Smooth Greeks with Chebyshev Interpolation

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SHAPING
THE FUTURE

Index

Problem Statement

- A **fast** and **accurate** evaluation of **Greeks** is a fundamental task for both Front Office (for effectively managing risks) and Risk Management (see sensitivity-based regulations such as FRTB, SIMM, SA-CVA).
- When a financial product is priced via **Monte Carlo** simulation, Greeks computation tend to be **lengthy** and **noisy** and particular care must be used. Second order derivatives (e.g. **Gamma**) suffer from the highest instabilities.
- **Adjoint Algorithmic Differentiation** is a powerful technique to speed-up and smooth first-order Greeks, but in general situations it cannot be applied to second order Greeks. Moreover, **singularities** in the payoff must be smoothed at the cost of adding a bias.

Problem Statement

- In [Maran et al.(2021)] we propose a methodology based on **Chebyshev interpolation**, able to tame the numerical instabilities of Monte Carlo Greeks, including second order Greeks, without increasing the overall computational burden.
- Chebyshev techniques have recently gained new interest in Finance, because of their ability to **boost** the **performance** of complex pricing tools. See e.g. [Gaß et al.(2018), Zeron-Ruiz(2018)].
- Typical **applications** focused, so far, on situations where a heavy pricing function has to be called many times. This is the case of calibration, XVA pricing, FRTB and CCR computations, etc. [Glau et al.(2020), Zeron-Ruiz(2020), Zeron-Ruiz(2021a), Zeron and Ruiz(2021b)].

Finite Differences: Bias-Variance Tradeoff

- **Finite Differences (FD)**: The original pricing function $f(x)$ is called on an equispaced grid given by the following 3 points

$$\{\, x_0 - h, \quad x_0, \quad x_0 + h \,\}$$

to obtain Greeks approximations with the following formulas

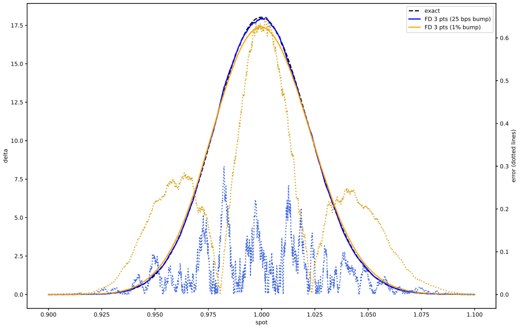
$$f'(x_0) = \frac{1}{2h}\Big[-f(x_0 - h) + f(x_0 + h)\Big]$$

$$f''(x_0) = \frac{1}{h^2}\Big[f(x_0 - h) - 2f(x_0) + f(x_0 + h)\Big]$$

- The **bump** h should be fine-tuned to minimize the FD **bias** without increasing too much the **variance** of the MC result. Note that:

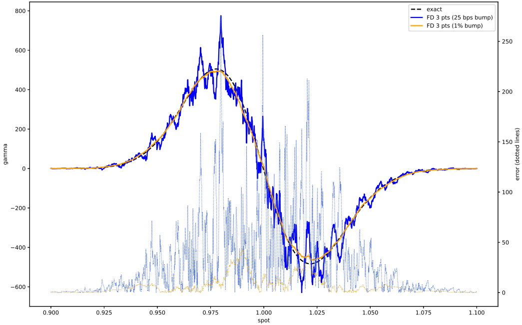
$$\mathrm{Bias}[f', f''] = \mathcal{O}(h^2), \qquad \mathrm{Var}[f'] = \mathcal{O}(h^{-1}), \qquad \mathrm{Var}[f''] = \mathcal{O}(h^{-3})$$

Finite Differences: Bias-Variance Tradeoff



Finite Difference **Delta** *of a digital option for different spot levels. Black-Scholes exact formulas are compared with MC results using 300K paths and two different bump sizes: the higher bump produces significant bias.*

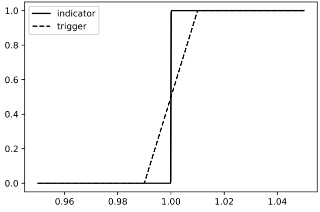
Finite Differences: Bias-Variance Tradeoff



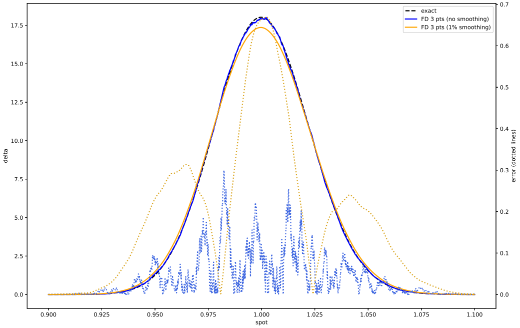
Finite Difference **Gamma** *of a digital option for different spot levels. Black-Scholes exact formulas are compared with MC results using 300K paths and two different bump sizes: the lower bump produces significant variance.*

Finite Differences: Smoothing Singularities

- Another source of numerical instabilities in MC Greeks is the presence of **discontinuities** in the price or in some of its derivatives.
- One trivial way to deal with this problem is to **smooth** the payoff function, however this adds another hyperparamter (the *smoothing* parameter) and leads to a bias in the results.
- In practice this is achieved by replacing indicator functions with **trigger** functions (tight call spreads):

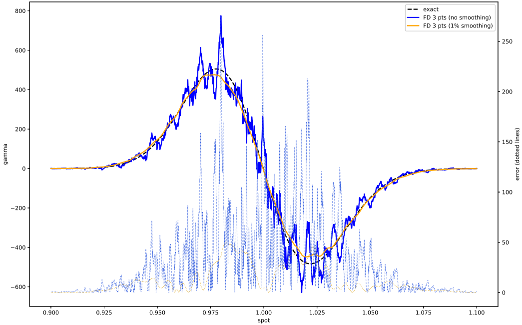


Finite Differences: Smoothing Singularities



Finite Difference ***Delta*** *of a digital option for different spot levels. Black-Scholes exact formulas are compared with MC results using 300K paths, 0.0025 bump and different smoothing parameters: the smoothed Greek is biased.*

Finite Differences: Smoothing Singularities



Finite Difference **Gamma** *of a digital option for different spot levels. Black-Scholes exact formulas are compared with MC results using 300K paths, 0.0025 bump and two different smoothing parameters: the smoothed Greek is biased.*

Lagrange Polynomial Interpolator

- The idea of using Chebyshev techniques stems from using as a good approximation of the sensitivity the **derivative** of a **polynomial** interpolating the price function.
- We start by defining the **Lagrange polynomial interpolant** of a function f on an interval $[a, b]$ based on interpolation grid $\{x_k\}_{k=0,\dots,n-1}$ as

$$p_{n-1}(x) = \sum_{k=0}^{n-1} f(x_k)\ell_k(x)$$

where

$$\ell_k(x) = \prod_{j \neq k} \frac{x - x_j}{x_k - x_j}$$

is the k-th **Lagrange polynomial**.

Barycentric Formula

- The Lagrange polynomial interpolant can be efficiently evaluated through the **barycentric formula**

$$p_{n-1}(x) = \frac{\sum_{k=0}^{n-1} \frac{w_k\, f(x_k)}{x - x_k}}{\sum_{k=0}^{n-1} \frac{w_k}{x - x_k}}$$

where $\{w_k\}$ are the **barycentric weights**

$$w_k := \frac{1}{\prod_{j \neq k}(x_k - x_j)}$$

- For details on Chebyshev interpolation techniques we refer to [Trefethen(2020)] and references therein.

Barycentric Derivatives

- Differentiating the barycentric formula on the interpolation nodes leads to

$$p_{n-1}^{(m)}(x_i) = \sum_{k=0}^{n-1} f(x_k)\ell_k^{(m)}(x_i) = \sum_{k=0}^{n-1} D_{ik}^{(m)} f(x_k)$$

where $D^{(m)}$ is called **differential matrix** of order m. They depend only on the grid points and can be computed by the following recursive formula

$$D_{ik}^{(0)} = \delta_{ik}, \quad D_{ik}^{(m)} = \begin{cases} \frac{m}{x_i - x_k}\left(\frac{w_k}{w_i} D_{ii}^{(m-1)} - D_{ik}^{(m-1)}\right) & \text{if } i \neq k \\ -\sum_{j\neq i} D_{ij}^{(m)} & \text{otherwise} \end{cases}$$

- The value of $p_{n-1}^{(m)}(x)$ at a generic point can then be obtained via the barycentric formula, replacing $f(x_k)$ with $p_{n-1}^{(m)}(x_k)$ as given above.

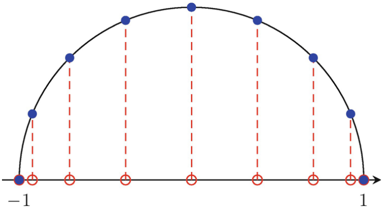
Chebyshev Points

- The optimal choice for the interpolation nodes is represented by **Chebyshev points**, which on the base interval $[-1, 1]$ are defined as

$$x_k = \mathrm{Re}\left(e^{\frac{i\,k\,\pi}{n-1}}\right) = \cos\left(\frac{k\,\pi}{n-1}\right), \quad k = 0, \ldots, n-1$$

- The corresponding barycentric weights are simply given by:

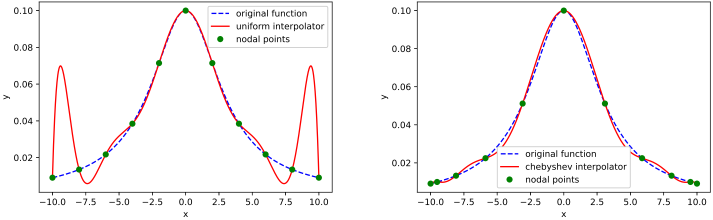
$$w_k = \begin{cases} \frac{(-1)^k}{2} & \text{if } k = 0,\ k = n-1 \\ (-1)^k & \text{otherwise} \end{cases}$$



Runge Phenomenon

Consider the polynomial interpolation of the following function:

$$f(x) = \frac{1}{10 + x^2}$$



Lagrange interpolators of $f(x)$ in $[-10, 10]$ for $\boldsymbol{n = 11}$ **points**. *Left panel: equispaced grid. Right panel: Chebyshev grid. As n increases, the uniform interpolator diverges close to the boundary of the interpolation domain, while the Chebyshev interpolator converges everywhere to $f(x)$.*

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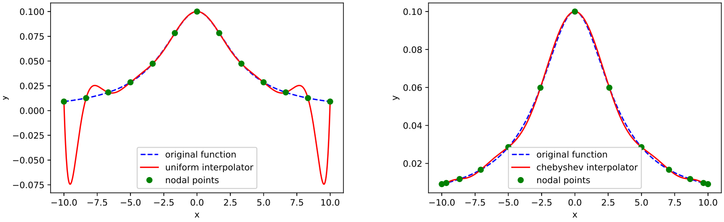

Lagrange interpolators of $f(x)$ in $[-10, 10]$ for $\boldsymbol{n = 13}$ **points**. *Left panel: equispaced grid. Right panel: Chebyshev grid. As n increases, the uniform interpolator diverges close to the boundary of the interpolation domain, while the Chebyshev interpolator converges everywhere to $f(x)$.*

Runge Phenomenon

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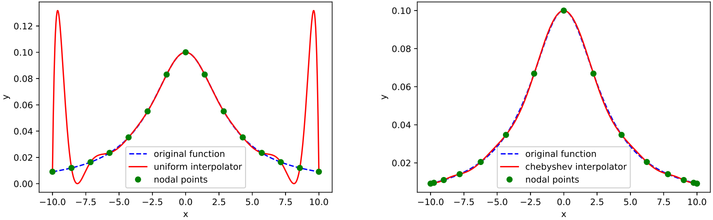


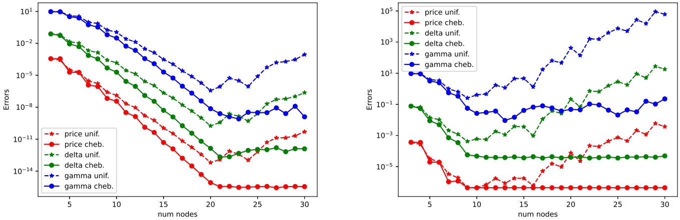
Lagrange interpolators of $f(x)$ in $[-10, 10]$ for $\boldsymbol{n = 15}$ **points**. *Left panel: equispaced grid. Right panel: Chebyshev grid. As n increases, the uniform interpolator diverges close to the boundary of the interpolation domain, while the Chebyshev interpolator converges everywhere to $f(x)$.*

Chebyshev Convergence

Theorem (Chebyshev Approximation)

Let f be an **analytic** *function on $[-1,1]$ which is analytically continuable to the closed Bernstein ellipse $\bar{E}(\rho)$ of radius $\rho > 1$. Then, $\forall m, n \in \mathbb{N}\ \exists C > 0$ s.t.*

$$||f^{(m)} - p_{n-1}^{(m)}||_\infty \leq C\rho^{-n}$$



L^∞ errors vs number of interpolation nodes for a call with $K = 1$, $T = 0.1$, $r = 0$, $\sigma = 7\%$ and $S_0 \in [0.94, 1.01]$. Left panel: **analytical** *pricer. Right panel:* **MC** *pricer.*

Chebyshev Greeks

- Let $f(x)$ be the price of a financial product as a function of the parameter x.
- We approximate $f^{(m)}$ at some point $\bar{x}$ through the derivative $p^{(m)}(\bar{x})$ of the Chebyshev interpolant of f in some interval including $\bar{x}$.
- The procedure can be be split out in the following phases:
  1. Choose the interpolation domain $H \ni \bar{x}$ and the number n of Chebyshev points.
  2. Build the Chebyshev interpolator. This amounts to:
     - i compute Chebyshev points $\{x_k\}_{k=0,\ldots,n-1}$ on H using an affine map;
     - ii compute barycentric weights $\{w_k\}_{k=0,\ldots,n-1}$;
     - iii compute the differential matrices up to the desired order m;
     - iv evaluate the original pricing function on the Chebyshev points to obtain the interpolation nodes $\{f(x_k)\}_{k=0,\ldots,n-1}$.
  3. Obtain the desired Greeks as $p^{(m)}(\bar{x})$.

Adaptive Domains

- Particular attention should be paid to handle possible **singularities**, e.g. barriers, in the price function or its derivatives.

- We leverage on the following **heuristics**: close to a singularity, the derivatives of the price function show some peaks, which are higher and more localized the more the barrier date approaches in time. In this case we reduce the interpolation domain. Away from the singularity, we can always use a wider domain.

- Therefore, we employ different methods to *adapt* the size of the interpolation domain $H := [\xi - a, \xi + a]$ (for some ξ) to the presence of a singularity at b:

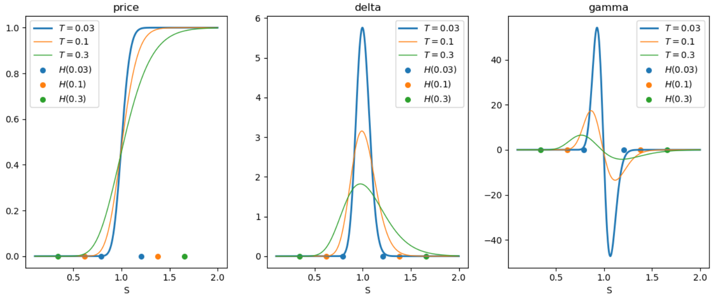
  1. **time adaptivity**: $a \propto \bar{x}\,\sigma\sqrt{\tau}$
  2. **space adaptivity**: $a \propto |\bar{x} - b|$

  where σ is the underlying ATM volatility, τ is the time to next singularity and a is the half-size of the interpolation domain, with appropriate bounds a_{min} and a_{max}. Moreover, if $\xi = \bar{x}$ we say that H is *centered*.

- If the singularity appears also in the price function, then we need to avoid that it falls inside H: in this case, we apply a **translation** to H, without reducing it (*uncentered* space-adaptivity).

Adaptive Domains

- The above formula for **time adaptivity** is motivated by the following argument: assuming that singularities are generated by digital features, for digital options in Black model the scale of the singularity is given by $\sigma\sqrt{\tau}$.

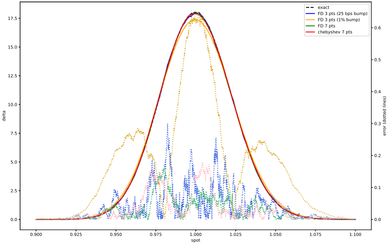


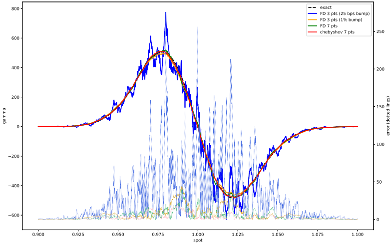
Graphical representation of the Chebyshev adaptive domain $H(T)$ in dependence on the time to next barrier T compared to the behaviour of digital price (left panel), delta (middle panel) and gamma (right panel). Results shown for a Black digital call with $K = 1$, $\sigma = 0.4$, $r = 0$.

Example

In order to assess the accuracy of Chebyshev methodology w.r.t. other techniques, we consider a **digital option** in **Black** model and evaluate the following numerical **errors**:

$$\varepsilon_\Delta(S) = \left|p'_{n-1}(S) - \Delta_{BS}(S)\right| \ , \qquad \varepsilon_\Gamma(S) = \left|p''_{n-1}(S) - \Gamma_{BS}(S)\right|$$





Method	# nodes	Avg ε_Δ	Std ε_Δ	Max ε_Δ	Avg ε_Γ	Std ε_Γ	Max ε_Γ
FD 25bps bump	3	0.04	0.05	0.3	30.3	39.8	275.4
FD 1% bump	3	0.17	0.17	0.65	6.6	8.6	43.9
FD 1% bump	7	0.03	0.03	0.17	5.19	6.75	40.3
adaptive Cheb.	7	0.03	0.04	0.18	3.03	3.93	20.6

Applications to Real Payoffs

We now assess the effectiveness of our methodology in the computation of spot Greeks **Delta** and **Gamma** for some exotic payoffs with different types of singularities:

1. FX **Target Redemption Forwards** (TARFs) under the *Stochastic Local Volatility* (SLV) model by [Tataru-Fisher(2010)]
2. Equity **Autocallable** options under multi-asset *Local Volatility* (LV) model by [Dupire(1994), Derman-Kani(1994)]

We compare Chebyshev Greeks with 3-point finite difference Greeks for different pricing dates and spot levels. The efficiency of the methodology can be assessed via the average **MC error**, defined as[1]:

$$\varepsilon_t(y) = \frac{1}{L} \sum_{p=1}^{L} SE_y(S_p)$$

[1] Here, t denotes the pricing date, $\{S_p\}_{p=1,\dots,L}$ is a grid of spot levels ($L = 200$ in our tests), SE denotes the standard error. Finally, y denotes price, delta or gamma.

Test 1: FX TARFs

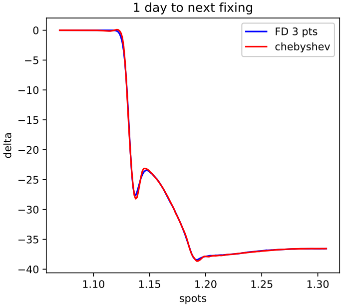
We consider a **TARF KO**, i.e. an FX product which pays, at each fixing date T_i, the following put-like coupons until a maximum payout θ (target) is reached:

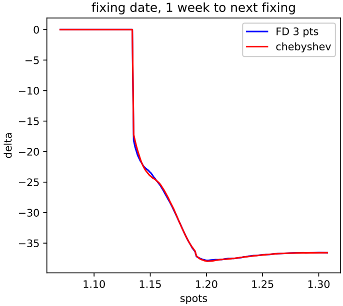
$$\frac{K - S_{T_i}}{K\, S_{T_i}} \left(\mathbb{1}_{\{S_{T_i} \leq K\}} + \mathbb{1}_{\{S_{T_i} > B_{KI}\}}\right) \mathbb{1}\left\{ \sum_{j=1}^{i} (K - S_{T_j})^+ < \theta \right\} \left(1 - \min\left\{1, \sum_{j=1}^{i} \mathbb{1}_{\{S_{T_j} < B_{KO}\}}\right\}\right) N_i$$

In particular, we specify the following contract data:

- underlying asset: $S =$ EUR/USD FX rate
- strike: $K = 1.15$
- knock-in barrier: $B_{KI} = 1.19$
- knock-out barrier: $B_{KO} = 1.135$
- number of coupons: 70
- coupon frequency: weekly
- residual target: $\theta = 0.2$
- coupon notionals: $N_i = 0.3 \cdot \mathbb{1}_{\{S_{T_i} \leq K\}} + 0.6 \cdot \mathbb{1}_{\{S_{T_i} > K\}}$

Test 1: FX TARFs - Delta

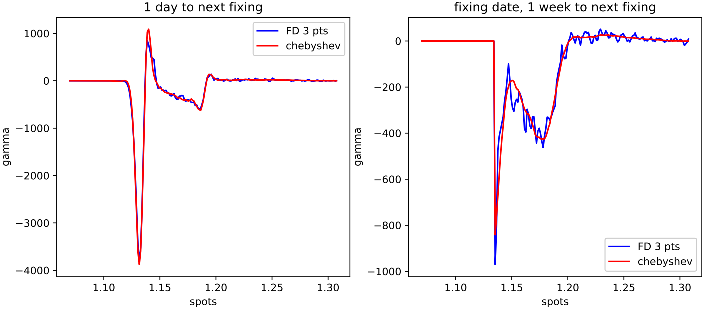




Greeks method	**MC paths**	**# nodes**	a_{min}	a_{max}	$\varepsilon_{1d}(\Delta)$	$\varepsilon_{1w}(\Delta)$
finite differences	1,000,000	3	0.25%	0.25%	0.03	0.03
adaptive* Chebyshev	300,000	7	0.75%	5%	0.04	0.03

* *centered space-time adaptivity*

Test 1: FX TARFs - Gamma



Greeks method	MC paths	# nodes	a_{min}	a_{max}	$\varepsilon_{1d}(\Gamma)$	$\varepsilon_{1w}(\Gamma)$
finite differences	1,000,000	3	0.25%	0.25%	21	18
adaptive* Chebyshev	300,000	7	0.75%	5%	14	4

* *centered space-time adaptivity*

Test 2: Equity Autocallables

We now consider an **autocallable** with memory on a basket of two stocks, i.e. an option paying, at each time T_i, the following coupons, unless the basket performance crosses a level B_{call}:

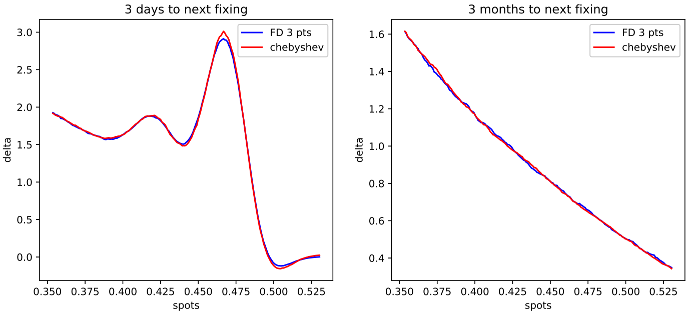
$$\Pi(T_i) = \mathbb{1}_{\{\tau > T_i\}} \left(\left[N_i + \sum_{j=1}^{i-1} \Big(N_j - \Pi(T_j) \Big) \right] \mathbb{1}_{\{P(T_i) \geq B_{coup}\}} + \delta_{iN} \left(P(T_N) - 1 \right) \mathbb{1}_{\{P(T_N) < B_{guar}\}} \right)$$

where $\tau = \min\{T_i : P(T_i) \geq B_{call}\}$.

In particular, we specify the following contract data:

- underlying assets: TELECOM and VODAFONE share prices
- performance type: "worst-of", i.e. $P(t) = \min\left\{ S_t^{TEL}/0.48, S_t^{VOD}/1.3 \right\}$
- call barrier: $B_{call} = 100\%$
- coupon barrier: $B_{coup} = 90\%$
- capital-guarantee barrier: $B_{guar} = 60\%$
- number of coupons: $N = 7$
- coupon frequency: quarterly
- coupon notionals: $N_i = 0.02$

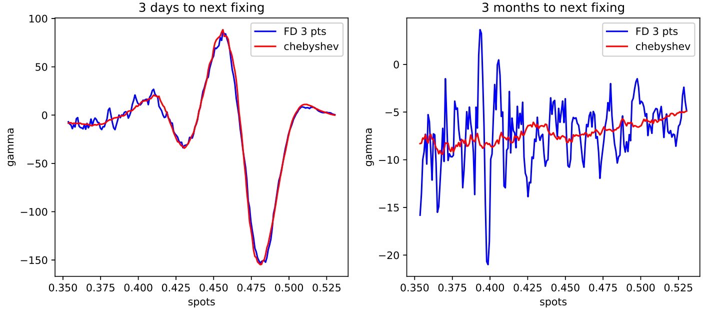
Test 2: Equity Autocallables - Delta



Greeks method	MC paths	# nodes	a_{min}	a_{max}	$\varepsilon_{3d}(\Delta)$	$\varepsilon_{3m}(\Delta)$
finite differences	1,000,000	3	1%	1%	0.02	0.01
adaptive* Chebyshev	300,000	7	3%	10%	0.02	0.01

* *centered space-time adaptivity*

Test 2: Equity Autocallables - Gamma



Greeks method	MC paths	# nodes	a_{min}	a_{max}	$\varepsilon_{3d}(\Gamma)$	$\varepsilon_{3m}(\Gamma)$
finite differences	1,000,000	3	1%	1%	7	5
adaptive* Chebyshev	300,000	7	3%	10%	5	1

* *centered space-time adaptivity*

Test 3 (Beyond MC Greeks): Window UOC

The benefits of Chebyshev Greeks are not limited to MC pricers: consider e.g. an **up-and-out window barrier** option under the SLV model by [Tataru-Fisher(2010)], priced with a **PDE approach** as described in [in 't Hout-Foulon(2010)]:

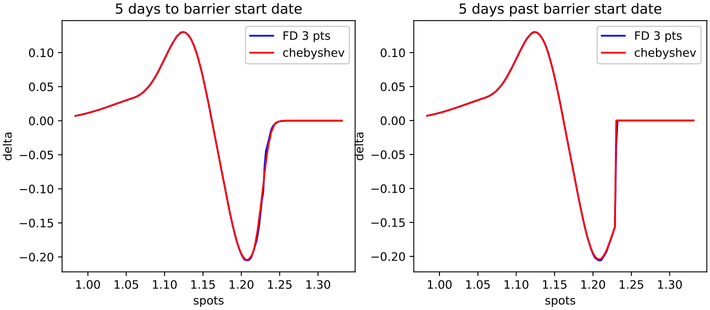
$$\Pi(T) = [\omega\,(S_T - K)]^+ \; \mathbb{1}\{S_t < B,\, \forall T_s \leq t \leq T_e\}$$

We perform a similar analysis as before[2] on the following contract:

- underlying asset: $S =$ EUR/USD FX rate
- call type: $\omega = 1$
- strike: $K = 1.1574$
- barrier level: $B = 1.23$
- barrier monitoring: continuous
- barrier end date T_e: 11M
- expiry T: 1Y

[2] Regarding the estimations of errors $\varepsilon_t(y)$, we consider here the average distance, on 200 spot levels, with values of y computed with a double-sized PDE grid and 7-point FD with a bump of 5bps. The barrier start date T_s can be either future or past, depending on the test case.

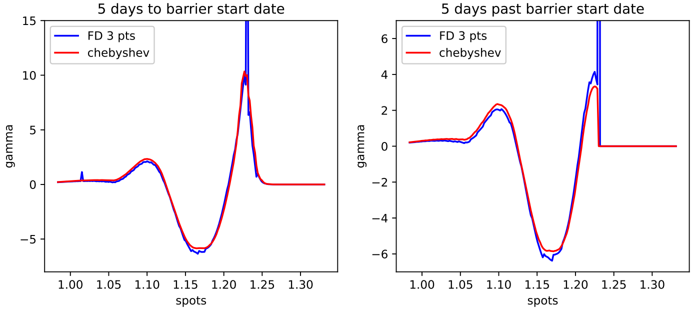
Test 3 (Beyond MC Greeks): Window UOC - Delta



Greeks method	**x-thickness**	**# nodes**	a_{min}	a_{max}	$\varepsilon_{-5d}(\Delta)$	$\varepsilon_{5d}(\Delta)$
finite differences	600	3	5bps	5bps	0.0007	0.0005
adaptive* Chebyshev	600	5	0.1%	1%	0.0002	0.00005

* *centered time-adaptivity before barrier start, uncentered space-adaptivity afterwards.*

Test 3 (Beyond MC Greeks): Window UOC - Gamma



Greeks method	x-thickness	# nodes	a_{min}	a_{max}	$\varepsilon_{-5d}(\Gamma)$	$\varepsilon_{5d}(\Gamma)$
finite differences	600	3	5bps	5bps	0.4	0.8
adaptive* Chebyshev	600	5	0.1%	1%	0.04	0.01

* *centered time-adaptivity before barrier start, uncentered space-adaptivity afterwards.*

Conclusions and Future Perspectives

- We presented a *simple* and *general* method, based on **Chebyshev interpolation** techniques, for Greeks computation of arbitrarily complex payoffs, with different numerical pricing techniques.
- The number of interpolation nodes can be kept *low* (typically under 10), while the size of interpolation domain can be *adapted* to the **time** and **space** distance from **singularities** in the price or its derivatives.
- In most situations, the methodology allows to significantly **improve** the **numerical stability** of Greeks and, at the same time, to **reduce** the **computational time**, even though the number of repricings is slightly increased w.r.t. standard FD.
- The multi-dimensional extension[3] of Chebyshev techniques would offer the possibility to effectively compute also **cross-gammas**.

[3] see e.g. [Glau et al.(2020)]

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