Pricing of Pension Bulk Annuities

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Content

Introduction

- Difficulties to cope with the unprecedented levels of DB pension liabilities
- Asset-liability management or De-risking strategies
- Pension buy-in and buy-out deals
- A benchmark pricing model based on the risk neutral market framework and independence assumption for pension bulk annuities *(Lin et al., 2014)*

Motivation

- Investigating the general set-up and assumptions to enhance the pricing mechanism of the pension buy-outs

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Aim of the Study

Pricing pension buy-outs under dependence assumption of financial and insurance markets

A Proposed Model for Pricing of the Pension Buy-outs under the Dependence Assumption

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- A combined modelling framework $(\Omega, \mathcal{G}, (\mathcal{G}_t), \mathbb{P})$ s.t. $\mathcal{G}_t = \mathcal{M}_t \vee \mathcal{F}_t$
- $\mathcal{M}_t$ is the filtration of mortality process μ and $\mathcal{F}_t$ is the filtration of short rate process r.

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- The deal guarantees to pay the pension liabilities and compensate any potential asset-liability mismatching.
- Adopt the suggested model by *Jalen and Mamon (2009)* to state the liability process of a hypothetical pension scheme
- Consider the difference between asset and liability processes as one year put option spreads where the strike prices are defined according to the pension liabilities on the valuation dates as offered by *Lin et al. (2014)*

Model Framework

- A new model to derive the price of a pure endowment policy under the dependence assumption using the *change of measure technique* by *Jalen and Mamon (2009)*
- The price of the pure endowment policy

$$\begin{aligned} B_S(t,T,C_T) &= E[exp(-\int_t^T r(s)ds)\,1_{\tau>T}\,C_T|\mathcal{G}_t] \\ &= 1_{\tau>t}E[exp(-\int_t^T (r(s)+\mu(s,x+s))ds)C_T|\mathcal{G}_t] \end{aligned}$$

turns into the following formula:

Model Framework (Continued)

$$B_S(t,T,C_T) = 1_{\tau>t} B(t,T)\tilde{p}(t,T,x)\tilde{E}^T[C_T|\mathcal{G}_t] \tag{1}$$

where

1. $B_S(t,T,C_T)$ represents the present value of the contract with a variable survival benefit C_T at the end of the maturity T.
2. $B(t,T) = E^{\mathbb{Q}}[exp(-\int_t^T r(s)ds)]$
3. $\tilde{p}(t,T,x) = E^T[exp(-\int_0^T \mu(s,x+s)ds)]$
4. From Bayes' rule $E^T[H|\mathcal{G}_T]$ is the expectation under the forward measure $\mathbb{P}^T$, for a contingent claim H

Model Framework (Continued)

- $E^T[H|\mathcal{G}_T]$ could be written more explicitly as below:

$$E^T[H|\mathcal{G}_T] = \frac{E[exp(-\int_t^T r(s)ds)H|\mathcal{G}_T]}{B(t,T)}$$

- The Radon-Nikodym derivative of $\mathbb{P}^T$ with respect to the risk-neutral measure $\mathbb{Q}$ as

$$\frac{d\mathbb{P}^T}{d\mathbb{Q}}|\mathcal{G}_T = \Lambda_{0,T} = \frac{exp(-\int_0^T r(s)ds)}{B(0,T)}$$

Model Framework (Continued)

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- Adopt $B_S(t, T, C_T)$ to derive the liability process of the pension scheme for a constant benefit amount C, i.e.

$$B_S(t, t_i, C) = 1_{t_i > t} B(t, t_i, r_t) E^{t_i}[C \times exp(-\int_t^{t_i} \mu(s, x+s)ds) \\ |\mathcal{G}_{t_i}]$$

where

$$\frac{d\mathbb{P}^{t_i}}{d\mathbb{Q}} = \Lambda_{0,t_i} = \frac{exp(-\int_0^{t_i} r(s)ds)}{B(0, t_i, r_t)}$$

- $a(t, C) = \sum_{t_i=t+1}^{T} B_S(t, t_i, C)$
- $L_t = N(t) \times a(t, C)$

Model Framework (Continued)

- The similar asset process suggested by *Lin et al. (2014)* may be applied, i.e.

$$dlogPA_t^* = (r - \frac{1}{2}\sigma_W^2(t))dt + \sum_{i=1}^{3}\pi_i(t)\sigma_i dW_{it} \qquad (2)$$

1. r is the risk free rate.
2. $\sigma_W^2(t) = \sum_{i,j=1}^{3}\pi_i(t)\pi_j(t)\rho_{ij}\sigma_i\sigma_j$ where ρ_{ij} is the correlation coefficient between asset i and j
3. $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))'$ are the weights of the portfolio at time t (*Fernholz (2002)*).

Model Framework (Continued)

- The risk-neutral price of the payoff associated with both risks, conditioning on $N(t)$, is given by

$$V(0,T) = \sum_{t=1}^{T} e^{-rt} E^{\mathbb{Q}}[(L_t - PA_t)^+ | N(t)] - v^{(\tau_N+1)} E^{\mathbb{Q}}[PA_{\tau_N+1}] \quad (3)$$

- The buy-out price under the dependence assumption

$$P_{buyout} = \frac{E_M[V(0,T)]}{L_0}$$

Lee-Carter Model

$$\ln(m_{x,t}) = \alpha_x + \beta_x \kappa_t + \xi_{x,t} \tag{4}$$

- $m_{x,t}$ = the observed central death rate of an x-aged individual at time t
- α_x = the average age specific death pattern
- $\xi_{x,t}$ = the residual term for x-aged individual in year t
- κ_t = the improvement of mortality for each age group in logarithmic scale
- β_x = the sensitivity of change in κ_t according to the relevant age
- The standardization conditions are

$$\alpha_x = \frac{1}{T}\sum_t \ln m_{xt} \qquad \sum_x \beta_x^2 = 1 \qquad \sum_t \kappa_t = 0 \tag{5}$$

Ornstein-Uhlenbeck Model

- The OU process as the unique solution of the following differential equation:

$$\begin{aligned} dX_t &= -cX_t dt + \sigma dW_t \\ X_0 &= x \end{aligned} \tag{6}$$

- The explicit solution of this SDE

$$X_t = xe^{-ct} + \sigma e^{-ct} \int_0^t e^{cs} dW_s \tag{7}$$

- $E(X_t) = xe^{-ct} + \sigma e^{-ct} E(\int_0^t e^{cs} dW_s)$
- $Var(X_t) = \sigma^2 \frac{1-e^{-2ct}}{2c}$

Vasicek Model

$$dr(t) = a(b - r(t))dt + \sigma dW(t) \tag{8}$$

- a, b and σ are non-negative constants.
- The explicit solution of this SDE

$$r(t) = r(0)e^{-at} + b(1 - e^{-at}) + \sigma e^{-at} \int_0^t e^{as} dW_s \tag{9}$$

- The price of zero coupon bonds,

$$\begin{aligned} P(t, T) &= E[e^{-\int_t^T r(s)ds} | \mathcal{F}_t] \\ &= \exp[-(T - t)R(T - t, r(t))] \end{aligned}$$

- $R(\theta, r) = R_\infty - \frac{1}{a\theta}[(R_\infty - r)(1 - e^{-a\theta})\frac{\sigma^2}{4a^2}(1 - e^{-a\theta})^2]$ with $R_\infty = lim_{\theta \to \infty R(\theta, r)}$.

Application

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  1. Short rate is assumed to follow Vasicek model.
  2. OU process and LC model are applied to mortality dynamics.

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Assumptions

1. Pensioners are aged at 65.
2. No annual contributions
3. No pension gap at inception

Case Study I: OU Process

- Assume affine structures for both interest and mortality rate dynamics as below:

$$dr_t = a^r(b^r - r_t)dt + c^r dW_t^r$$
$$d\mu_t = -a^\mu \mu_t dt + c^\mu dW_t^\mu$$

- Short rate model, i.e. Vasicek model, is hold for both scenarios.

Case Study I: OU Process (Continued)

- Zero coupon bond prices can be obtained as

$$\begin{aligned} B(t,t_i,r_t) &= E^{\mathbb{Q}}[exp(-\int_t^{t_i} r_u du)|\mathcal{F}_t] \\ &= exp(E[-\int_t^{t_i} r_u(r_t)du] + \frac{1}{2}Var[-\int_t^{t_i} r_u(r_t)du]) \\ &= exp(-A_r(t,t_i)r_t + B_r(t,t_i)) \end{aligned}$$

where

$$\begin{aligned} A_r(t,t_i) &= \frac{1-e^{-a^r(t_i-t)}}{a^r} \\ B_r(t,t_i) &= (b^r - \frac{\sigma^{2r}}{2a^{2r}})[A_r(t,t_i)-(t_i-t)] - \frac{\sigma^{2r}A_r(t,t_i)^2}{4a^r} \end{aligned}$$

Mamon (2004).

Case Study I: OU Process (Continued)

- The survival rates can be calculated as

$$\tilde{p}(t, t_i, x) = E^{t_i}[exp(-\int_t^{t_i} \mu(s, x+s)ds)|\mathcal{G}_t]$$

under the t_i-forward measure $\mathbb{P}^{t_i}$.

- The relevant mortality rate dynamics under $\mathbb{P}^{t_i}$

$$d\mu_t = -a^{\mu}(-\frac{c^{\mu}c^{r}A_r(t, t_i)}{a^{\mu}} - \mu_t)dt + c^{\mu}dW_t^{\mathbb{P}^{t_i}}$$

where $W_t^{\mathbb{P}^{t_i}}$ is a Wiener process under $\mathbb{P}^{t_i}$ and

$$dW_t^{\mathbb{P}^{t_i}} = dW_t^{\mathbb{Q}} - c^{r}A_r(t, t_i)dt$$

Case Study I: OU Process (Continued)

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- $\tilde{p}(t, t_i, x)$ could be calculated using the following formula:

$$\tilde{p}(t, t_i, x) = exp(-A_\mu(t, t_i)\mu_t + B_\mu(t, t_i))$$

where

$$B_\mu(t, t_i) = (-\frac{c^r c^\mu A_r(t, t_i)}{a^\mu} - \frac{c^{2\mu}}{2a^{2\mu}})[A_\mu(t, t_i) - (t_i - t)] + \frac{c^{2\mu} A_\mu(t, t_i)^2}{4a^\mu}$$

and

$$A_\mu(t, t_i) = \frac{e^{a^\mu (t_i - t)} - 1}{a^\mu}$$

Case Study I: OU Process (Continued)

- We are ready to calculate the liability process for a constant benefit C using annuity factors, i.e.

$$B_S(t, T, C_T) = 1_{\tau > t} B(t, T)\tilde{p}(t, T, x)C$$

and

$$a(t, C) = \sum_{t_i = t+1}^{T} B_S(t, t_i, C)$$

$$L_t = N(t) \times a(t, C)$$

Case Study II: LC Model

- The generalized LC model is stated as

$$\mu_t = \exp(\alpha(x+t) + \beta(x+t)\kappa_t) \tag{10}$$

- κ parameter is assumed to have an SDE in the following form by *Biffis and Denuit (2006)*

$$d\kappa_t = \delta(t, \kappa_t)dt + \sigma(t, \kappa_t)dW_t \tag{11}$$

- Under the assumption of a *single stopping time* τ, the SDE of the process μ under $\mathbb{P}$

$$d\mu_t = \mu_t(\delta_t^{\mu} dt + \sigma_t^{\mu} dW_t) \tag{12}$$

Case Study II: LC Model (Continued)

- The stochastic intensity μ under $\mathbb{Q}$

$$\tilde{\mu}_t^i = \mu_t^i(1+\phi_t^i) = \exp(\tilde{\alpha}^i(x_i+t) + \tilde{\beta}^i(x_i+t).\kappa_t) \quad (13)$$

where $\tilde{\alpha}^i = \alpha + a^i$, $\tilde{\beta}^i = \beta + b^i$ using a transformation based on ϕ^i s.t.

$$\phi_t^i = \exp(a^i(x_i+t) + b^i(x_i+t).\kappa_t) - 1$$

for some functions $(a^i)_{x\in\mathcal{I}}$ and $(b^i)_{x\in\mathcal{I}}$ (*Biffis et al. (2010)*).

Case Study II: LC Model (Continued)

- The SDE dynamics of μ under $\mathbb{Q}$ would be

$$d\mu_t = \mu_t(\tilde{\delta}^{\mu} dt + \sigma_t^{\mu} d\tilde{W}_t) \tag{14}$$

with $\tilde{\delta}^{\mu} = \delta^{\mu} - \eta\sigma^{\mu}$
Biffis and Denuit (2006); Biffis et al. (2010).

- The SDE dynamics of μ under $\mathbb{P}^T$

$$\begin{aligned} d\mu_t &= \mu_t(\tilde{\delta}^{\mu} dt + \sigma_t^{\mu} \times [dW_t^{\mathbb{P}^T} + c^r A_r(t,T)dt]) \\ &= \mu_t([\tilde{\delta}^{\mu} + c^r A_r(t,T)\sigma_t^{\mu}]dt + \sigma_t^{\mu} dW_t^{\mathbb{P}^T}) \end{aligned}$$

- The solution of this SDE

$$\mu_T = \mu_t \times \exp\left(\int_t^T [\tilde{\delta}^{\mu} + c^r A_r(s,T)\sigma_t^{\mu} - \frac{1}{2}\sigma_s^{2\mu}]ds + \int_t^T \sigma_s^{\mu} dW_s^{\mathbb{P}^T}\right) \tag{15}$$

Case Study II: LC Model (Continued)

- The explicit solution of Equation (15) is equal to

$$\mu_T = \mu_t \times \exp(\tilde{\delta}^{\mu}(T-t) + \sigma^{\mu} c^r \int_t^T A_r(s,T)ds - \frac{1}{2}\sigma^{2\mu}(T-t)) \times \exp(\sigma^{\mu}[W_T^{\mathbb{P}^T} - W_t^{\mathbb{P}^T}]) \quad (16)$$

where $\int_t^T A_r(s,T)ds$ is obtained as

$$\begin{aligned} \int_t^T A_r(s,T)ds &= \int_t^T \frac{1-e^{a^r(T-s)}}{a^r} ds \\ &= \frac{1}{a^r}(T-t) - \frac{1}{a^{2r}}[1-e^{-a^r(T-t)}] \end{aligned}$$

according to the short rate model, i.e. Vasicek model.

Case Study II: LC Model (Continued)

- Finally we get

$$\mu_T = \mu_t \times f(t,T) \times e^{\sigma^{\mu}[W_T^{\mathbb{P}^T} - W_t^{\mathbb{P}^T}]} \tag{17}$$

where

$$f(t,T) = \exp([\tilde{\delta}^{\mu} + \frac{\sigma^{\mu} c^r}{a^r} - \frac{\sigma^{2\mu}}{2}](T-t) - \frac{\sigma^{\mu} c^r}{a^{2r}}[1 - e^{-a^r(T-t)}])$$

- Therefore the relevant survival rates are drived as

$$\begin{aligned} \tilde{p}(t,t_i,x) &= E^{t_i}[\exp(-\int_t^{t_i} \mu_s ds)|\mathcal{F}_t] \\ &= E^{t_i}[\exp(-\int_t^{t_i} \mu_t \times f(t,s) \times e^{\sigma^{\mu}[W_s^{\mathbb{P}^{t_i}} - W_t^{\mathbb{P}^{t_i}}]} ds)|\mathcal{F}_t] \end{aligned}$$

Numerical Results

- The plan funds are assumed to be invested in the *S&P 500 index* $A_{1,t}$, the *Merrill Lynch corporate bond index* $A_{2,t}$ and the *3-month T-bill* $A_{3,t}$.

Table 1: Parameter Estimates of Three Pension Assets

Parameter	Estimate	Parameter	Estimate
α_1	0.1097	σ_1	0.1458
α_2	0.0959	σ_2	0.0770
α_3	0.0631	σ_3	0.0286

- Estimated correlation coefficients

$$\Sigma_{WW} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} \\ \rho_{12} & 1 & \rho_{23} \\ \rho_{13} & \rho_{23} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0.2905 & 0.0615 \\ 0.2905 & 1 & 0.0129 \\ 0.0615 & 0.0129 & 1 \end{bmatrix}$$

Numerical Results for Case Study I

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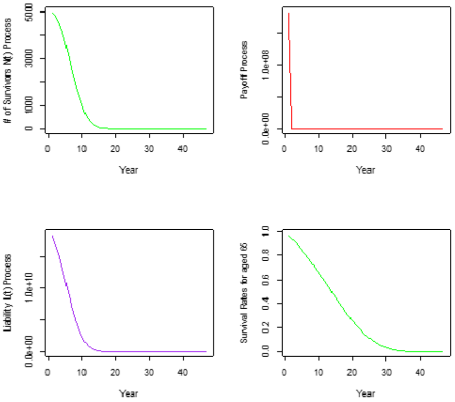
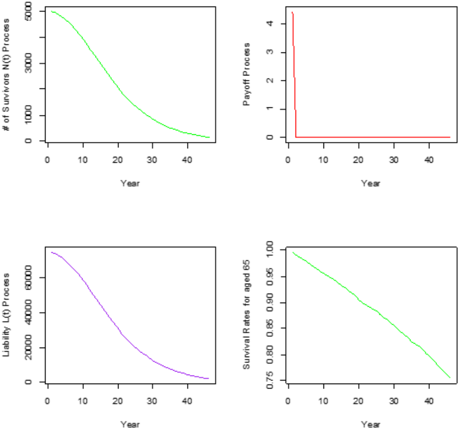
Figure 1: *Number of Survivors, Liability and Payoff Processes for Case Study I*



Numerical Results for Case Study II

Figure 2: *Number of Survivors, Liability and Payoff Processes for Case Study II*



Numerical Results (Continued)

Table 2: Buy-out Premiums under Dependence Assumption

	OU	LC
Premium	2.106578e-04	1.294719e-06

Assumptions

1. Benefit= 60 000 and N(0)= 5000.
2. 1000 scenarios.
3. Zero coupon bonds, i.e. discount rates are under Vasicek model.

Summary

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Summary

- Changing the independence assumption $\equiv$ using change of measure technique for pricing
- Adopt the pure endowment pricing formula suggested by *Jalen and Mamon (2009)* to state the liabilities of the pension plans
- Using the options, which have variable strike prices related to these liabilities
- Obtaining the risk premium of a buy-out deal as the expected value of one year put option spreads

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Thanks for your attention.