Gaussian Process Models for Mortality Rates and Improvement Factors: An Interactive R Markdown Approach

Longevity 14

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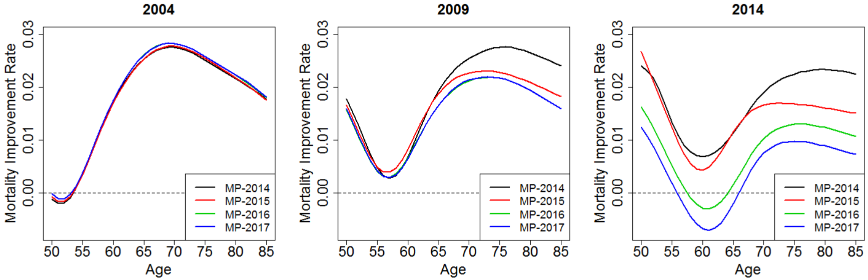
US Mortality Improvement

MP-2014 / MP-2015

- Published by SOA, uses “RPEC_2014" model
- US CDC Data
- MP-2014 uses years 1950-2009
- Plans to update scales at least triennially; two years of additional CDC data shows drastic change in later years
  - MP-2015 emerges
  - (Also showing MP-2016, MP-2017, came after this paper finished)



Goal:

Model US Mortality data using Gaussian Process (GP) regression

- Bayesian
- Provides posterior Gaussian distribution for input of any age and year
- Offers easy analysis of both mortality and mortality improvement simultaneously
- Gaussian distribution implies one-year mortality improvement factors remains Gaussian
- Differentiable: can provide instantaneous mortality improvement (still Gaussian)
- Spatial approach inherently handles missing and edge data
- Provide simple to use code with output through an R notebook

Typical Regression Assumption

Hypothesis:

$$y = f(x) + \varepsilon$$

- Observe $\boldsymbol{y} = y^{1:N}$ for input locations $\boldsymbol{x} = x^{1:N}$
- Want to understand the function f
  - e.g. $f(x) = \beta_0 + \beta_1 x$ (simple linear regression)
- ε is noise
  - e.g. measurement error
  - can't observe $f(x)$ directly
- Assume $\varepsilon \sim \mathcal{N}\left(0, \sigma^2(x)\right)$ (often $\sigma(x) \equiv \sigma \in \mathbb{R}^+$)
- Often in mortality modeling: $f(x)$ is based on an ARIMA process(es) or on splines
- Our assumption: f is a Gaussian Process (modeling log-mortality, $x = (x_{ag}, x_{yr})$)

Gaussian Process

- Defined as a collection of random variables $\{f(x)|x \in \mathbb{R}^d\}$
- Any finite subset has a multivariate Gaussian distribution with covariance $C(\cdot, \cdot)$:

$$f(x_1), \ldots, f(x_n) \sim \mathcal{N}\left((m(x_1), \ldots, m(x_n)), C(\boldsymbol{x}, \boldsymbol{x}^T)\right).$$

- Fix mean function m and covariance kernel C; this provides a prior distribution

Modeling with Gaussian Processes

1. Declare prior mean function and covariance kernel
    - Mean function can also be parametric and fitted with data; useful in extrapolation
    - Covariance kernel governs spatial relation between points
    - Hyperparameters can be specified using expert knowledge or fitted from data

Modeling with Gaussian Processes

1. Declare prior mean function and covariance kernel
   - Mean function can also be parametric and fitted with data; useful in extrapolation
   - Covariance kernel governs spatial relation between points
   - Hyperparameters can be specified using expert knowledge or fitted from data
2. Output can be easily evaluated at any location
   - Output is a random variable with mean and covariance depending on neighboring inputs

Posterior

- Observe pairs $(\boldsymbol{y}, \boldsymbol{x}) = ((y, x)^{1:N})$
  - (e.g. $y =$ historic log–mortality and $x =$ (age, year))
- Gaussian assumptions imply that marginally for any input x

$$f(x)|(\boldsymbol{y}, \boldsymbol{x}) \sim \mathcal{N}\left(m_*(x), s_*^2(x)\right)$$

Posterior

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  - (e.g. y = historic log–mortality and x = (age, year))
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$$f(x)|(\boldsymbol{y}, \boldsymbol{x}) \sim \mathcal{N}\left(m_*(x), s_*^2(x)\right)$$

- m_* and s_*^2 are the posterior mean and variance functions

$$\begin{cases} m_*(x) \doteq \boldsymbol{c}(x)^T(\boldsymbol{C} + \Sigma)^{-1}\boldsymbol{y}; \\ s_*^2(x) \doteq C(x,x) - \boldsymbol{c}(x)^T(\boldsymbol{C} + \Sigma)^{-1}\boldsymbol{c}(x), \end{cases} \tag{1}$$

where

$$\begin{cases} \boldsymbol{c}(x) \doteq \left(C(x, x^i)\right)_{1\le i\le N} & \text{(covariances between } x \text{ and inputs } \boldsymbol{x}) \\ \boldsymbol{C} \doteq \left(C(x^i, x^j)\right)_{1\le i,j\le N} & \text{(covariances between inputs } \boldsymbol{x}) \\ \Sigma \doteq \operatorname{diag}\left(\sigma^2(x^i)\right) & \text{(diagonal matrix of noise variance)} \end{cases}$$

Covariance Kernels & Parameter Estimation

- Common choice is squared-exponential (or Gaussian) covariance kernel

$$C(x, x') = \eta^2 \exp\left(-\frac{(x_{ag} - x'_{ag})^2}{2\theta_{ag}^2} - \frac{(x_{yr} - x'_{yr})^2}{2\theta_{yr}^2} \right).$$

- Knowing mortality at x will greatly influence mortality at “neighboring" x’s
  - e.g. knowing mortality for a 80 year old in 2015 greatly aids in prediction of a 85 year old’s mortality in 2016; knowing a 50 year old’s mortality in 2000 has a nearly non-existent effect
- Implies hyperparameter family of $\Theta \doteq (\theta_{ag}, \theta_{yr}, \eta^2, \sigma^2)$
  - Also mean function hyperparameters (if included)
- Estimates are fit using MLE; likelihood can be written out explicitly due to Gaussian assumptions
  - Done using `R` package `DiceKriging`
- Alternatively, can use Bayesian approach with priors on Θ
  - Separate package using STAN language
  - Leads to non-Gaussian posterior

Illustrative Example

Goal: Learn $f(x) = \sin(x)$ over domain $[0, 1.5\pi]$

- Observe realizations of

$$y = \sin(x) + \varepsilon$$

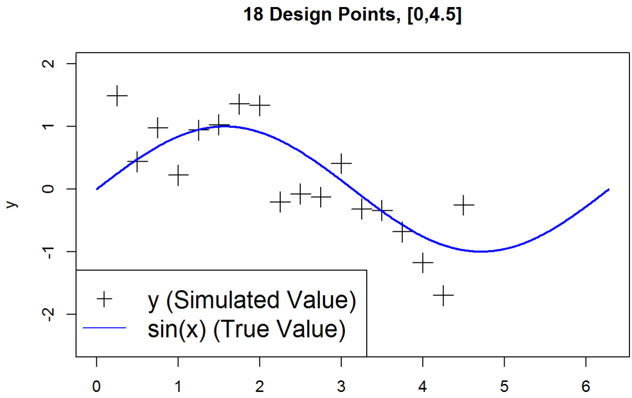
  where $\varepsilon \sim N(0, \sigma = 0.5)$
- Try:
  - $x = 0.25, 0.5, 0.75, \ldots, 2.75, 4.5$ ($N = 18$)
- Then update model with data on $(1.5\pi, 2\pi]$ to see how the overall fit changes
  - add $x = 4.75, \ldots, 6$ (total $N = 24$)

Illustrative Example

$$y = \sin(x) + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, 0.01x)$$

1. Generate data from random process

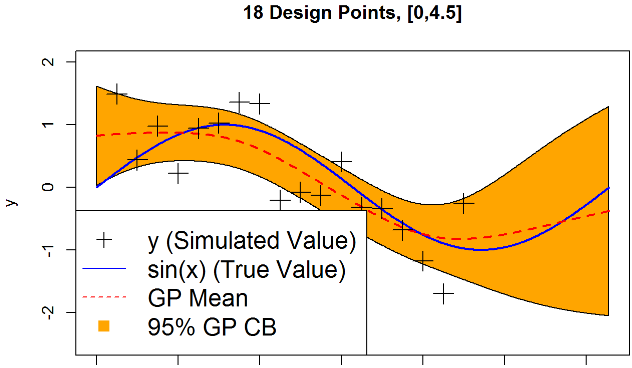
$$y = \sin(x) + \varepsilon, \qquad e \sim N(0, 0.5)$$



Illustrative Example

Fit GP to $N = 18$ (x, y) pairs

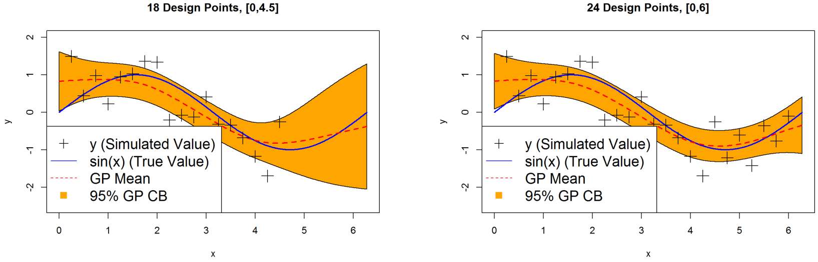
- Estimate hyperparameters (θ, η, σ)
- Produce posterior mean, covariance matrix (provides credible intervals)
- Observe naturally increasing uncertainty at edge data



Illustrative Example

Update GP with $N_{new} = 6$ (x, y) pairs on $[4.75, 6]$

- (Optional) update hyperparameters (θ, η, σ)
- Produce posterior mean, covariance matrix (provides credible intervals)



R Notebook Code

```
x <- seq(0.25,4.5,0.25)
n <- length(x)
y <- sin(x)+rnorm(n,0,0.5)
```

```
library(DiceKriging)
fit_nug <- km(formula = ~1,
              design = data.frame(x=x), response = y,
              nugget.estim = TRUE,
              covtype= "gauss",
              optim.method="gen")
```

```
nug <- fit_nug@covariance@nugget

fit <- km(formula = ~1,
          design = fit_nug@X, fit_nug@y,
          noise.var = rep(nug, fit_nug@n),
          coef.trend = fit_nug@trend.coef, #re-use obtained hyperparameters
          coef.cov = fit_nug@covariance@range.val)
```

```
##
## Call:
## km(formula = ~1, design = fit_nug@X, response = fit_nug@y, coef.trend = fit_nug@trend.coef,
##     coef.cov = fit_nug@covariance@range.val, noise.var = rep(nug,
##         fit_nug@n))
##
## Trend  coeff.:
##
##  (Intercept)    -0.0346
##
## Covar. type  : matern5_2
## Covar. coeff.:
##
##     theta(x)     1.3373
##
## Variance: 0.5803238
```

Comments

- $m(x) = m$ assumed *(clearly not true)*
- In practice,
  - Data is usually detrended, or
  - Parametric trend function e.g. $f(x) = \beta_0 + \beta_1 x$ assumed (and fitted alongside)
- Example is one-dimensional ($x \in \mathbb{R}$)
  - Framework naturally extends to multi-dimensional case ($x \in \mathbb{R}^d$), for example
    - $f(x, y) = \sin(x)\cos(y) + 2xy$
    - $f(\text{age}, \text{year}) = (\text{mortality rate depending on age, year})$

R Notebook For GPs on Mortality Data

- Provide R “code blocks" along with explanation of what it does and discussion of results
- Practitioner can choose to modify as much as needed
  - For example, simply change `cdcMale.csv` to `myInsuranceCompanyData.csv`
  - Plots have changeable ranges (easy to choose what years to plot)
  - ALL code is available, so programmers can easily modify as needed

Data

CDC Data

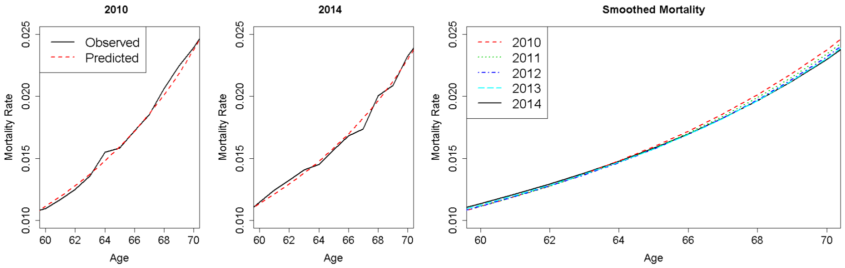
- United States
- Ages 50–84, Years 1999–2014
  - 1360 Data Points ($x = (x_{ag}, x_{yr})$)
  - 84 is maximal age for CDC data
  - 50 chosen as cutoff to minimize mixing lower age behavior
  - 1999 earliest year available on wonder.cdc.gov
  - Could add earlier years, but our analysis suggests they have little effect
  - Most relevant for longevity risk

GP Model Assumptions

- Observe central mortality rate $e^{-\mu(x_{ag},x_{yr})} = D(x_{ag,yr})/E(x_{ag,yr})$
- Fit log-mortality rate y to $x = (x_{ag}, x_{yr})$ pairs
- Can try $\sigma(x)$ based on Binomial assumption
  - Overdispersion issues ($\mu_{ag,yr}$ is unknown)
  - Minimal change in final model from simply choosing $\sigma := \sigma(x)$
- Use Gaussian covariance kernel
  - Implies f is differentiable
  - Minimal change in final model from other kernel choices
- Changed $m(x)$ trend function based on application
  - In-sample analysis generally used $m(x) = \beta_0$
  - Out-of-sample generally used $m(x) = \beta_0 + \beta_1 x_{ag} + \beta_2 x_{ag}^2 + \beta_3 x_{yr}$ (like Gompertz)

Posterior Predicted Mortality Rates

- Showing $m_*(x)$ for each ages 60–70
- Left panels include historic observations
- Right panel suggests mortality improvement



Goals

- In-sample smoothing
- Extrapolation (both in calendar year and age)
- Mortality Improvement

$$MI^{obs}_{back}\left(x_{ag}; yr\right) \doteq 1 - \frac{\exp\left(\mu(x_{ag}, yr)\right)}{\exp\left(\mu(x_{ag}, yr-1)\right)}$$

compare with SOA MP-2015 results

Loading Data

- Input data should be an R data frame with
  - age, calendar year, deaths, exposure
- The corresponding log mortality rates are computed as

$$y^n = \log(D^n/L^n)$$

  - D^n is the number of deaths and for the nth age/year pair $x^n = (x^n_{ag}, x^n_{yr})$,
  - L^n midyear count of lives

```
mortData <- read.csv("cdcMale.csv",header=T)
mortData$rate <- mortData$D / mortData$L
mortData$y <- log(mortData$rate)
head(mortData)
```

```
##   X age year     D       L        rate         y
## 1 1  50 1999  9775 1847555 0.005290776 -5.241790
## 2 2  51 1999 10470 1762492 0.005940452 -5.125970
## 3 3  52 1999 11509 1900702 0.006055131 -5.106849
## 4 4  53 1999  9885 1355175 0.007294261 -4.920667
## 5 5  54 1999 10717 1413117 0.007583944 -4.881722
## 6 6  55 1999 11728 1390616 0.008433673 -4.775523
```

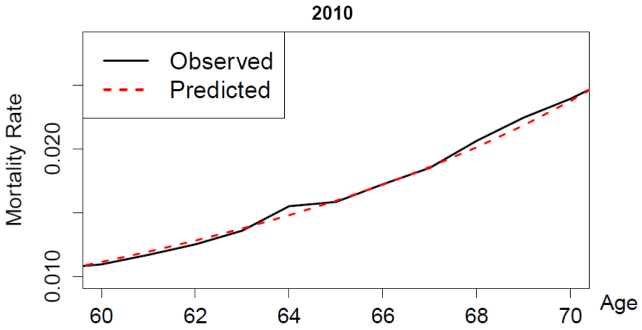
Fitting the Model

```
xMort <- data.frame(age = mortData$age, year = mortData$year)
yMort <- mortData$y
mortModel_nug <- km(formula = ~1,
                    design = data.frame(x = xMort), response = yMort,
                    nugget.estim=TRUE,
                    covtype="gauss",
                    optim.method="gen",
                    # the "control" parameters below handle speed versus risk of
                    # converging to local minima.  See "rgenoud" package for details
                    control=list(max.generations=100,pop.size=100,wait.generations=8,
                                 solution.tolerance=1e-5))
mortModel_nug
```

```
##
## Call:
## km(formula = ~1, design = data.frame(x = xMort), response = yMort,
##     covtype = "gauss", nugget.estim = TRUE, optim.method = "gen",
##     control = list(max.generations = 100, pop.size = 100, wait.generations = 8,
##         solution.tolerance = 1e-05))
##
## Trend  coeff.:
##                Estimate
##  (Intercept)    -3.8710
##
## Covar. type  : gauss
## Covar. coeff.:
##                 Estimate
##   theta(x.age)    15.8250
## theta(x.year)    15.5361
##
## Variance estimate: 1.842994
##
## Nugget effect estimate: 0.0002808436
```

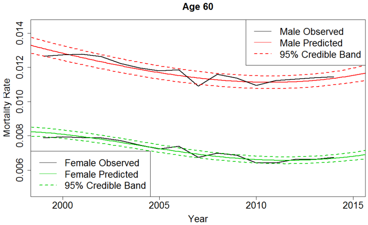
In-Sample Smoothing

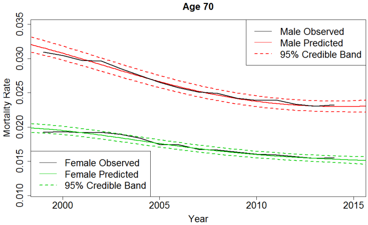
```
agesForecast <- seq(58,72,1)
yearsForecast <- 2010

rateForecastTrue <- dplyr::filter(mortData,age %in% agesForecast, year %in% yearsForecast)$rate
# build data frame for desired forecasts, then call predict
xPred <- data.frame(age=agesForecast,year=yearsForecast)
mortPred <- predict(mortModel, newdata=data.frame(x=xPred),
                    cov.compute=TRUE,
                    se.compute=TRUE,type="UK")

# plot mortality as a function of age
plot(agesForecast,rateForecastTrue,type="l",lwd=2, main="2010",
     xlab="Age",ylab="Mortality Rate",cex.axis=1.3,cex.lab=1.3,cex=1.3,
     xlim=c(60,70))
lines(agesForecast,exp(mortPred$mean),col=2,lty=2,lwd=2)
legend("topleft",c("Observed","Predicted"),lwd=c(2,2),lty=c(1,2),col=c(1,2),cex=1.5)
```

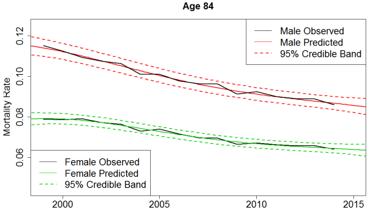


Mortality Over Time with Credible Bands

- Posterior mean and 95% credibility bands for f_* over calendar year
- Can observe increasing uncertainty at edges
- Observe mortality improvement then decline





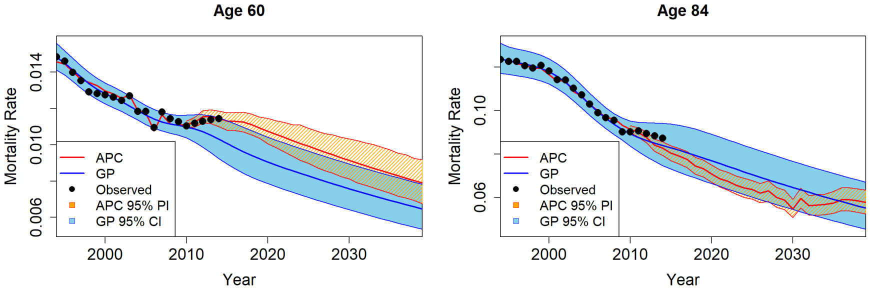


Extrapolation

- Compare to `apc` model from `R` package `stmomo`

$$\mu(x,t) = \alpha(x) + \kappa(t) + \gamma(t-x), \qquad \sum_c \gamma(c) = 0, \sum_c c\gamma(c) = 0$$

- GP produces similar forecasts with more desirable smoothness properties



Mortality Improvement

- Typical way is to look at the annual backward improvement

$$MI^{obs}_{back}\left(x_{ag}; yr\right) \doteq 1 - \frac{\exp\left(\mu(x_{ag}, yr)\right)}{\exp\left(\mu(x_{ag}, yr-1)\right)}$$

- $f_*(x_{ag}, yr)$ is a random variable, so we have the predicted mean improvement

$$m^{GP}_{back}\left(x_{ag}, yr\right) = \mathbb{E}\left[MI^{GP}_{back}\left(x_{ag}, yr\right)\right] \doteq \mathbb{E}\left[1 - \frac{\exp\left(f_*(x_{ag}, yr)\right)}{\exp\left(f_*(x_{ag}, yr-1)\right)}\right]$$

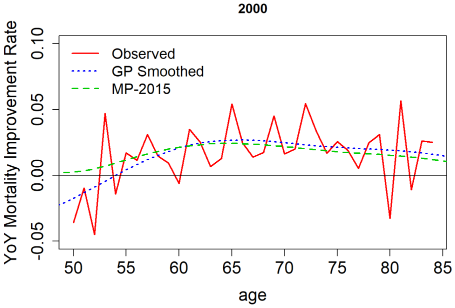
  - Available in closed form (lognormal distribution)
- Also have $MI^{MP}_{back}\left(x_{ag}; yr\right)$ (published MP-2015 improvement factors)

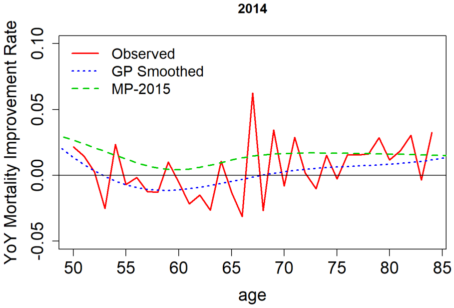
YoY Mortality Improvement Plots

```
# compare to MP-2015 tables
mp2015 <- read.csv("mp2015.csv")

agesForecast <- 48:86
agesObserved <- 50:84
yearsForecast <- c(2000,2014)
yearsPred <- c(1999,2000,2013,2014) # need to add one year back for improvement

nYr <- length(yearsPred)
nAg <- length(agesForecast)

# predict
xPred <- data.frame(age=rep(agesForecast,each=nYr),year=rep(yearsPred,nAg))
mortPred <- predict(mortModel, newdata=data.frame(x=xPred),cov.compute=TRUE,
             se.compute=TRUE,type="UK")

xPred$m <- mortPred$mean
for(yr in yearsForecast){
  forwardObs <- dplyr::filter(mortData, age %in% agesObserved, year == yr)$rate
  backwardObs <- dplyr::filter(mortData, age %in% agesObserved, year == yr-1)$rate
  MIbackobs <- 1-forwardObs/backwardObs    # raw observed improvement rates

  forwardPred <- filter(xPred, age %in% agesForecast, year == yr)$m
  backwardPred <- filter(xPred, age %in% agesForecast, year == yr-1)$m
  # smoothed improvement rates using the GP model
  mibackgp <- 1-exp(forwardPred)/exp(backwardPred)

  plot(agesObserved,MIbackobs, type="l", lwd=2, main = yr, ylim=c(-0.05, 0.1), xlab="age",
       ylab="YoY Mortality Improvement Rate", cex.axis=1.5,cex.lab=1.5,cex=1.5, col=2)
  lines(agesForecast,mibackgp, col=4, lwd=2, lty=3)
  lines(c(0,100),c(0,0))

  # MP-2015 rates
  mpRate <- dplyr::filter(mp2015, gender=="male", age %in% agesForecast, year == yr)$improvement
  lines(agesForecast, mpRate, col=3, lwd=2, lty=2)

  legend("topleft",c("Observed","GP Smoothed","MP-2015"), col=c(2,4,3),lwd=rep(2,3),
```

Comparing Mortality Improvement Methods

- Raw improvements extremely noisy (unsurprising)
- Smoothed methods both follow data well
- GP implies a stronger decline
  - Additional data suggests mortality deceleration

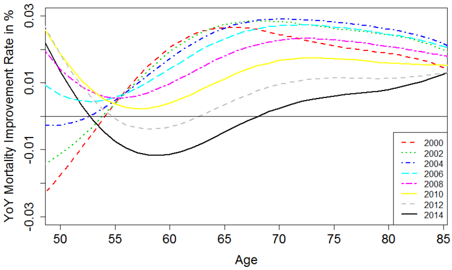




GP Improvement Over Time

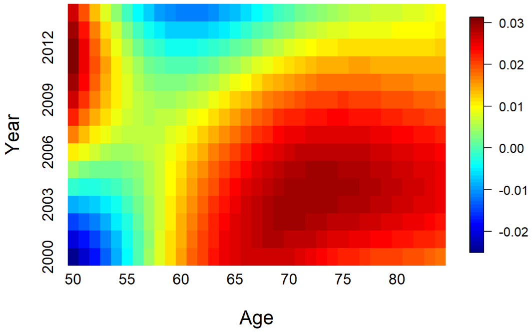
GP Improvements from 2000–2014 (in 2 year increments)

- Shape changes (flips) over time
- Consistent with MP-2015
- Generally decelerating after age 55



GP Mortality Improvement Heatmap

- Heatmap indicates possible cohort type relation with mortality improvement



Backward Difference & Derivatives

f_* denotes the fitted GP

$$1 - \left(\frac{\exp\left(f_*(x_{ag}, yr)\right)}{\exp\left(f_*(x_{ag}, yr - h)\right)} \right)^{1/h} \approx -\frac{f_*(x_{ag}, yr) - f_*(x_{ag}, yr - h)}{h} \quad (3)$$

- As defined, the typical annual mortality improvements are backward differences with $h = 1$
- Right side remains a GP by linearity
- Taking limit as $h \to 0$ yields derivative
  - Exists (depending on covariance kernel)
- Closed form expressions for distribution of $\frac{\partial f_*}{\partial x_{yr}}$

GP Derivative

Proposition

For the Gaussian Process f_* with a twice differentiable covariance kernel C, the limiting random variables

$$\frac{\partial f_*}{\partial x_{yr}}(x_{ag}, yr) \doteq \lim_{h \to 0} \frac{f_*(x_{ag}, yr+h) - f_*(x_{ag}, yr)}{h} \tag{4}$$

exist in mean square and form a Gaussian process $\frac{\partial f_*}{\partial x_{yr}} \sim GP(m_{diff}, s_{diff})$. Given the training set $\mathcal{D} = (\mathbf{x}, \mathbf{y})$, the posterior distribution of $\frac{\partial f_*}{\partial x_{yr}}(x_*)$ has mean and variance

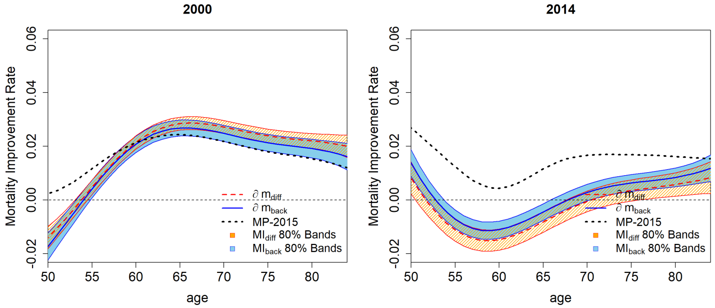
$$\begin{cases} m_{diff}(x_*) = \mathbb{E}\left[\frac{\partial f_*}{\partial x_{yr}}(x_*) \middle| \mathbf{x}, \mathbf{y} \right] = \frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*)(\mathbf{C} + \Sigma)^{-1}\mathbf{y}, \\ s^2_{diff}(x_*) = Var\left(\frac{\partial f_*}{\partial x_{yr}}(x_*) \middle| \mathbf{x}, \mathbf{y} \right) = \frac{\partial^2 C}{\partial x_{yr} \partial x'_{yr}}(x_*, x_*) - \frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*)(\mathbf{C} + \Sigma)^{-1} \frac{\partial C}{\partial x_{yr}}(x_*, \mathbf{x}), \end{cases}$$

where $\frac{\partial C}{\partial x'_{yr}}(\mathbf{x}, x_*) = \left[\frac{\partial C}{\partial x'_{yr}}(x^1, x_*), \ldots, \frac{\partial C}{\partial x'_{yr}}(x^N, x_*) \right]$ and each component is computed as the partial derivative of $C(x, x')$.

See Theorem 2.2.2 in Adler (2010) for more details/proof.

Comparing Other Methods with GP Derivative

- Blue is backwards mortality difference (as before); red is GP derivative; black is MP-2015
- Analysis of other years shows deceleration begins around 2010
  - Implies mortality evolution is convex
    - Justifies accelerating divergence between yearly difference and derivative methods
  - MP-2014 and MP-2015 begin to diverge around 2010
    - Suggests that later years are crucial to mortality forecasts



R Notebook Comments

- R notebook approach illustrates ease of use of GP's
- Practitioners have many options:
  - can simply change `.csv` file to their own data
  - can change output ranges for plots (e.g. plot 2016 instead of 2015)
  - have access to each plot and piece of code so programmers can specialize if needed
- Applied to US Females, Japan Male/Female, UK Male/Female data
  - Showed plausible forecasts for mortality and mortality improvement

Conclusions

- GP's provide a variety of benefits to modeling mortality and mortality improvement
  - Bayesian approach (data driven)
  - Posterior distribution for any location
    - Including distribution of mortality improvement (both yearly difference and instantaneous)
    - Credible bands (historic and forecasting)
- Relatively consistent results with MP-2015
  - Four years of additional data pushes GP results in the direction that MP-2015 took compared to MP-2014 (and MP-2016, 2017 found later)
  - Differences in results is likely due to data differences than model issues
- GP framework easily handles joint analysis of mortality rates and mortality improvement

Future Work

- Modeling annual mortality improvement directly with GP
- Monotonicity constraint: $f \Big|_{\frac{\partial f}{\partial f_{age}} > 0}$
- Multiple populations
  - Jointly modeling male & female mortality
  - Multivariate GP of multiple countries and factors
- Modeling by cause of death

References

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THANK YOU!