)$ when population basis risk is present and only one instrument is being used $(m = 1)$.

A Hypothetical Situation: When Basis Risk is Present

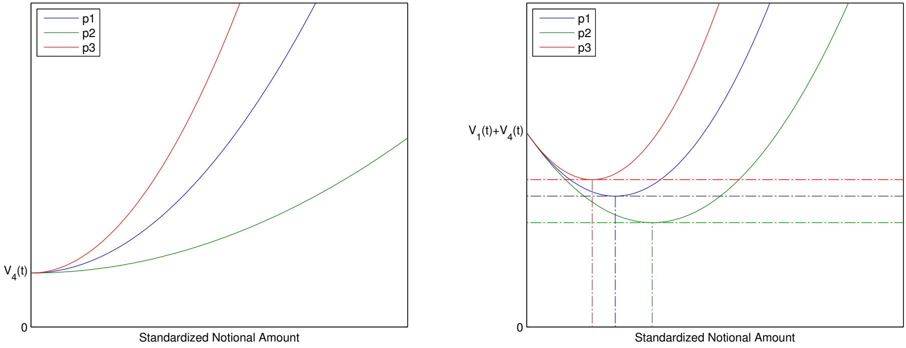


Figure: The hypothetical curves of $V_4(t) + V_5(t), V_1(t) + V_2(t) + V_3(t)$ and $V_1(t) + V_2(t) + V_3(t) + V_4(t) + V_5(t)$ when population basis risk is present and only one instrument is being used $(m = 1)$.

Outline

A Numerical Illustration

The Assumed Mortality Model

- ACF model

Data

- P_L: Canada
- P_H: United States (US)
  England and Wales (EW)
  Netherlands (NE)
  West Germany (WG)

- Age range: 60 to 89
- Sample period: 1961 to 2009
- Gender: Male

A Numerical Illustration: When Basis Risk is Absent

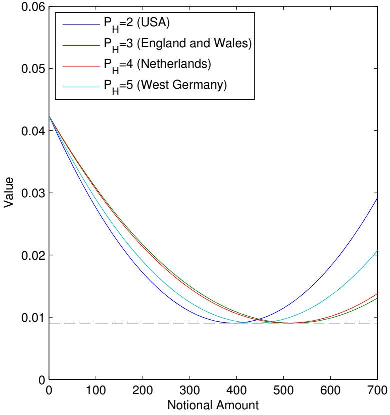


Figure: Plot of $V_1(t_0) + V_2(t_0) + V_3(t_0)$ for different reference populations.

A Numerical Illustration: When Basis Risk is Absent

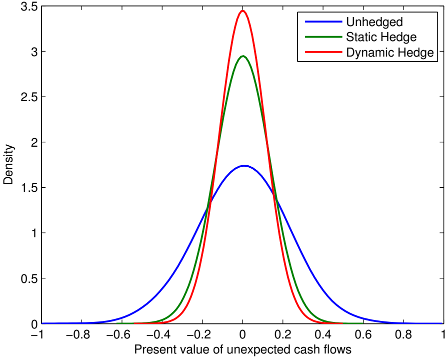


Figure: The distribution of the present value of unexpected cash flows when population basis risk is absent.

A Numerical Illustration: When Basis Risk is Present

	US	EW	NE	WG
$BRP(x_1, T_1, P_H)$	0.0020	0.0019	0.0003	0.0005

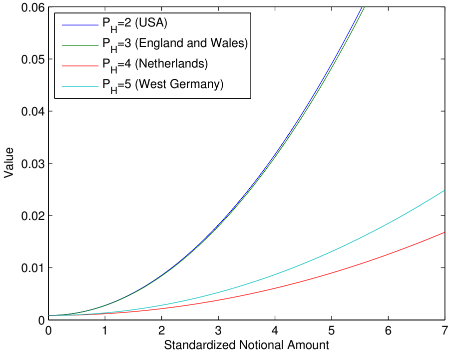
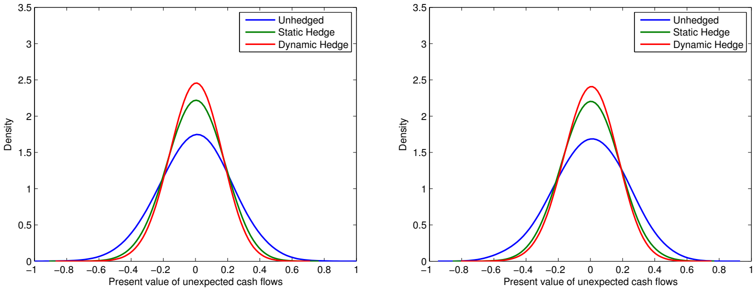


Figure: Value of $(V_4(t) + V_5(t))$ as a function of standardized notional amount

A Numerical Illustration: When Basis Risk is Present

	US	EW	NE	WG
HE (static)	0.5085	0.5143	0.7016	0.6761
HE (dynamic)	0.6243	0.6279	0.8404	0.8213



(a) US

(b) EW

Figure: The distribution of the present value of unexpected cash flows when population basis risk is present.

A Numerical Illustration: When Basis Risk is Present

	US	EW	NE	WG
HE (static)	0.5085	0.5143	0.7016	0.6761
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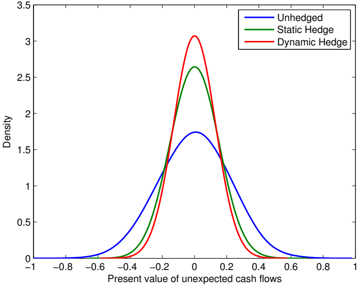
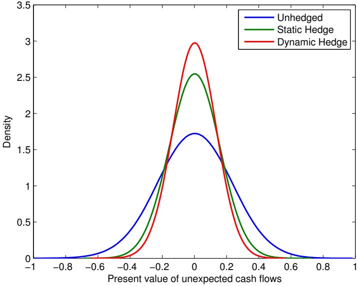


(a) NE



(b) WG

Figure: The distribution of the present value of unexpected cash flows when population basis risk is present.

Outline

Concluding Remarks

- The generalized state space hedging method is proposed for use when the populations associated with the hedging instruments and the liability being hedged are different.
- The GSS hedging method can also be applied to mortality models with cohort effect, as long as the model can be written in state space form.
- A quantity called standardized basis risk profile $BRP(x_1, T_1, P_H)$ has been developed.
- A numerical illustration has been provided to demonstrate the use of $BRP(x_1, T_1, P_H)$.