

On Randomization of Affine Diffusion Processes
with Application to Pricing of Options on VIX and S&P 500
(and more...)

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Motivation

- Recent studies have shown that the standard models **do not offer sufficient flexibility** in pricing advanced derivatives, e.g., options on S&P and VIX.
- In Carr and Wu (2007), where the problem of **insufficient skew** was reported, and the remedy in terms of randomization was suggested: *“it would be tempting to try to capture stochastic skewness by **randomizing** the mean jump size parameter (...) However, randomizing either parameter is not amenable to analytic solution techniques that greatly aid econometric estimation.”*
- The concept of randomizing is more fundamental, i.e., it represents the incorporation of the **uncertainty of potentially hidden states** that are not adequately captured by deterministic parameters.
- We can also consider randomization as a **regime-switching** method, with the states determined by the randomizing random variable.

In the Nutshell

- We consider [an affine model](#) for which pricing is performed efficiently, e.g., Heston, Bates, etc.
- Take a certain model parameter to be random and follow some random variable $\vartheta(p_1, p_2)$, with parameters p_1 and p_2 .
- By the application of the [Randomized-Affine-Diffusion \(RAnD\)](#) method [Grzelak \(2022a\)](#), the randomized ChF is equivalent to taking a linear combination of constituent ChFs, i.e.,

$$\phi_{\mathbf{x}}(\mathbf{u}; t, T) = \sum_{n=1}^N \omega_n \phi_{\mathbf{x}|\vartheta=\theta_n}(\mathbf{u}; t, T), \quad \sum_{n=1}^N \omega_n = 1.$$

- Or, at the price level:

$$V(t) = \sum_{n=1}^N \omega_n V(t, S(t; \theta_n)), \quad \sum_{n=1}^N \omega_n = 1,$$

where θ_i are some realizations of the random parameter, ϑ , and where ω_n indicates appropriate weight.

- The pairs $\{\omega_i, \theta_i\}_{i=1}^N$ are computed based on moments of $\vartheta(p_1, p_2)$.

Stochastic vs. Random

- It is very important to distinguish between a **stochastic process** and a **random variable**.

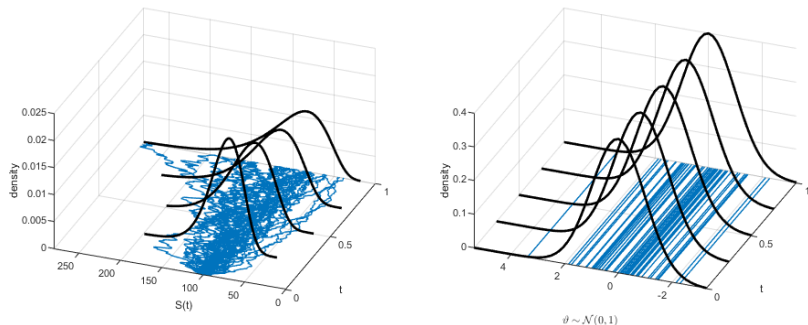


Figure: Left: Stock Paths, Right: Paths for a random parameter.

Affine Models

- The stochastic model of interest can be expressed by the following stochastic differential form:

$$d\mathbf{X}(t) = \boldsymbol{\mu}(t, \mathbf{X}(t))dt + \boldsymbol{\sigma}(t, \mathbf{X}(t))d\tilde{\mathbf{W}}(t) + \mathbf{J}(t)^\top d\mathbf{X}_{\mathcal{P}}(t),$$

where $\tilde{\mathbf{W}}(t)$ is a column vector of *independent* Brownian motions, $\boldsymbol{\mu}$, is a drift, $\boldsymbol{\sigma}$ corresponds to volatility, and $\mathbf{X}_{\mathcal{P}}(t)$ is a vector of orthogonal Poisson processes characterized by an intensity vector $\boldsymbol{\xi}$.

- We consider an orthogonal vector $\Theta = [\vartheta_1, \dots, \vartheta_n]^\top$, $n \in \mathbb{N}$, where each ϑ_i is an independent, time-invariant, random variable¹.
- A realization of ϑ_i we indicate by θ_i , $\vartheta_i(\omega) = \theta_i$.
- We assume the model is affine for a realization of a random parameter.

¹We consider here $n \in \mathbb{N}$ stochastic parameters, this is however not a necessary constraint.

Affine Models

- Affinity conditions require the following linearity of the model:

$$\begin{aligned}\boldsymbol{\mu}(t, \mathbf{X}(t)) &= a_0(\theta) + a_1(\theta)\mathbf{X}(t), \\ \mathbf{r}(t, \mathbf{X}(t)) &= r_0(\theta) + r_1(\theta)^\top \mathbf{X}(t), \\ (\boldsymbol{\sigma}(t, \mathbf{X}(t))\boldsymbol{\sigma}(t, \mathbf{X}(t))^\top)_{i,j} &= (c_0(\theta))_{i,j} + (c_1(\theta))_{i,j}^\top \mathbf{X}_j(t), \\ \xi(t, \mathbf{X}(t)) &= l_0(\theta) + l_1(\theta)\mathbf{X}(t).\end{aligned}$$

- For a given realization of Θ , θ , we consider $\mathbf{X}_\theta(t) := \mathbf{X}(t)|\Theta = \theta$, $\mathbf{J}_\theta(t) := \mathbf{J}(t)|\Theta = \theta$, the discounted characteristic function is also of the following form (Duffie et al., 2000):

$$\phi_{\mathbf{X}_\theta}(\mathbf{u}; t, T) = \mathbb{E}_t \left[e^{-\int_t^T r(s)ds + i\mathbf{u}^\top \mathbf{X}_\theta(T)} \right] = e^{A(\mathbf{u}; \tau, \theta) + \mathbf{B}^\top(\mathbf{u}; \tau, \theta) \mathbf{X}_\theta(t)},$$

with the expectation under risk-neutral measure $\mathbb{Q}$ for $\tau = T - t$.

Affine Models and Randomization

- The coefficients $A := A(\mathbf{u}; \tau, \theta)$ and $\mathbf{B} := \mathbf{B}^\top(\mathbf{u}; \tau, \theta)$, satisfy complex-valued *Riccati* ODEs (Duffie et al., 2000):

$$\begin{aligned}\frac{dA}{d\tau} &= -r_0(\theta) + \mathbf{B}^\top a_0(\theta) + \frac{1}{2} \mathbf{B}^\top c_0(\theta) \mathbf{B} + l_0^\top \mathbb{E} \left[e^{\mathbf{J}_\theta(\tau) \mathbf{B}} - 1 \right], \\ \frac{d\mathbf{B}}{d\tau} &= -r_1(\theta) + a_1(\theta)^\top \mathbf{B} + \frac{1}{2} \mathbf{B}^\top c_1(\theta) \mathbf{B} + l_1(\theta)^\top \mathbb{E} \left[e^{\mathbf{J}_\theta(\tau) \mathbf{B}} - 1 \right],\end{aligned}$$

where the expectation, $\mathbb{E}[\cdot]$, is taken with respect to the jump amplitude $\mathbf{J}_\theta(t)$.

- Then, for stochastic parameter ϑ , the ChF is given by:

$$\phi_{\mathbf{x}}(\mathbf{u}; t, T) := \mathbb{E}_t \left[e^{-\int_t^T r(s) ds + i \mathbf{u}^\top \mathbf{x}(T)} \right] = \mathbb{E}_t \left[\mathbb{E}_t \left[e^{-\int_t^T r(s) ds + i \mathbf{u}^\top \mathbf{x}(T)} \mid \Theta = \theta \right] \right].$$

- The inner expectation can be recognized as the conditional ChF; thus, by definition of the ChF and integration over all the parameter space, we find,

$$\phi_{\mathbf{x}}(\mathbf{u}; t, T) = \mathbb{E}_t [\phi_{\mathbf{x}|\Theta}(\mathbf{u}; t, T)] = \int_{\mathbb{R}^n} \phi_{\mathbf{x}|\Theta=\theta}(\mathbf{u}; t, T) f_\Theta(\theta) d\theta.$$

- We aim to provide numerically efficient methods for the computation of randomized ChF.

Affine Models and Randomization

- To determine the ChF of an affine model with a *randomized parameter*, one needs to integrate the parameter's probability density function- **computationally expensive!**
- This can be avoided, i.e., the complicated integrand can be factored into a set of pairs $\{\omega_n, \theta_n\}_{n=1}^N$, $N \in \mathbb{N}$, with a nonnegative “weights” function, $\omega_n \geq 0$, such that $\sum_{n=1}^N \omega_n = 1$ and specific, collocation, points θ_n .
- Once the number of evaluations, N , is low, **we can significantly reduce the computational cost**. The key element here, however, is that the pairs, $\{\omega_n, \theta_n\}_{n=1}^N$, cannot be chosen arbitrarily but need to be computed based on the parameter's distribution, ϑ .
- We consider a random parameter ϑ with its PDF, $f_\vartheta(\cdot)$, such that for a fixed number $N \in \mathbb{N}$, moments are finite, i.e., $\mathbb{E}[\vartheta^{2N}] < \infty$ (that is the only requirement).

Affine Models and Randomization

- We determine a mapping, often called *quadrature rule*, function:

$$\zeta(\vartheta) : \mathbb{R} \rightarrow \{\omega_n, \theta_n\}_{n=1}^N.$$

- We follow the approach presented in (Golub and Welsch, 1969) where ω_n are the quadrature weights determined based on the moments of the random parameter, ϑ .
- Pairs $\{\omega_n, \theta_n\}_{n=1}^N$ are based on a three-term recurrence relation which is well-known for orthogonal polynomials generated by $f_\vartheta(x)$, or equivalently, by the moments² of ϑ .
- The computation of the Gauss quadrature points and the associated weights $\{\omega_n, \theta_n\}_{n=1}^N$ is based on orthogonal polynomials.

²The accompanying Python and MATLAB codes can be found at <https://github.com/LechGrzelak/Randomization>.

$$\zeta(\vartheta) : \mathbb{R} \rightarrow \{\omega_n, \theta_n\}_{n=1}^N$$

- These polynomials can be associated with moments of the underlying variable ϑ , [Golub and Welsch \(1969\)](#).
- In a nutshell, we need to construct a tridiagonal matrix J :

$$J := \begin{bmatrix} \alpha_1 & \sqrt{\beta_1} & 0 & 0 & 0 \\ \sqrt{\beta_1} & \alpha_2 & \sqrt{\beta_2} & 0 & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & 0 & \sqrt{\beta_{N-2}} & \alpha_{N-1} & \sqrt{\beta_{N-1}} \\ 0 & 0 & 0 & \sqrt{\beta_{N-1}} & \alpha_N \end{bmatrix} \in \mathbb{R}^{N \times N},$$

where α_i and β_i are based on moments of ϑ ; with spectral factorization:

$$J = W \Lambda W^T, \quad \Lambda = \text{diag}[\lambda_1^*, \lambda_2^*, \dots, \lambda_N^*], \quad WW^T = I.$$

- Moreover, it is well known [Golub and Welsch \(1969\)](#) that the nodes and weights, $\{x_n, w_n\}_{n=1}^N$, of the Gauss rule are, for $n = 1, \dots, N$, given by $x_n = \lambda_n^*$ and $w_n = (\epsilon_1^T W \epsilon_n)^2$, where ϵ_j is the j th axis vector³.

³The accompanying Python and MATLAB codes can be found

Randomized Affine Models

Theorem (ChF for Randomized Affine Jump Diffusion Processes)

Consider a random variable ϑ , with its PDF, $f_{\vartheta}(x)$, CDF, $F_{\vartheta}(x)$ and a realization θ , $\vartheta(\omega) = \theta$ such that for some $N \in \mathbb{N}$ the moments are finite, $\mathbb{E}[\vartheta^{2N}] < \infty$. Assuming that the corresponding ChF, $\phi_{\mathbf{X}|\vartheta=\theta}(\cdot)$, is well defined and $2N$ times differentiable w.r.t. θ , the unconditional ChF for the randomized $\mathbf{X}$, exists and is given by:

$$\phi_{\mathbf{X}}(u; t, T) = \sum_{n=1}^N \omega_n \phi_{\mathbf{X}|\vartheta=\theta_n}(u; t, T) + \epsilon_N = \sum_{n=1}^N \omega_n e^{A(u; \tau, \theta_n) + \mathbf{B}^T(u; \tau, \theta_n) \mathbf{X}(t)} + \epsilon_N,$$

where

$$\epsilon_N = \frac{1}{(2N)!} \frac{\partial^{2N}}{\partial \xi^{2N}} \phi_{\mathbf{X}|\vartheta=\xi}(u; t, T),$$

for $a < \xi < b$ and where the pairs $\{\omega_n, \theta_n\}_{n=1}^N$ are the Gauss-quadrature weights and the nodes based on the parameter distribution, $f_{\vartheta}(\cdot)$, determined by $\zeta(\vartheta) : \mathbb{R} \rightarrow \{\omega_n, \theta_n\}_{n=1}^N$.

- We report exponential convergence!

Affine Models and Randomization

- The ChF of the randomized AD model is a weighted sum of a set of conditional ChFs evaluated at certain realizations, θ_n , of the underlying random parameter ϑ .
- The theorem shows the **exponential decay of the error** in terms of N -suggesting **high precision** for low N .
- When analytical moments are available, the computation of the corresponding points only requires the computation of a Cholesky decomposition and certain eigenvalues; it is, therefore, **computationally cheap**.
- Variables under closed under linear transformations allow for the tabulation of the corresponding quadrature points!

Table: Selected distributions for the stochastic parameters.

name	raw moment	domain
$\vartheta \sim \mathcal{U}([\hat{a}, \hat{b}])$	$\mathbb{E}[\vartheta^n] = \frac{\hat{b}^{n+1} - \hat{a}^{n+1}}{(n+1)(\hat{b} - \hat{a})}$	$[\hat{a}, \hat{b}]$
$\vartheta \sim \exp(\hat{a})$	$\mathbb{E}[\vartheta^n] = \frac{n!}{\hat{a}^n}$	$\mathbb{R}^+$
$\vartheta \sim \mathcal{N}(0, 1)$	$\mathbb{E}[\vartheta^n] = (n-1)!!$ if n even; 0 otherwise	$\mathbb{R}$
$\vartheta \sim \Gamma(\hat{a}, \hat{b})$	$\mathbb{E}[\vartheta^n] = \hat{b}^n \Gamma(n + \hat{a}) / \Gamma(\hat{a})$	$\mathbb{R}^+$
$\vartheta \sim \chi^2(\hat{a}, \hat{b})$	$\mathbb{E}[\vartheta^n] = 2^{n-1} (n-1)! (\hat{a} + n\hat{b}) + \sum_{j=1}^{n-1} \frac{(n-1)! 2^{j-1}}{(n-j)!} (\hat{a} + j\hat{b}) \mathbb{E}[\vartheta^{n-j}]$	$\mathbb{R}^+ \cup \{0\}$

Pricing with Randomized Models

- The pricing will rely on a Fourier inversion method, namely the COS method ([Oosterlee and Grzelak, 2019](#)).
- The generic pricing equation is given by:

$$V(t_0) = e^{-r(T-t_0)} \sum_{k=0}^{N_c-1} \Re \left[\phi_{\mathbf{x}} \left(\frac{k\pi}{b-a}; t_0, T \right) \exp \left(-ik\pi \frac{a}{b-a} \right) \right] \cdot H_k + \epsilon_{c_1},$$

where H_k for $k \geq 0$ are known in **closed-form coefficients** corresponding to the payoff function.

- We will use the H_k coefficients derived for European-style call/put options and options on VIX.
- Parameters, a and b are the *tuning* parameters used to determine the integration range; while the error ϵ_{c_1} , is exponentially decaying in N_c .

Randomized Black-Scholes Model

- We consider the **randomized Black-Scholes**, with σ random, and follow a uniform distribution, $\sigma \sim \mathcal{U}([\cdot, \cdot])$.
- The randomized Black-Scholes model follows the following SDE:

$$dS(t) = rS(t)dt + \sigma S(t)dW(t), \quad \sigma \sim \mathcal{U}([\hat{a}, \hat{b}]), \quad \hat{a}, \hat{b}, \in \mathbb{R}^+.$$

- The corresponding ChF for $X(t) = \log S(t)$, reads:

$$\phi_X(u; t_0, T) = \sum_{n=1}^N \omega_n \exp \left(iuX(t_0) + \left(r - \frac{1}{2}\sigma_n^2 \right) iu(T - t_0) - \frac{1}{2}\sigma_n^2 u^2 (T - t_0) \right) + \epsilon_N.$$

- In the experiment we take: $\sigma \sim \mathcal{U}([0.1, 0.45])$.
- We note that $\{\omega_i, \theta_i\}_{i=1}^N$ can be computed for $\mathcal{U}([0, 1])$ and then scaled appropriately (the weights ω_i).

Implied Volatility Surface for the Randomized BS Model

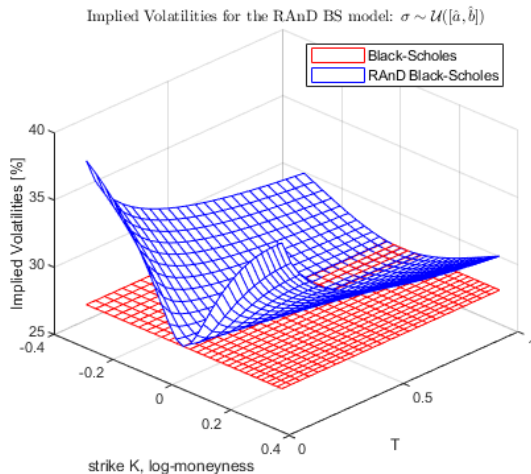


Figure: Right: Implied volatility surface for the RAnD BS model for $\sigma \sim \mathcal{U}([\hat{a}, \hat{b}])$.

- The application of Fourier inversion to the randomized ChF yields,

$$f_{\mathbf{X}}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \sum_{n=1}^N \omega_n \phi_{\mathbf{X}|\vartheta=\theta_n}(u; t, T) du = \sum_{n=1}^N \omega_n f_{\mathbf{X}|\vartheta=\theta_n}(x).$$

Since $\omega_1 + \dots + \omega_N = 1$, $\omega_n \geq 0$, for $n = 1, \dots, N$, which implies the density of the affine, randomized, system of SDEs, $\mathbf{X}(t)$ can be expressed as a, **possibly multi-modal, mixture distribution**.

- Mixture distribution models have been studied in **Brigo and Mercurio (2002)**, where the sum of (log)normal PDFs was analyzed. The model, although very flexible, was limited by a large number of model parameters.

Mixture Distributions

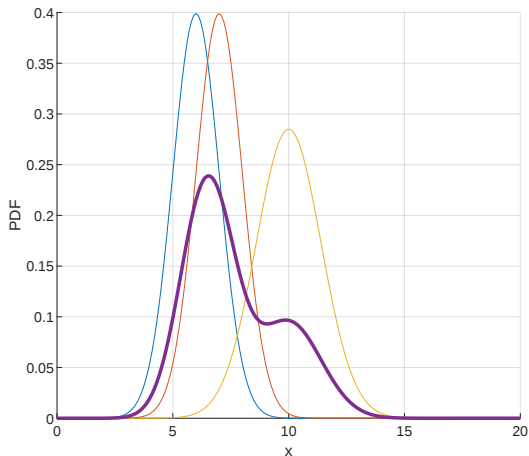


Figure: mixture PDF with three PDFs: $X \sim \mathcal{N}(6, 1)$, $Y \sim \mathcal{N}(7, 1)$, $Z \sim \mathcal{N}(10, 1.4)$.

The RAnd Bates Model

- The Bates model [Bates \(1996\)](#), under the $\mathbb{Q}$ measure, is described by the following system of SDEs:

$$\begin{aligned}dS(t)/S(t) &= \left(r - \lambda \mathbb{E}\left[e^J - 1\right]\right) dt + \sqrt{v(t)} dW_x(t) + (e^J - 1) dX_{\mathcal{P}}(t), \\dv(t) &= \kappa (\bar{v} - v(t)) dt + \gamma \sqrt{v(t)} dW_v(t),\end{aligned}$$

with Poisson process $X_{\mathcal{P}}(t)$, intensity λ , and normally distributed jump sizes, $J \sim \mathcal{N}(\mu_J, \sigma_J^2)$, with $\mathbb{E}[e^J] = e^{\mu_J + \frac{1}{2}\sigma_J^2}$, and $\rho dt = dW_x(t)dW_v(t)$.

- $X_{\mathcal{P}}(t)$ is assumed to be independent of the Brownian motions and the jump sizes.
- Under this model, the variance process follows the non-central chi-square distribution, $\chi^2(\delta, \bar{\kappa}(\cdot, \cdot))$, with δ degrees of freedom and non-centrality parameter $\bar{\kappa}(t_0, t)$,

$$v(t)|v(t_0) \sim \bar{c}(t_0, t)\chi^2(\delta, \bar{\kappa}(t_0, t)),$$

where

$$\bar{c}(t_0, t) = \frac{\gamma^2}{4\kappa}(1 - e^{-\kappa(t-t_0)}), \quad \delta = \frac{4\kappa\bar{v}}{\gamma^2}, \quad \bar{\kappa}(t_0, t) = \frac{4\kappa e^{-\kappa(t-t_0)}v(t_0)}{\gamma^2(1 - e^{-\kappa(t-t_0)})}.$$

Implied Volatility Surface for r-Bates Model

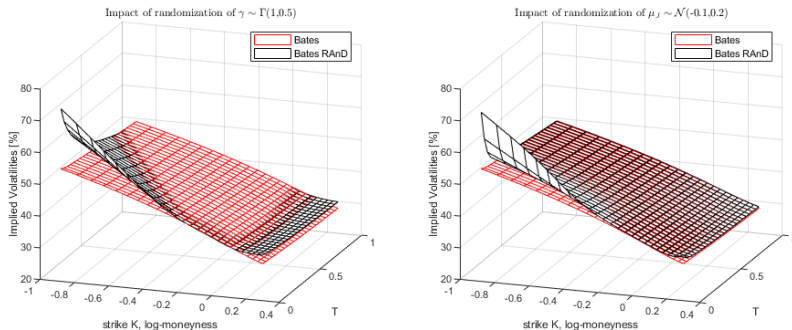


Figure: Implied volatility surface for RAnD Bates model. Left panel: randomized *vol-vol*, $\gamma \sim \Gamma(1, 0.5)$. Right panel: randomized jump's mean, $\mu_J \sim \mathcal{N}(-0.1, 0.2)$. Other model parameters are $r = 0$, $\mu_J = -0.1$, $\sigma_J = 0.06$, $\lambda = 0.08$, $\kappa = 0.5$, $\gamma = 0.5$, $\bar{v} = 0.13$, $\rho = -0.7$, $T = 1/12$, and $v_0 = 0.13$.

Random Parameters and Impact on IVs

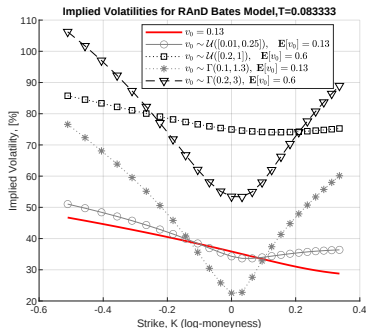
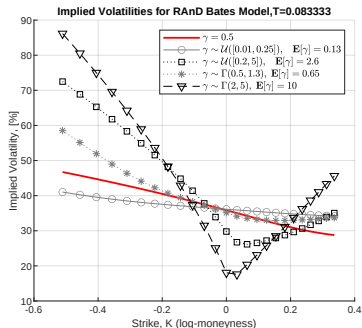


Figure: Impact of randomized parameters on implied volatilities. Left: randomized *vol-vol*, γ . Right: randomized *initial vol*, v_0 .

Random Parameters and Impact on IVs

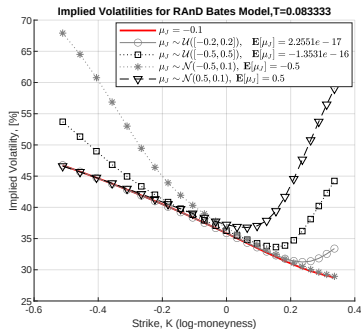
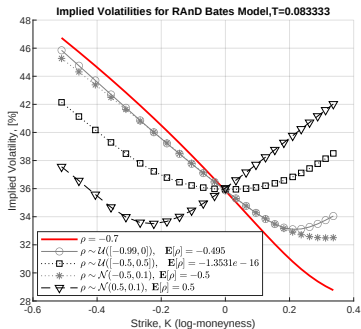


Figure: Impact of randomized parameters on implied volatilities. Left: randomized *correlation*, ρ . Right: randomized *jump's mean*, μ_J .

Pricing of Options on VIX

- For a given fixed time-horizon $[t, T]$, the volatility index of an asset $S(t)$, denoted as $\overline{\text{vix}}(t, T)$, is defined as:

$$\overline{\text{vix}}^2(t, T) = 100^2 \times \frac{-2}{T-t} \mathbb{E}_t \left[\log \frac{S(T)}{S(t)} \right],$$

where $\mathbb{E}_t[\cdot]$ indicates the expectation taken under the risk-neutral measure $\mathbb{Q}$ and the natural filtration $\mathcal{F}(t)$.

- Under the Bates model, the VIX is expressed by:

$$\begin{aligned}\overline{\text{vix}}^2(t, T) &= 100^2 \times \text{vix}^2(t, T), \\ \text{vix}^2(t, T) &= a(t, T)v(t) + b(t, T) + c,\end{aligned}$$

with deterministic functions $a(t, T)$, $b(t, T)$ and c .

- A call option on VIX is then given as:

$$\begin{aligned}V_{\text{vix}}(t) &= e^{-r(T-t)} \mathbb{E}_t \left[\max(\overline{\text{vix}}(T, T + \delta T) - K, 0) \right] \\ &= 100 \times e^{-r(T-t)} \int_{\mathbb{R}^+} \max(\sqrt{v} - K, 0) f_{\text{vix}^2}(v; T, T + \delta T) dv.\end{aligned}$$

- At this point we need to derive coefficients H_k for the COS method.

Pricing of Options on VIX

- Since we can utilize the analytically known distribution for $\overline{\text{vix}}$, the pricing may also be performed by directly integrating the payoff function and employing the PDF:

$$V_{\text{vix}}(t) = 100 \times 2\alpha_1 e^{-r(T-t)} \int_{\overline{K}} x(x - \overline{K}) f_{\chi^2(\delta, \bar{\kappa}(t, T))} \left(\alpha_1(x^2 - \alpha_2) \right) dx.$$

- With one of the model parameters stochastic, the RAnD pricing equation reads:

$$V_{\text{vix}}(t) = 100 \times 2\alpha_1 e^{-r(T-t)} \sum_{n=1}^N \omega_n \int_{\overline{K}} x(x - \overline{K}) f_{\chi^2(\delta, \bar{\kappa}(t, T))} \left(\alpha_1(x^2 - \alpha_2); \theta_n \right) dx,$$

where θ_n in $f_{\chi^2(\cdot, \cdot)}(\cdot; \theta_n)$ indicates a particular realization of the model parameter and ω_n corresponds to its weight.

- Once the pricing equations are known, we can perform the model calibration.

Calibration to Options on SPX and VIX

Table: Parameters determined in calibration of S&P and VIX

Calibrated RAnD Bates parameters								
date	κ	v_0	$\bar{v}$	ρ	μ_J	σ_J	λ	γ
02/02/2022	0.5	0.170^2	0.23	-0.65	-0.25	0.05	0.25	$\gamma \sim \mathcal{U}([0.01, 2.3])$
13/05/2022	0.14	0.267^2	0.28	-0.8	-0.25	0.02	0.1	$\gamma \sim \mathcal{U}([0.002, 2.1])$
14/07/2022	0.5	0.250^2	0.10	-0.85	-0.25	0.05	0.15	$\gamma \sim \mathcal{U}([0.05, 1.4])$

Calibration to Options on SPX and VIX

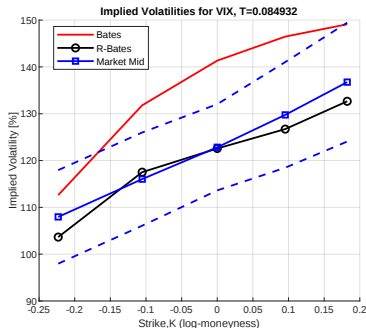
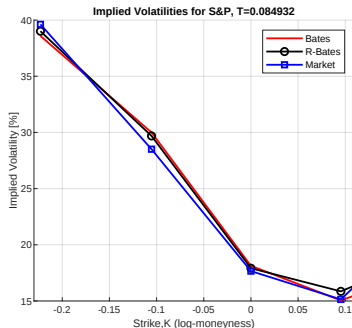


Figure: Calibration results of the RANd Bates model. The implied volatilities for S&P and ViX were obtained on 02/02/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Calibration to Options on SPX and VIX

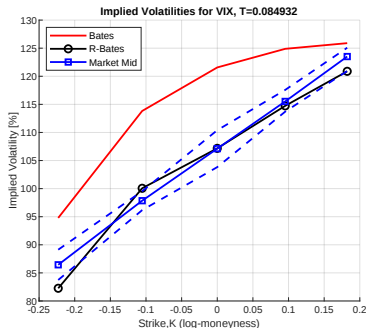
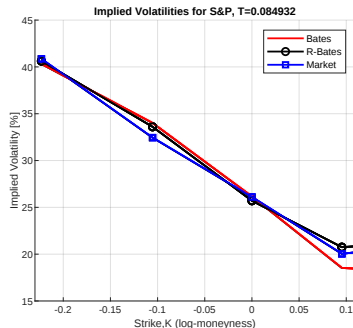


Figure: Calibration results of the RANd Bates model. The implied volatilities for S&P and ViX were obtained on 13/05/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Calibration to Options on SPX and VIX

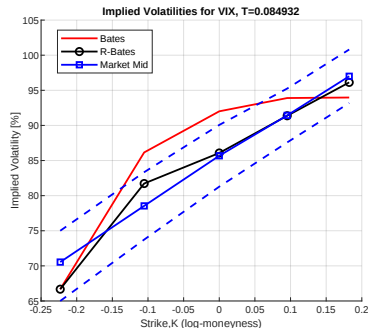
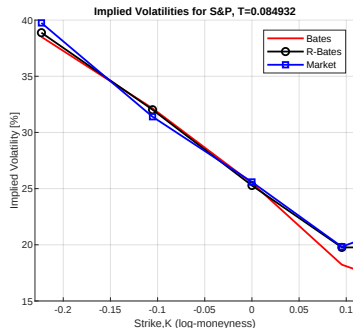


Figure: Calibration results of the RAN Bates model. The implied volatilities for S&P and ViX were obtained on 14/07/2022. Dotted lines indicate bid-ask spreads. Left: S&P, Right: VIX.

Pricing Equations for Randomized Short-Rates

- The RAnD method can also be applied in the world of interest rates, in Grzelak (2022b), *"Randomization of Short-Rate Models, Analytic Pricing and Flexibility in Controlling Implied Volatilities"*.
- The randomization does not need to occur at the ChF level, but it can be applied to any conditional expectation.
- Consider a random variable ϑ , defined on some finite domain $D_\vartheta := [a, b]$, with its PDF, $f_\vartheta(x)$, CDF, $F_\vartheta(x)$ and a realization θ , $\vartheta(\omega) = \theta$ such that for some $N \in \mathbb{N}$ the moments are finite, $\mathbb{E}[\vartheta^{2N}] < \infty$.

$$V(t, r(t; \vartheta)) = \sum_{n=1}^N \omega_n V(t, r(t; \theta_n)) + \epsilon_N,$$

where the error ϵ_N is defined as:

$$\epsilon_N = \frac{1}{(2N)!} \frac{\partial^{2N}}{\partial \xi^{2N}} V(t; r(t, \vartheta = \xi)), \quad a < \xi < b.$$

PDF of Randomized Models

- Under the HJM framework and the arbitrage-free condition for the drift in, the Hull-White model is specified by:

$$\gamma(t, T) = \eta \cdot e^{-\lambda(T-t)}, \quad t < T,$$

- We consider three different randomization cases: the randomization of the volatility parameter, η , the mean-reversion, λ , or the randomization of both parameters using bivariate distribution:

$$\eta \stackrel{d}{=} \vartheta_1, \quad \text{or} \quad \lambda \stackrel{d}{=} \vartheta_2, \quad \text{or} \quad \lambda|\eta \stackrel{d}{=} \vartheta_2|\vartheta_1.$$

- For constant realizations of the randomized parameter, the SDE reads:

$$dr(t) = \lambda(\psi(t) - r(t))dt + \eta dW(t), \quad r_0 \equiv f(0, 0),$$

with

$$\psi(t) = f(0, t) + \frac{1}{\lambda}f(0, t) + \frac{\eta^2}{2\lambda^2} \left(1 - e^{-2\lambda t}\right), \quad f(0, t) = -\frac{\partial \log P(0, t)}{\partial t},$$

- As before, we can show that the PDF of the randomized HW model will be a convex sum of constituent PDFs:

$$f_{r(T)}(x) = \sum_{n=1}^N \omega_n f_{r(T; \theta_n)}(x) + \epsilon_N^F.$$

Dynamics of the Randomized HW model

- We consider a sequence of HW model processes, $\bar{r}_1(t), \dots, \bar{r}_N(t)$, corresponding to parameter realizations, and the probability density relation,

$$f_{r(T)}(x) = \sum_{n=1}^N \omega_n f_{r(T; \theta_n)}(x).$$

- We want to determine the corresponding SDE for the rHW process, $\bar{r}(t)$. Formally, we seek an SDE, with the solution and where each of the constituent processes, $\bar{r}_n(t)$, is driven by the HW model.
- Thus, we consider the following process,

$$d\bar{r}(t) = \bar{\lambda}(t, \bar{r}(t))dt + \bar{\eta}(t, \bar{r}(t))dW(t), \quad \bar{r}(t_0) = f(0, 0),$$

with some state-dependent drift, $\bar{\lambda}(t, \bar{r}(t))$, and volatility, $\bar{\eta}(t, \bar{r}(t))$, and where Brownian motion $W(t)$ is common for all the underlying HW processes.

Dynamics of the Randomized HW model

Proposition (Local volatility process for the HW model with randomized volatility parameter, η)

Let us assume a sequence of positive constants η_n , $n = 1, \dots, N$. Then, the SDE

$$d\bar{r}(t) = \bar{\lambda}(t, \bar{r}(t))dt + \bar{\eta}(t, \bar{r}(t))dW(t), \quad \bar{r}(t_0) = f(0, 0),$$

with

$$\bar{\lambda}(t, y) = \sum_{n=1}^N \bar{\Lambda}_n(t, y) \lambda(\bar{\psi}_n(t) - y), \quad \bar{\eta}^2(t, y) = \sum_{n=1}^N \eta_n^2 \bar{\Lambda}_n(t, y),$$

where:

$$\bar{\Lambda}_n(t, y) = \frac{\omega_n f_{\bar{r}(t; \eta_n)}(y)}{\sum_{n=1}^N \omega_n f_{\bar{r}(t; \eta_n)}(y)}, \quad f_{\bar{r}(t)}(y) = \sum_{n=1}^N \omega_n f_{\bar{r}(t; \eta_n)}(y),$$

where $\sum_{n=1}^N \omega_n = 1$ for $\omega_n \geq 0$, $n = 1, \dots, N$ with $f_{\bar{r}(t; \eta_n)}(x)$ the PDF of the HW model with dynamics, given by:

$$d\bar{r}_n(t) = \lambda(\bar{\psi}_n(t) - \bar{r}_n(t))dt + \eta_n dW(t), \quad \bar{r}_n(t_0) = f(0, 0),$$

where $\bar{r}_n(t) := \bar{r}_n(t; \eta_n)$ with $\bar{\psi}_n(t) = f(0, t) + \frac{1}{\lambda} f(0, t) + \frac{\eta_n^2}{2\lambda^2} (1 - e^{-2\lambda t})$.

Pricing of Swaptions under the rHW Model

Lemma (Pricing of Swaptions under randomized Hull-White model)

Consider the rHW model, with parameters $\{\lambda, \eta\}$ and the randomizing random variable ϑ , which randomizes either of the model parameters. For a unit notional, a constant strike, K , option expiry $T = T_{i-1}$ and a strip of swap payments $\mathcal{T} = \{T_i, \dots, T_m\}$, with $T_i > T_{i-1}$ and accruals $\tau_i = T_i - T_{i-1}$, the prices of swaption payer and receiver, $P/R := \text{Payer/Receiver}$, are given by:

$$V_{P/R}^{\text{Swpt}}(t_0, T, \mathcal{T}, K; \vartheta) = \sum_{n=1}^N \omega_n \sum_{k=i}^m c_k V_{\chi}^Z(t_0, T, T_k, \hat{K}_k(\theta_n); \theta_n),$$

with a swaption payer, P , for $\chi = -1$, swaption receiver, R , with $\chi = 1$, where $V_{\chi}^Z(\cdot)$ is the option on the ZCB and where the strike price

$\hat{K}_k(\theta_n) = \exp(A(T, T_k; \theta_n) + B(T, T_k; \theta_n)r_n^*)$. Here, r_n^* is determined by solving, for each parameter realization θ_n , the following equation:

$$1 - \sum_{k=i}^m c_k \exp\left(A(T, T_k; \theta_n) - B(T, T_k; \theta_n)r_n^*\right) = 0, \quad n = 1, \dots, N,$$

where $A(T, T_k; \theta_n)$ and $B(T, T_k; \theta_n)$ are known in the closed-form.

Hull-White vs. Randomized Hull-White Models

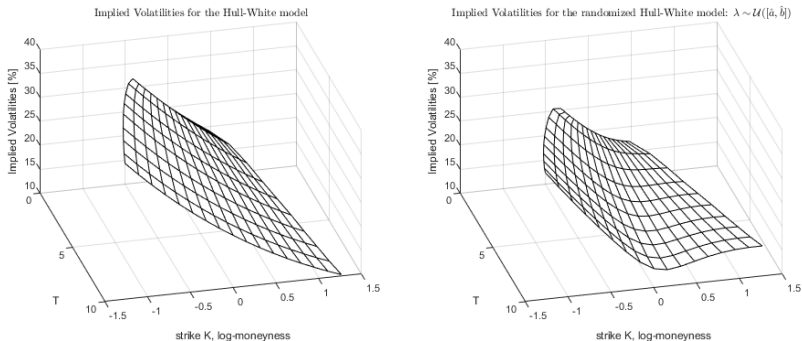


Figure: Swaption volatility evolution for the HW and rHW models implied by the shifted Black's model. The simulation was performed for varying swaption option expiry, T , and a fixed tenor of 1y. The parameters specified in the experiment are: for the HW model: $\eta = 0.005$, $\lambda = 0.001$ and for the rHW model: $\eta = 0.005$ and $\lambda \sim \mathcal{U}([-0.15, 0.6])$. In the experiment, the implied volatilities are computed with zero shift parameter, $s = 0$.

Swaptions: Calibration Quality

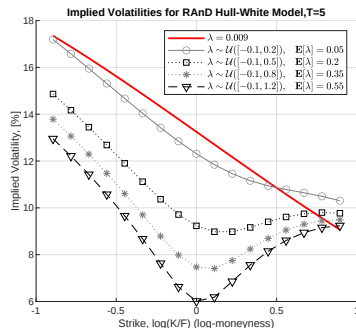
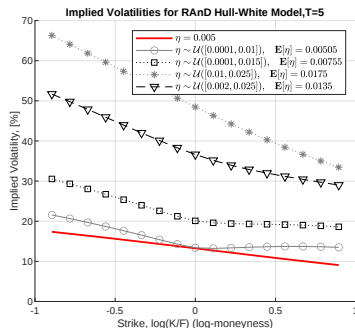


Figure: LHW: randomized volatility parameter η ; RHS: randomized mean-reversion parameter λ

Swaptions: Calibration Quality

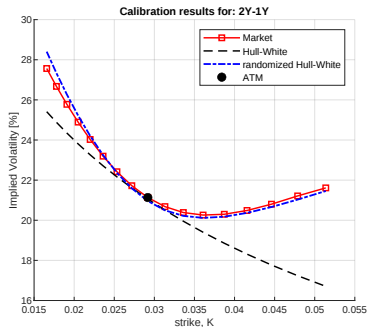
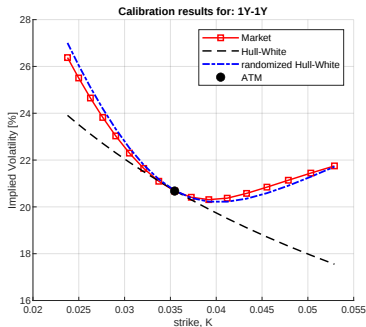


Figure: Calibration results of the HW and the rHW models. The market implied volatilities for swaptions were obtained on 18/08/2022 for the USD market. Option expiry: $T = 1y$ and $T = 2y$ and the implied volatility shift: $s = 1\%$. Calibrated parameters are presented in Table 3.

Swaptions: Calibration Quality

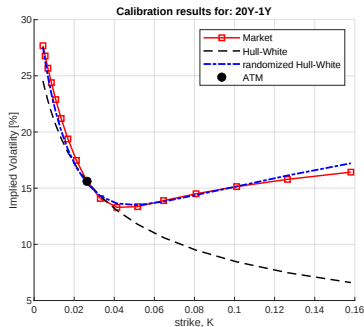
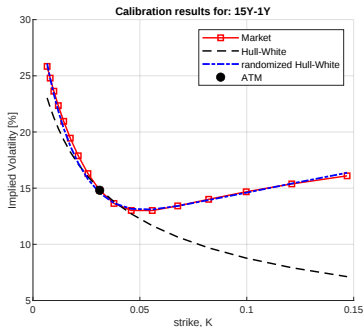


Figure: Calibration results of the HW and the rHW models. The market implied volatilities for swaptions were obtained on 18/08/2022 for the USD market. Option expiry: $T = 15y$ and $T = 20y$ and the implied volatility shift: $s = 1\%$. Calibrated parameters are presented in Table 3.

Swaptions: Calibrated Parameters

Table: Calibration of the HW and rHW model: parameters determined in swaption calibration.

T , expiry	Hull-White		RAnD Hull-White	
	η	λ	η	λ
1y	0.0094	0.0090	0.0091	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.45^2)$
2y	0.0082	0.0035	0.0080	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.33^2)$
5y	0.0069	0.0020	0.0079	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.16^2)$
8y	0.0067	0.0095	0.0080	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.12^2)$
10y	0.0067	0.0090	0.0082	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.11^2)$
15y	0.0064	0.0080	0.0085	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.09^2)$
20y	0.0060	0.0080	0.0086	$\lambda \sim \mathcal{N}(\mathbf{0.1}, 0.08^2)$

- Note that the mean for λ has been fixed! Therefore the number of degrees of freedom is equal to the case for the standard Hull-White model.

PDF of Randomized Models

- We have introduced the RAnD method for efficient computation of the affine models with random parameters.
- The proposed framework is generic and can be applied to any stochastic model, even outside the class of affine diffusions.
- As long as the randomizing random variable gives rise to finite, preferably closed-form, moments, one can price European-style options efficiently.
- The heart of the method is formed by a few *critical* collocation points to recover the characteristic function.
- Fast computation of the characteristic function is possible because the method converges exponentially in the number of expansion terms.
- We have shown that the randomization of stochastic models provides a breeze of fresh air to the class of affine models.
- The application of the RAnD method to the Bates model shows that randomization allows for simultaneous calibration to S&P and VIX options—a heavily desired feature of modern models.
- Finally, we have illustrated that the model randomized Hull-White model allows for almost perfect calibration to swaption implied volatilities, while the model stays analytic and computationally efficient.

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