Analysis of Optimal Hedging Strategies for Dealing Longevity Risk and Catastrophic Mortality Risk

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Outline

- Introduction
- Hedging Framework
- Modeling Mortality Dynamics for Different Product Lines
- Derivation of the Optimal Solution
- Example
- Conclusion

Introduction

- The uncertainty of mortality risk has become a new risk to be managed in the 21 Century.
- The insurer may involve two types of risk: longevity risk or mortality risk.
- Hedging strategy can be categorized as an internal or external method.
- Most of existing literature deal with either internal method or external method.

Introduction

- Internal method (Natural Hedging)
  - Insurer can hedge longevity risks with their own business products between life insurance and annuity because these two types of products are sensitive in opposing ways to the changes in mortality rates.
  - Natural Hedging Strategy & Immunization model (Wang et al 2010; Huang et al. 2013)

Introduction

- External method-Using capital market solution
  - Longevity bond & Q-forward contract & Kortis note
  - Securitization of longevity risk (Lin and Cox 2005; Cox et al. 2006; Dowd et al. 2006; Blake et al. 2006; Denuit et al. 2007; Biffis and Blake, 2009; Blake et al. 2010; Dawson et al., 2010)
  - Mortality Bonds
    - Swiss Reinsurance Company, the world's second-largest reinsurance company, first issued a three-year catastrophic mortality bond in 2003 (Vita Capital I).
    - The second bond (Vita Capital II) was issued in 2008

Introduction

- Restriction of internal hedging
  - Need to adjust sales volume of life and annuity product
    - Cox and Lin (2007) suggest that natural hedging is good but may be too expensive to be effective in the context of internal life insurance and annuity products.
  - The annuity providers usually has more annuity policies than life insurance policies in their liability-->suffering more longevity risk
  - The insurer may sell more life insurance policies i.e. in Taiwan →suffering more mortality risk

→They can't hedge their risk fully using natural hedge and has to deal the remaining risk using external hedging instruments.

Purpose of this research

- We attempt to deal with the type of insurer with more insurance policies.
- Need a hedging strategy that can combine both natural hedging and external hedging.
- To offset the remaining catastrophic mortality risk, we need to find a proper hedging instrument.
- We use life settlement products as an example.
- Using the mortality experience for different product lines that can avoid basis risk.

Hedging Framework

- The profit function-Liability Side
  - Annuity(Female)

$$\pi^{fa}(x, r_t, m^{fa}_{x,t}) = V^{fa}_{reference}(x, r_t, m^{fa}_{x,t}) - V^{fa}_{actual}(x, r_t, m^{fa}_{x,t})$$

  - Annuity(Male)

$$\pi^{ma}(x, r_t, m^{ma}_{x,t}) = V^{ma}_{reference}(x, r_t, m^{ma}_{x,t}) - V^{ma}_{actual}(x, r_t, m^{ma}_{x,t})$$

  - Life(Female)

$$\pi^{fl}(x, r_t, m^{fl}_{x,t}) = V^{fl}_{reference}(x, r_t, m^{fl}_{x,t}) - V^{fl}_{actual}(x, r_t, m^{fl}_{x,t})$$

  - Life(Male)

$$\pi^{ml}(x, r_t, m^{ml}_{x,t}) = V^{ml}_{reference}(x, r_t, m^{ml}_{x,t}) - V^{ml}_{actual}(x, r_t, m^{ml}_{x,t})$$

+→Loss

-→Profit

Hedging Framework

- The profit function-Asset Side
  - Life Settlement

$$\pi^S(x, r_t, m_{x,t}) = \overline{V}^S(x, r_t) - V^S(x, r_t, m^S_{x,t})$$

  - Bonds

$$\pi^P(r_t, T) = P(r_t, T) - \frac{1}{\prod_{i=1}^{T}(1 + r_i)})$$

+→Profit

- →Loss

Hedging Framework

- Total profit=profit(Asset)-profit(liability)

$$\pi(t) = \sum_{i=1}^{n_B} N_i \pi^B(r_t, T_i) + \sum_i^{n_S} M_i \pi^S(x_i, r_t, m_{m,t}^S) - \sum_i c_i^{fl} \pi^{fl}(x_i, r_t, m_{x,t}^{fl}) - \sum_i c_i^{ml} \pi^{ml}(x_i, r_t, m_{x,t}^{ml}) - \sum_i c_i^{fa} \pi^{fa}(x_i, r_t, m_{x,t}^{fa}) - \sum_i c_i^{ma} \pi^{ma}(x_i, r_t, m_{x,t}^{ma})$$

- Mean Variance Optimization

$$\max_{N_1,\dots,N_{n_B},M_1,\dots,M_{n_S}} E[\pi(t)] - \theta Var[\pi(t)]$$

$$N_i, M_j \geq 0 \ \forall i, j$$

Hedging Framework

- Price of life settlement product with unit benefit

$$\bar{V}^{S}(x, r_t, m_{x,t}) = \prod_{i=1}^{ET} \frac{1}{1 + r_i}$$

- We follow Brockett(1991) to construct the life time distribution for life settlement products.

According to standard life table the probability mass function of K(x) is

$$(g_0, g_1, \dots, g_{\omega-x})$$

where $g_i = \Pr\,(K(x) = i)$

We want to find adjusted mortality table with curtate life time of (x) as

$$(f_0, f_1, \dots, f_{\omega-x})$$

$$\text{s.t} \sum_k f_k = 1 \text{ and } \sum_k k f_k = m$$

where $f_i = \Pr\,(K(x) = i)$ under adjusted mortality table and m is the expectation of lifetime based on newly obtained information.

- Lagrange multiplier method

optimization problem

$$\min_{\mathrm{f_k}} \sum_k f_k \ln\left(\frac{f_k}{g_k}\right)$$

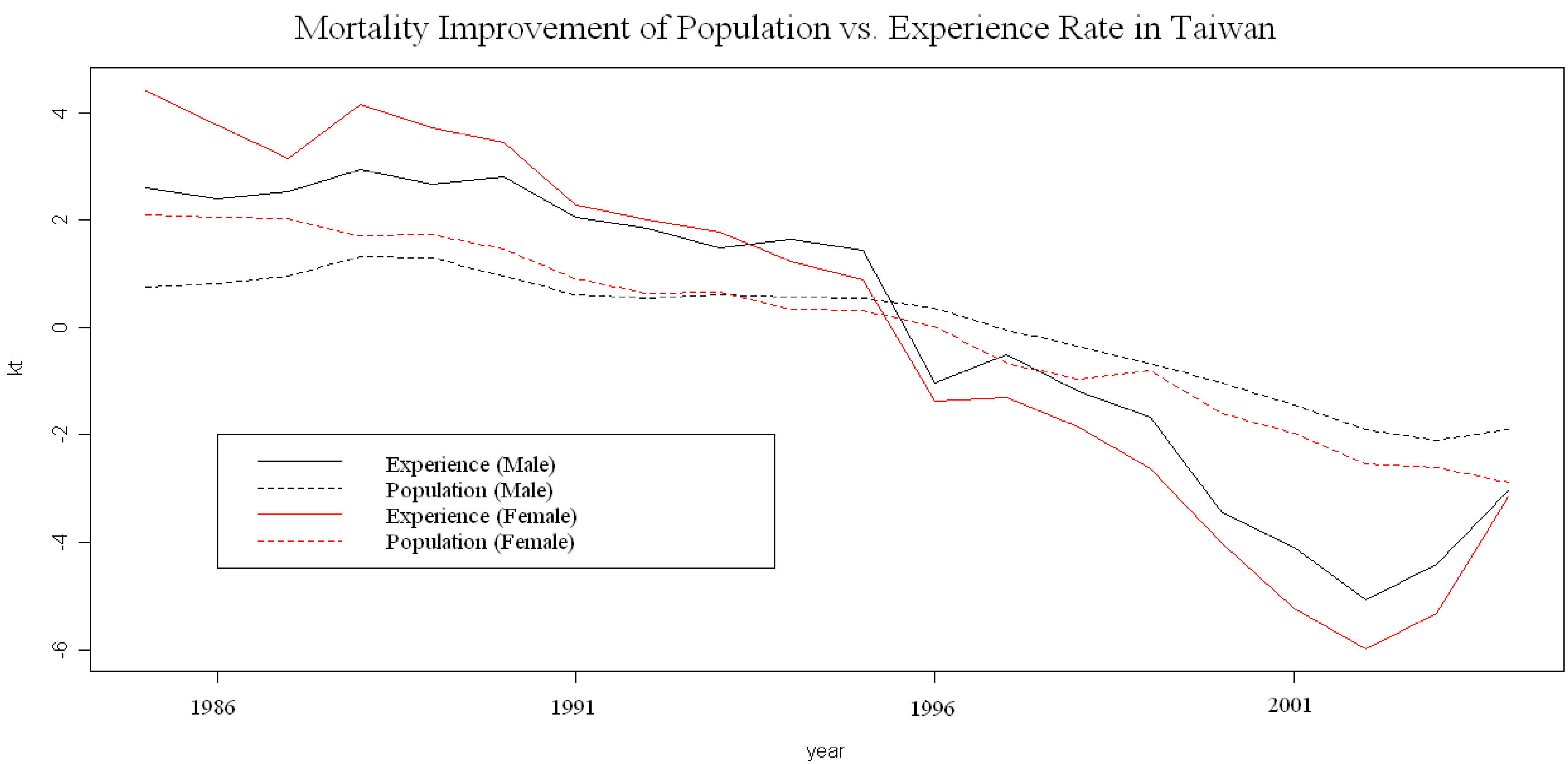
subject to

$$\sum_{\mathrm{k}} \mathrm{f_k} = 1$$

and

$$\sum_{\mathrm{k}} \mathrm{kf_k} = \mathrm{m}$$

Modeling Mortality Dynamics for Different Product Lines



Modeling Mortality Dynamics for Different Product Lines

- Following Yang and Wang (2013)'s VECM multi-population mortality framework.
- The future mortality rates for a person of age *x* at time *t* in matrix form

$$\begin{bmatrix} \ln m_{x,t}^{1} \\ \ln m_{x,t}^{2} \\ \vdots \\ \ln m_{x,t}^{N} \end{bmatrix} = \begin{bmatrix} a_x^{1} \\ a_x^{2} \\ \vdots \\ a_x^{N} \end{bmatrix} + \begin{bmatrix} b_x^{1} & 0 & \cdots & 0 \\ 0 & b_x^{2} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & b_x^{N} \end{bmatrix} \begin{bmatrix} k_t^{1} \\ k_t^{2} \\ \vdots \\ k_t^{N} \end{bmatrix} + \begin{bmatrix} e_{x,t}^{1} \\ e_{x,t}^{2} \\ \vdots \\ e_{x,t}^{N} \end{bmatrix}$$

$$\begin{pmatrix} \Delta k_t^{(1)} \\ \Delta k_t^{(2)} \\ \Delta k_t^{(3)} \\ \Delta k_t^{(4)} \end{pmatrix} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{pmatrix} + \Sigma^{\frac{1}{2}} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

Modeling Mortality Dynamics for Different Product Lines

- The data covers more than 50,000,000 policies issued by the life insurance companies in Taiwan from the period of 1972 to 2008.

- Summary of policy numbers derived from Taiwan insurance data

Insurance Type	Female	Male
With Endowments	7,175,200	7,509,730
No Endowments	18,254,681	18,072,776

Derivation of the Optimal Solution

Let $u = \left(M_1, \ldots, M_{n_S}, N_{n_S+1}, \ldots, N_{n_B+n_S}\right)' = (u_1, \ldots, u_n)'$ be the units column vectors, the first n_S components are the units we need to buy life settlements with different ages, gender and life expectancies and the last n_B components are the units we need to buy bonds with different maturities.

Our target is to solve the problem:

$$\max_u \left\{[m', \bar{m}]\begin{bmatrix} u \\ -1 \end{bmatrix} - \theta[u', -1]\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} u \\ -1 \end{bmatrix}\right\}$$

$$\text{s.t. } u'a = 1 \text{ and } u' \geq 0$$

$$a = \begin{bmatrix} \frac{B(r_t, T_1)}{L} \\ \vdots \\ \frac{B(r_t, T_{N_B})}{L} \\ \frac{\bar{V}^S(x_1, r_t, m_{x,t}^S)}{L} \\ \vdots \\ \frac{\bar{V}^S(x_{N_S}, r_t, m_{x,t}^S)}{L} \end{bmatrix}$$

Denote

$$f(u) = -[m', \bar{m}]\begin{bmatrix} u \\ -1 \end{bmatrix} + \theta[u', -1]\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}\begin{bmatrix} u \\ -1 \end{bmatrix}$$

$$= -m'u + \bar{m} + \theta[u'\Sigma_{11}u - \Sigma_{21}u - u'\Sigma_{12} + \Sigma_{22}]$$

$$= -m'u + \bar{m} + \theta[u'\Sigma_{11}u - 2u'\Sigma_{12} + \Sigma_{22}]$$

Our optimization problem becomes

$$\min_{u} f(x)$$

$$\text{s.t. } u'a - 1 = 0 \text{ and } -u' \leq 0$$

The optimal solution is

$$u = \frac{1}{2\theta}\Sigma_{11}^{-1}(\mu + m - \lambda a) + \Sigma_{11}^{-1}\Sigma_{12}$$

where

$$\lambda = \frac{a'\Sigma_{11}^{-1}m - 2\theta\left(1 - a'\Sigma_{11}^{-1}\Sigma_{12}\right)}{a'\Sigma_{11}^{-1}a}$$

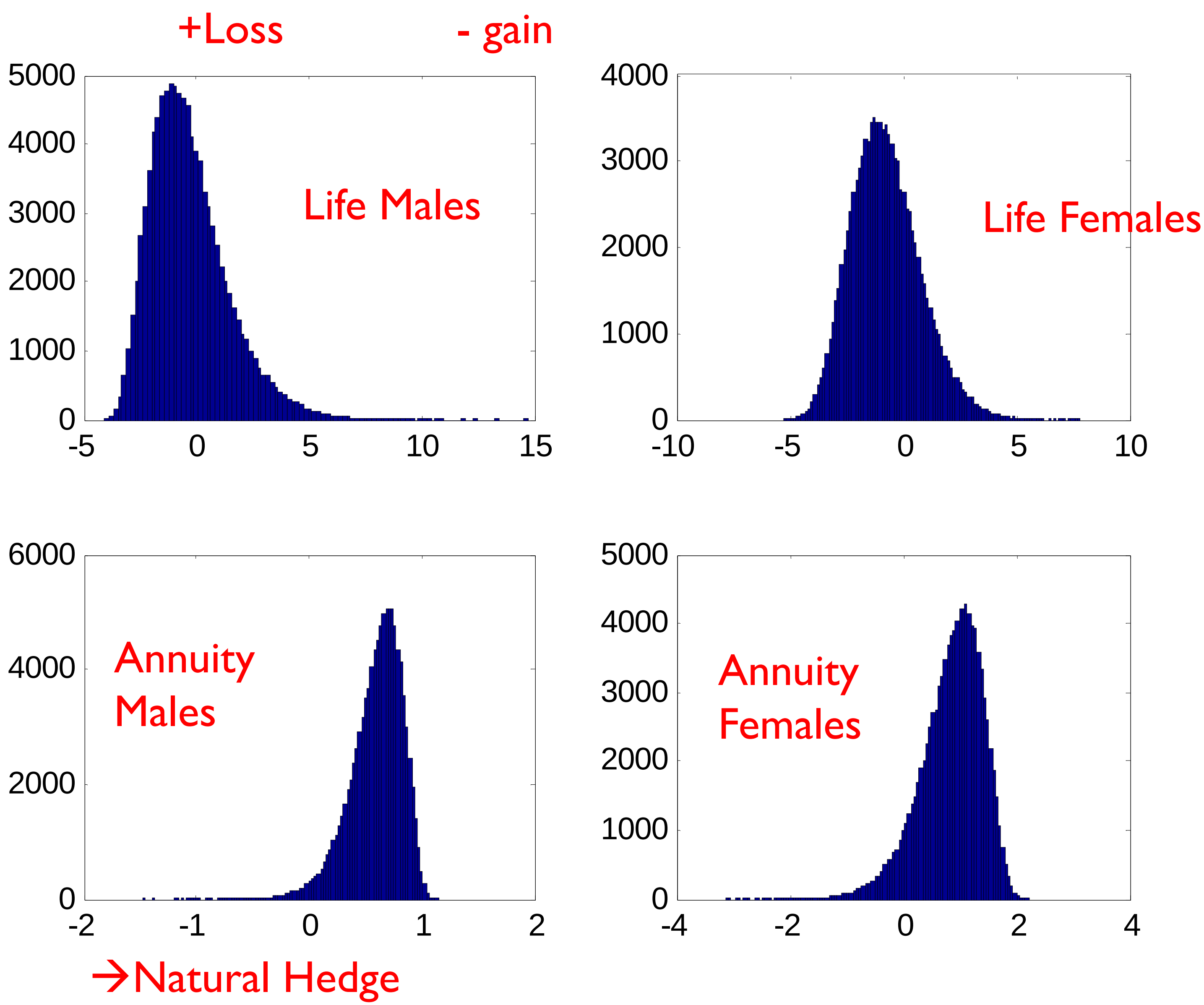
Numerical example

- Assumption of portfolios :
- Life insurance policies
  - one male insured aged 55 with face amount $5
  - one female insured aged 55 with face amount $5
- Annuity policies
  - one male insured aged 55 with face amount $1
  - one female insured aged 55 with face amount $1
- Q-forward
  - British male insured aged 65

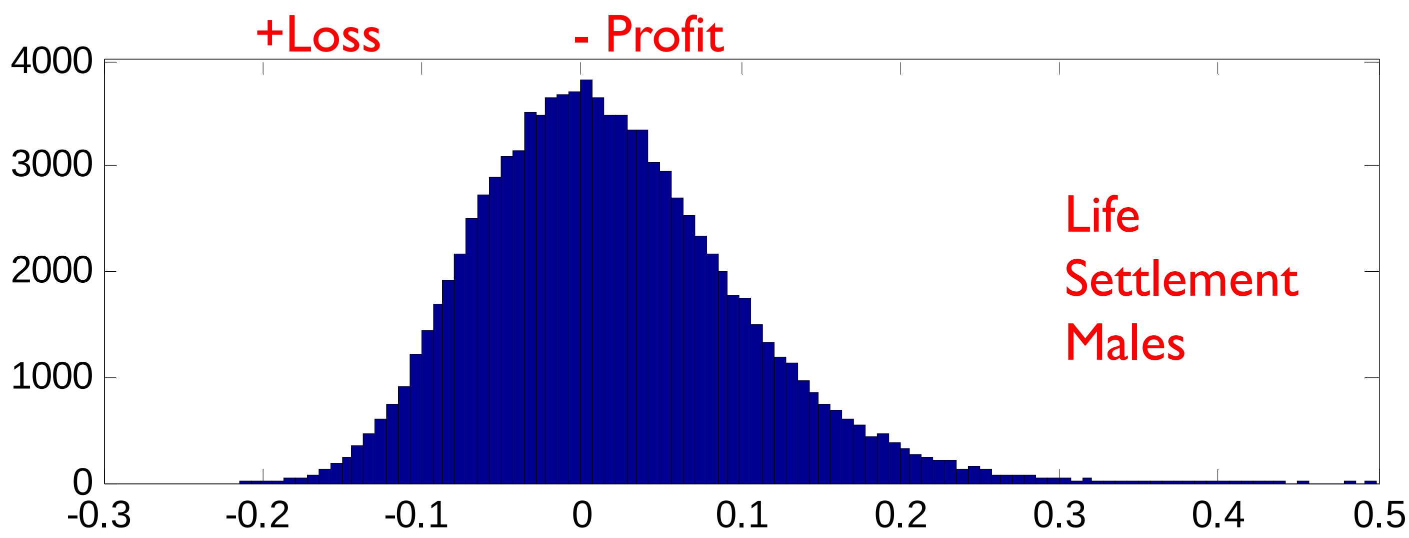
Numerical example

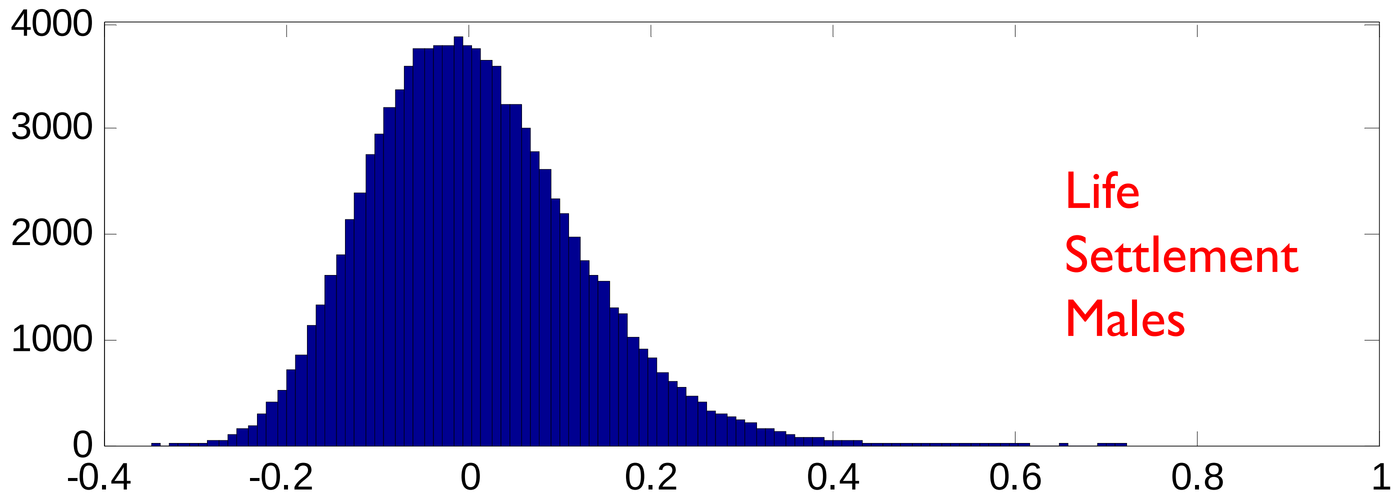
Asset	Liability
Life settlement: Male 65 (suggested life expectancy=10) Female 65 (suggested life expectancy=10)	**Life: (benefit=100)** Female 50 Male 65 **Annuity:** Female 55 Male 65

Loss Distribution-Liability



Loss Distribution-Asset



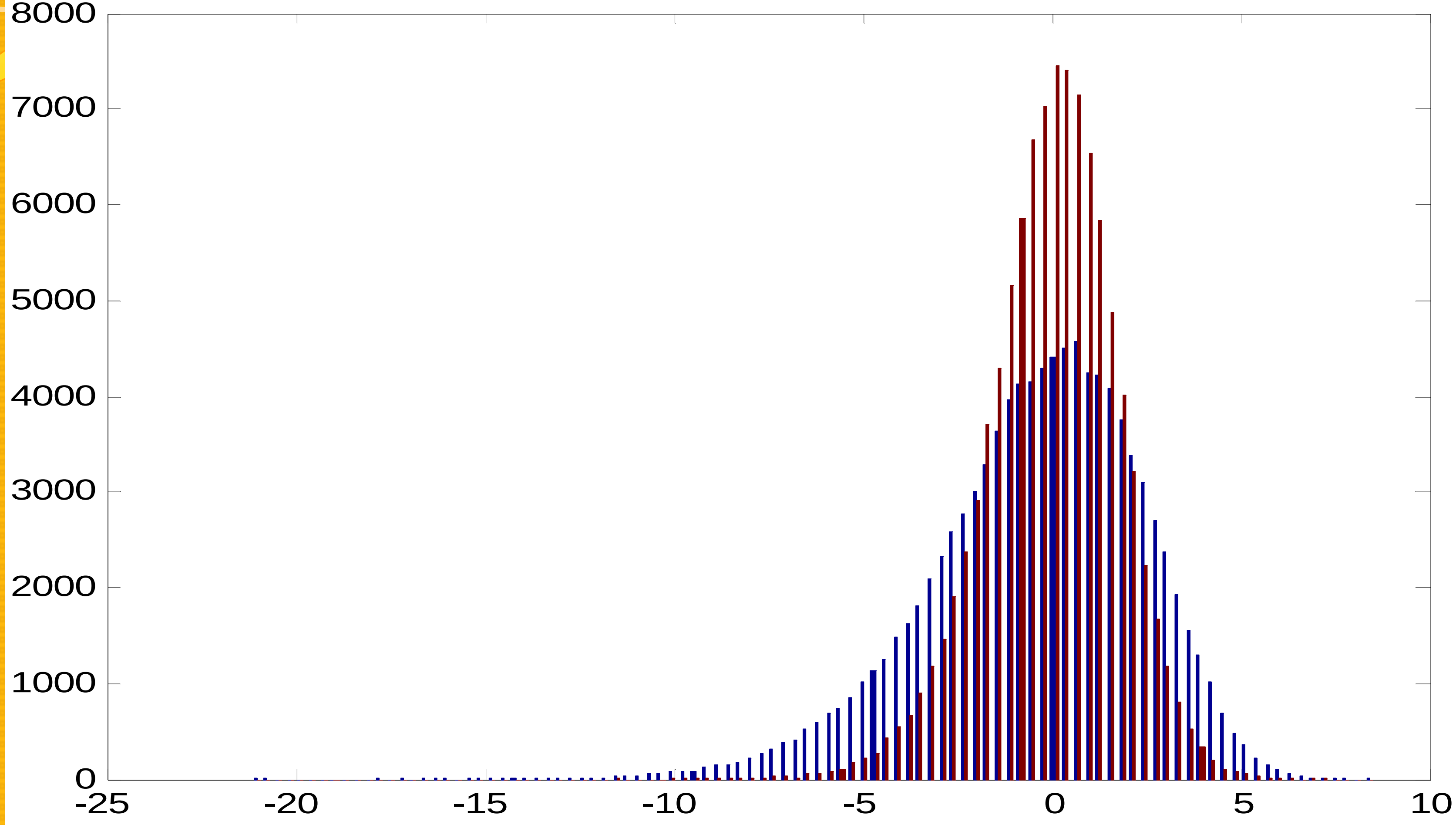


→External Hedge

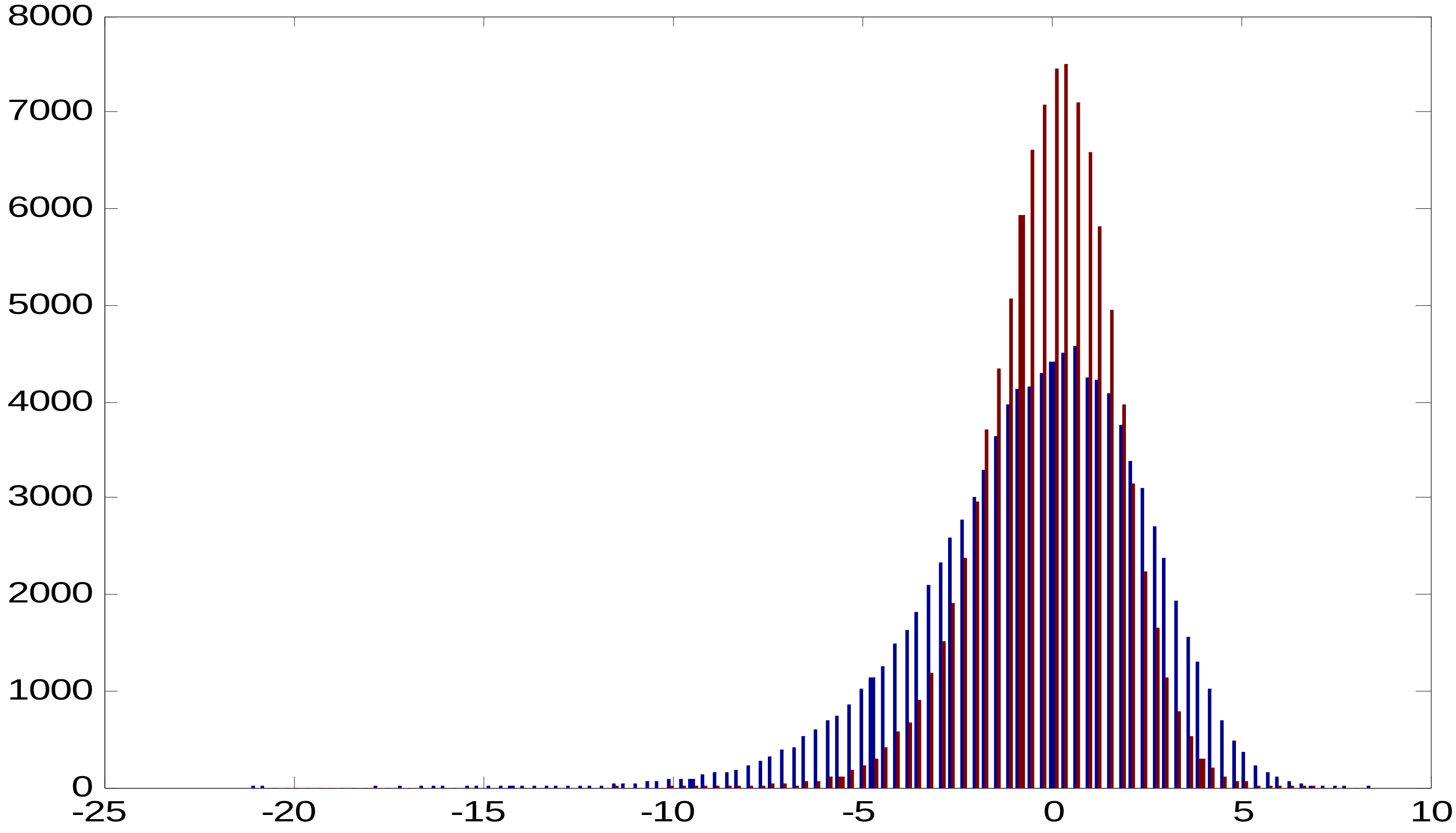
Optimal hedging Units

	Life settlement male 65	Life settlement female 65
MV θ = 1		
Units	17.7247	11.2857
Weight	0.1010	0.0746
MV θ = 2		
Units	17.0029	11.3729
Weight	0.0969	0.0752
VaR(0.05)		
Units	22.2499	11.8484
Weight	0.1268	0.0783
CTE(0.05)		
Units	21.9820	13.8237
Weight	0.1253	0.0914

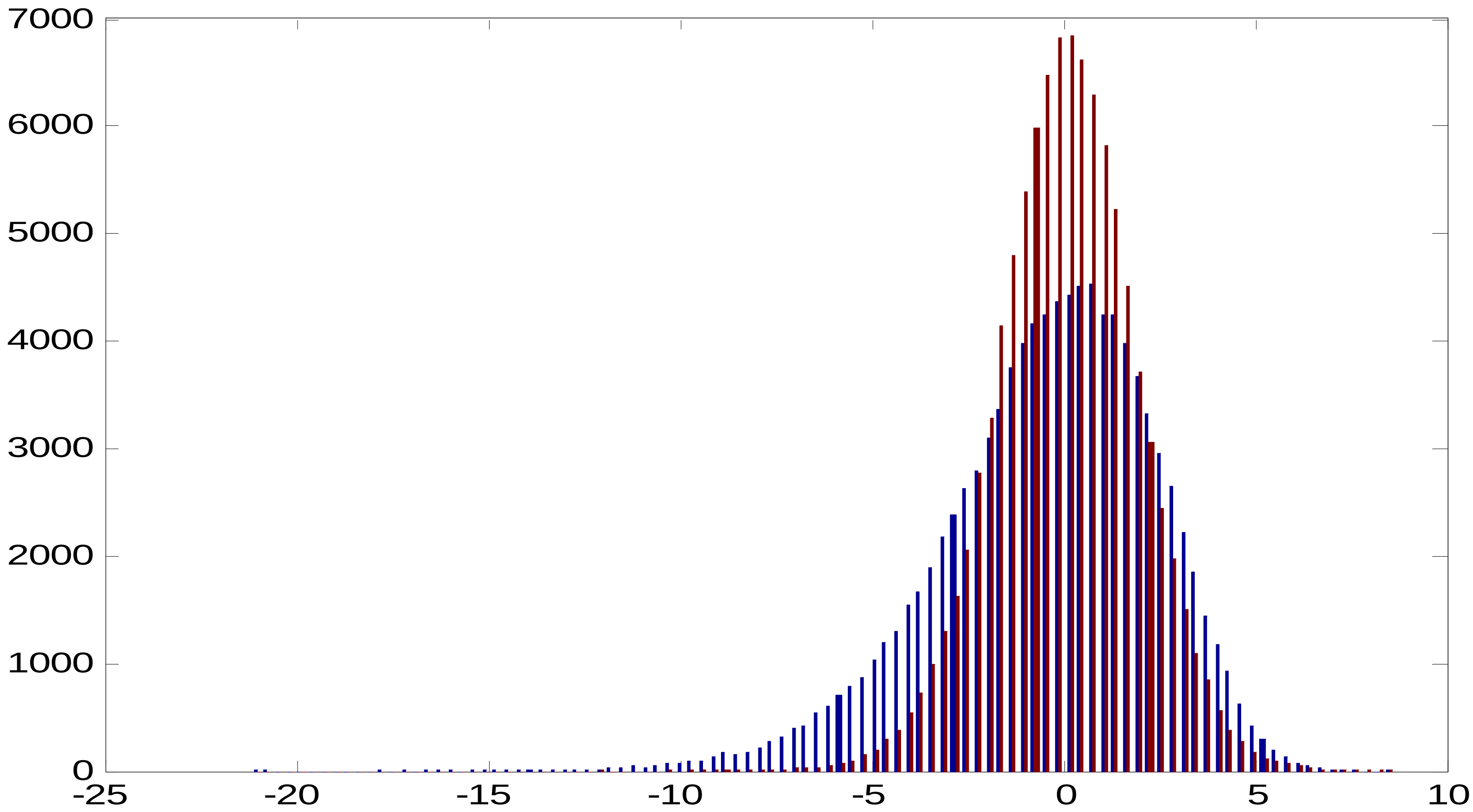
Hedging effectiveness-Mean Variance Approach (θ=1)



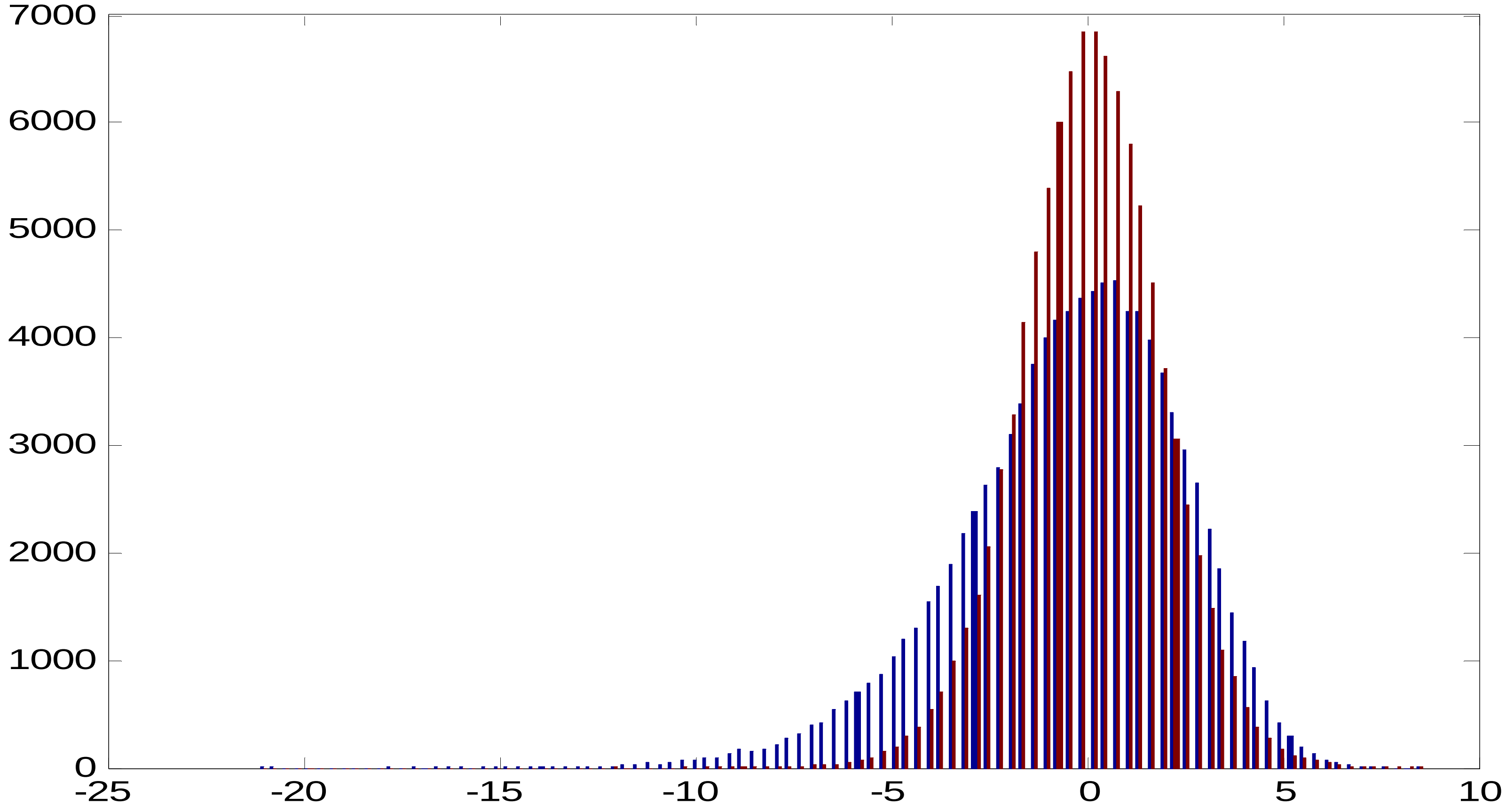
Hedging effectiveness-Mean Variance Approach(θ=2)



Hedging effectiveness-VaR



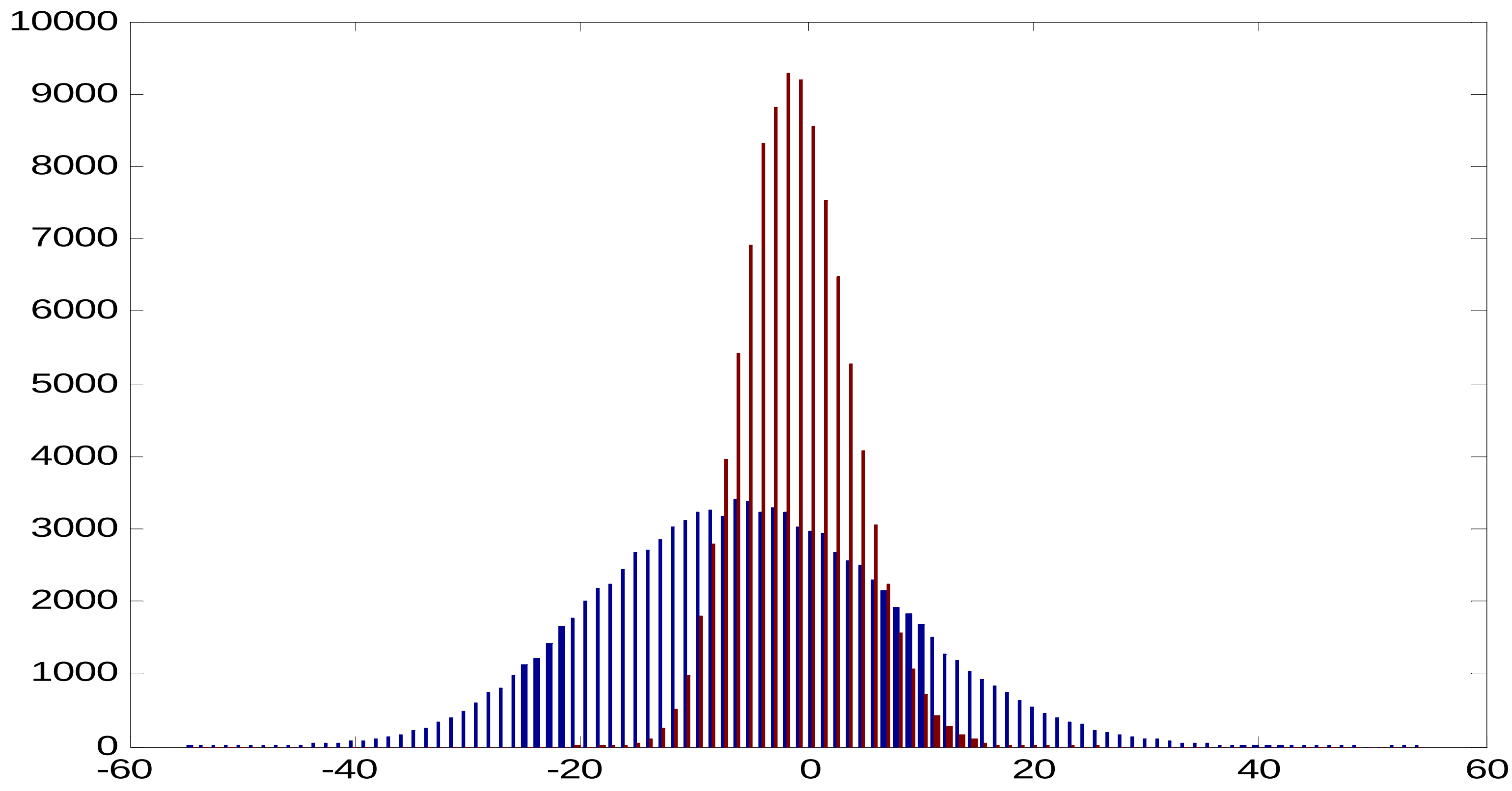
Hedging effectiveness-CTE



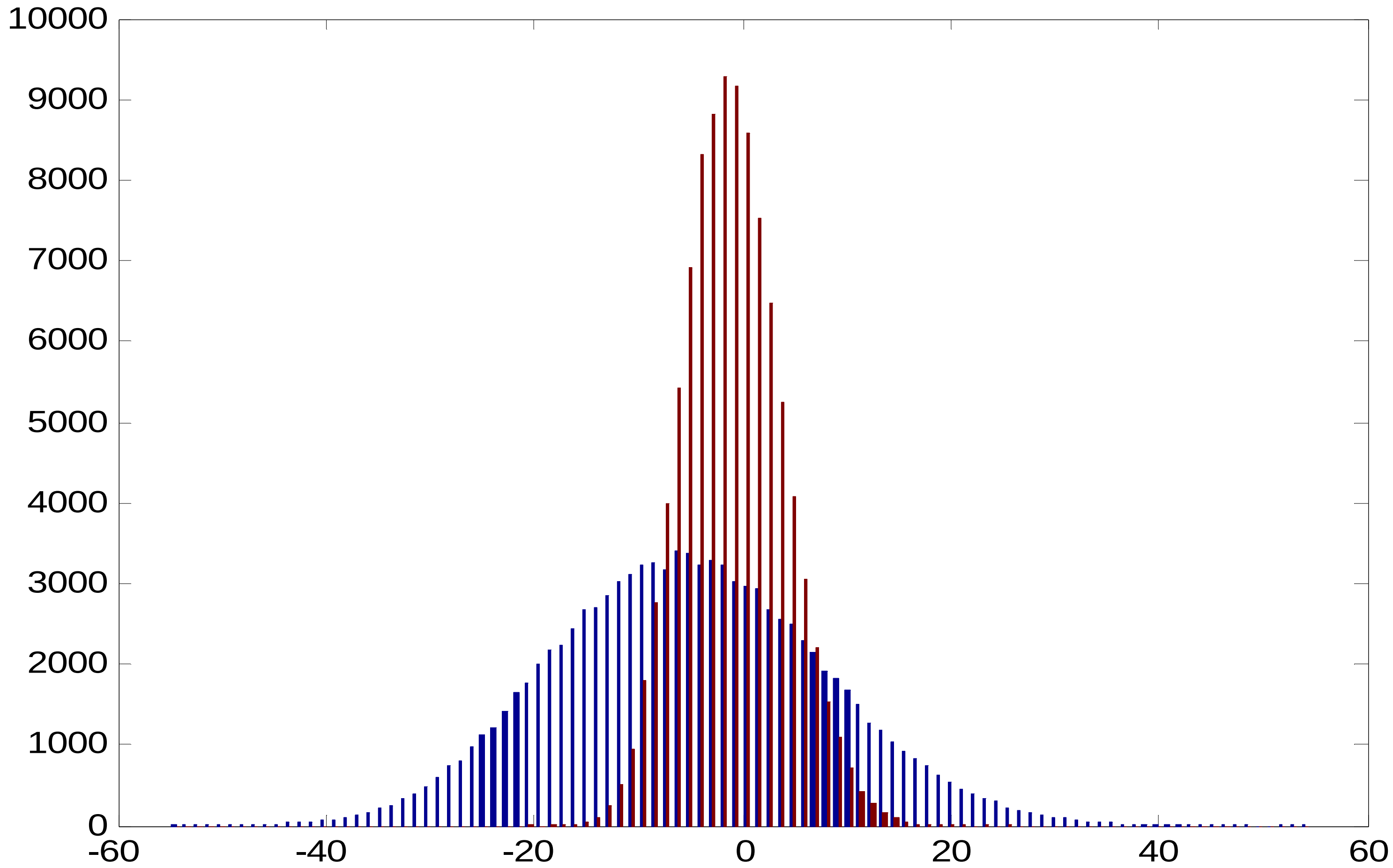
Optimal hedging strategies-Interest Rate Risk

	Zero coupon bond T=20	Life settlement male 65	Life settlement female 65
MV $\theta = 1$			
Units	183.5250	20.1810	12.7361
Weight	0.8008	0.1151	0.0842
MV $\theta = 2$			
Units	183.4617	19.7923	13.1131
Weight	0.8005	0.1128	0.0867
VaR(0.05)			
Units	188.1349	24.4197	6.0362
Weight	0.8209	0.1392	0.0399
CTE(0.05)			
Units	189.3252	20.0600	9.0111
Weight	0.8261	0.1144	0.0596

Hedging effectiveness-Mean Variance Approach (θ=1)



Hedging effectiveness-Mean Variance Approach ($\theta=2$)



Conclusion

- We provide a hedging framework that can offset the remaining mortality risk for the insurer.
- We also consider the basis risk and derive an analytic solution.
- The hedge effectiveness is examined in the numerical example..
- Further Research
  - Using different internal hedge instruments.