Pricing an Inflation-Linked Annuity Considering Interest Rate and Longevity Risk in the HJM Framework

Fen-Ying Chen, Shih Hsin University, Taiwan

Hong-Chih Huang, National Chengchi University , Taiwan

Sharon S. Yang, National Central University , Taiwan

Motivation and Contribution 1

1. For a long-term policy holder, increase in price levels will force him to withdraw more money from his investment portfolio to meet the rising expenses, leaving he with lesser money for later years. This may subject the holder to longevity risk — the risk that we may outlive our investments.
2. Investors would buy inflation-linked bonds to moderate inflation risk. However, the bonds have longevity risk such as TIPS. Therefore, this paper deals with the securitization of longevity risk and inflation risk through the inflation-linked longevity bonds.

Motivation and Contribution 2

3. Unlike the standard inflation-indexed derivatives that are based on a single inflation-linked underlying such as inflation bonds or forwards (Jarrow and Yildirim, 2003; Mercurio, 2005; Hinnerich, 2008; Kruse, 2011), the inflation-linked longevity bonds with underlyings from more than one risk are typically challenging to price.

4. The main contribution of this paper is to show that, under a no arbitrage Heath–Jarrow–Morton (HJM) type of framework where interest rate term structures are assumed to be Gaussian, closed form pricing results can be obtained for these exotic structures.

Literature Review

1. Blake and Burrows (2001) contend that ongoing improvement in life expectancy is a major concern for annuity providers; they recommend the use of survivor bonds as a hedging tool against longevity risk.
2. Lin and Cox (2005) present a model for the securitization of mortality risk in life annuity.
3. Blake et al. (2006) examine a few main characteristics of longevity bonds and the valuation issues in incomplete markets.
4. Tiong (2013) prices inflation-linked variable annuities under the HJM model.

Model Formulation 1

Consider an inflation-linked longevity bond. The payments of the bond at time t are linked to inflation rate at time t as follows:

$$\text{payment}_t = £50\text{m} \times S(t, x_0) \times P_t$$

which P_t represents CPI inflation rate at time t; $S(t, x_0)$ denotes the survival probability at time t aged x_0 at time 0.

Figure 2 clearly exhibits the payment structure of the bond.

Model Formulation 2

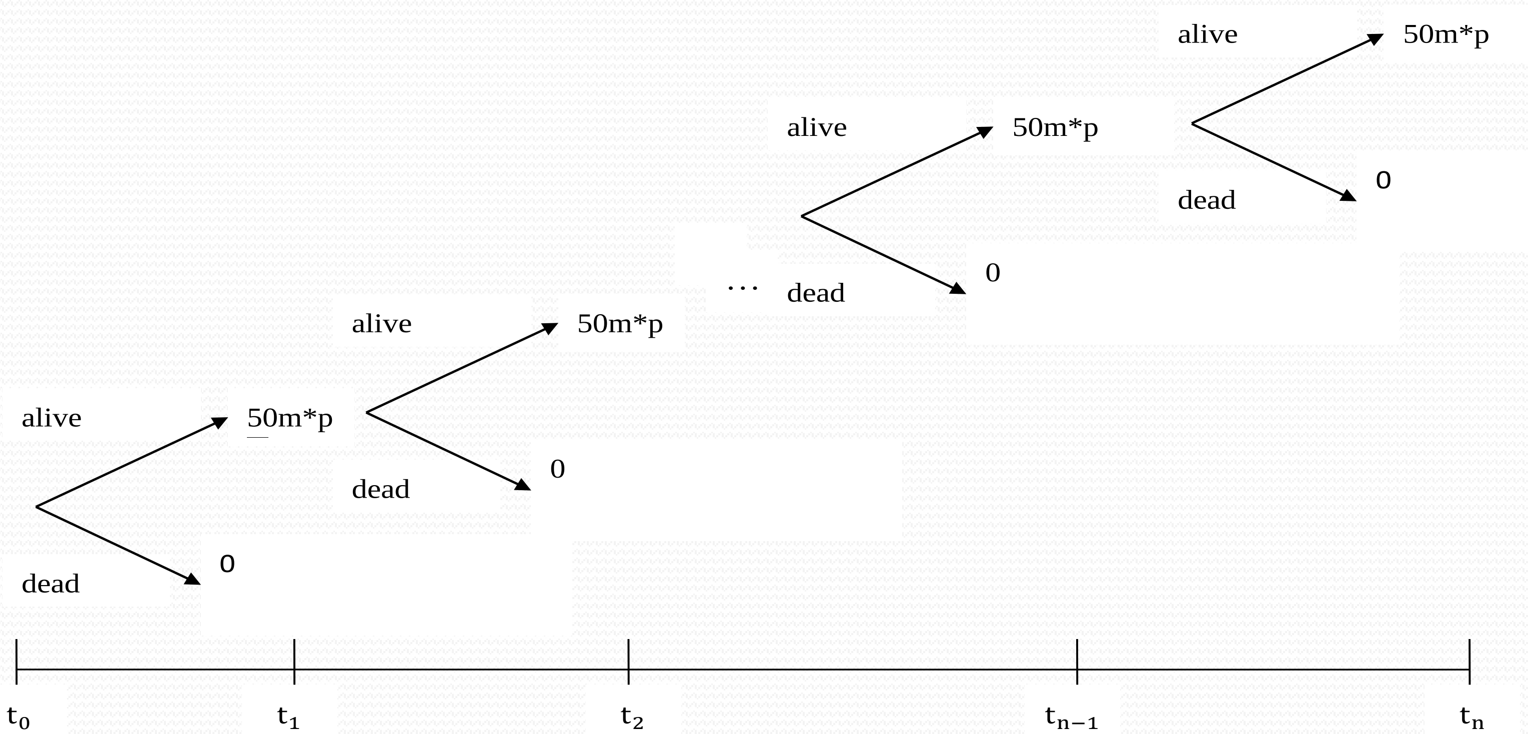

Figure 2 Payoffs of Inflation-Linked Longevity Bond

Model Formulation 3

Thus, we use equation (3) to describe the cash flow of the bond at each period.

$$LB^{CF}_{t_{j-1},\, t_j}(t_j) = \begin{bmatrix} £50m * P_{t_j}, & \text{if} \quad \text{the holder is alive} \\ 0, & \text{if} \quad \text{the holder is dead.} \end{bmatrix}, \qquad (3)$$

for all $j=1, 2, ..., n$, $t_0 = 0$, and $t_n = T$.

Pricing Inflation-Linked Longevity Bond 1

Let the dynamic processes of forward interest rate, inflation index, survivor probability and zero coupon bond under original probability measure be as follows:

$$df(t,T)=\mu(t,T)dt+\sigma(t,T)dW_t\ , \quad (4)$$

$$\frac{dP(t)}{P(t)}=\mu_P dt+\sigma_P dW_t, \quad (5)$$

$$\frac{dS(t,\ x_0)}{S(t,\ x_0)}=\mu_s dt+\sigma_s dW_{s,t}, \quad (6)$$

$$\frac{dB(t,T)}{B(t,T)}=r_t dt+b(t,T)dW_t, \quad (7)$$

$f(t,T)$: forward interest rate at time t with a maturity of T;

$\mu(t,T)$: an instantaneous drift term of forward interest rate;

$\sigma(t,T)$: volatility of;

$P(t)$: inflation index at time t;

μ_P : instantaneous drift term of inflation index;

σ_P : volatility of inflation index;

μ_s : instantaneous drift term of the survivor probability;

σ_s : volatility of the survivor probability;

Pricing Inflation-Linked Longevity Bond 2

$W_{s,t}, W_t$: one-dimensional Brownian motion at time t under the original probability measure, P

$\rho_{1,2} = corr(dW_t, dW_{s,t})$.

$b(t,T) = -\int_t^T \sigma(u,T)du$: the volatility of zero coupon bond.

Pricing Inflation-Linked Longevity Bond 3

Proposition 1: Given the dynamic processes as shown in equations (5), (6) and (7), the dynamic processes of $P(t)\times S(t, x_0)$, and $\frac{P(t)\times S(t, x_0)}{B(t,T)}$ can be obtained under spot original probability measure, Q, respectively. Let $G(t)=P(t)\times S(t, x_0)$, and $X(t)=\frac{P(t)\times S(t, x_0)}{B(t,T)}$.

Therefore,

$$\frac{dG(t)}{G(t)}=r(t)dt+\sigma_G(t)dW_{G,t}^{Q}, \qquad (8)$$

$$\frac{dX(t)}{X(t)}=\sigma_X dW_{X,t}^{PT}, \qquad (9)$$

with $\sigma_G(t)=\sqrt{\sigma_s^2+\sigma_P^2+2\rho_{1,2}\sigma_s\sigma_P}$,

$\sigma_X(t)=\sqrt{\sigma_G^2+b_t^2-2\rho_{1,2}\sigma_G b_t}$,

and $\rho_{1,2}=corr(dW_t^Q\ ,dW_{s,t}^Q)$.

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Thus, the time- t_{n-1} and time- t_{n-2} value of the inflation-linked longevity bond under a forward risk neutral probability, PT, can be obtained in equation (13).

$$LB_{t_{n-1},\,t_n}(t_{n-1}) = 50m \times B(t_{n-1},\, t_n) \times X(t_{n-1}),$$

$$LB_{t_{n-2},\,t_{n-1}}(t_{n-2}) = 50m \times B(t_{n-2},\, t_{n-1}) \times \left[X(t_{n-2}) + E^{PT}_{t_{n-2}}\left[G(t_{n-1})F_S(t_{n-1}, t_{n-1},\, x_0)\right]\right], \tag{13}$$

with $F_S(t_{n-1}, t_{n-1},\, x_0) = \dfrac{S(t_{n-1},\, x_0)}{B(t_{n-1},\, t_{n-1})}$.

$$LB_{t_{n-3},\,t_{n-2}}(t_{n-3}) = 50m \times B(t_{n-3},\, t_{n-2}) \times \begin{bmatrix} X(t_{n-3}) + E^{PT}_{t_{n-3}}\left[G(t_{n-2})F_S(t_{n-2}, t_{n-2},\, x_0)\right] + \\ E^{PT}_{t_{n-3}}\left[B(t_{n-2},\, t_{n-1})G(t_{n-1})F_S(t_{n-2}, t_{n-2},\, x_0)F_S(t_{n-1}, t_{n-1},\, x_0)\right] \end{bmatrix},$$

$$LB_{t_{n-4},\,t_{n-3}}(t_{n-4}) = 50m \times B(t_{n-4},\, t_{n-3}) \times \begin{bmatrix} X(t_{n-4}) + E^{PT}_{t_{n-4}}\left[G(t_{n-3})F_S(t_{n-3}, t_{n-3},\, x_0)\right] \\ + E^{PT}_{t_{n-4}}\begin{bmatrix} B(t_{n-3},\, t_{n-2})G(t_{n-2}) \\ F_S(t_{n-3}, t_{n-3},\, x_0)F_S(t_{n-2}, t_{n-2},\, x_0) \end{bmatrix} \\ + E^{PT}_{t_{n-4}}\begin{bmatrix} B(t_{n-3},\, t_{n-2})B(t_{n-2},\, t_{n-1})G(t_{n-1}) \\ F_S(t_{n-3}, t_{n-3},\, x_0)F_S(t_{n-2}, t_{n-2},\, x_0)F_S(t_{n-1}, t_{n-1},\, x_0) \end{bmatrix} \end{bmatrix}$$

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Thus, the time-0 value of the bond with maturity 25 is below

$$LB_{t_0,t_1}(t_0) = 50m \times B(t_0, t_1)$$

$$\times \begin{bmatrix} X(t_0) + E_0^{PT}\left[G(t_1)F_S(t_1,t_1, x_0)\right] \\ +E_{t_0}^{PT}\begin{bmatrix} B(t_1, t_2)G(t_2) \\ F_S(t_1,t_1, x_0)F_S(t_2,t_2, x_0) \end{bmatrix} \\ +E_{t_0}^{PT}\begin{bmatrix} B(t_1, t_2)B(t_2, t_3)G(t_3) \\ F_S(t_1,t_1, x_0)F_S(t_2,t_2, x_0)F_S(t_3,t_3, x_0) \end{bmatrix} \\ +\cdots+E_{t_0}^{PT}\begin{bmatrix} B(t_1, t_2)B(t_2, t_3)\ldots B(t_{23}, t_{24})G(t_{24}) \\ F_S(t_1,t_1, x_0)F_S(t_2,t_2, x_0)F_S(t_3,t_3, x_0)\ldots F_S(t_{24},t_{24}, x_0) \end{bmatrix} \end{bmatrix},$$

which

$$E_0^{PT}\left[G(t_1)F_S(t_1,t_1, x_0)\right] = X_G(t_0,t_1)F_S(t_0,t_1, x_0)e^{\rho_{1,2}\sigma_X\sigma_t(t_1-t_0)},$$

$$E_{t_0}^{PT}\begin{bmatrix} B(t_1, t_2)G(t_2) \\ F_S(t_1,t_1, x_0)F_S(t_2,t_2, x_0) \end{bmatrix} = B(t_0, t_1, t_2)X_G(t_0,t_2)$$

$$\times F_S(t_0,t_1, x_0)F_S(t_0,t_2, x_0)$$

$$\times e^{\rho_{1,2}\left[-b(t,T_2)+b(t,T_1)\right]\sigma_t(t_2-t_0)}$$