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An Econometric Analysis of Fractional Models to Credit Risk Pricing

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An Econometric Analysis of Fractional Models to Credit Risk Pricing

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Abstract: We propose a fractional version of two well-known credit risk pricing structural models: the Merton and Black and Cox models. We assume that the value of the firm obeys to a Geometric Fractional Brownian Motion. Prices for the equity, the bond and credit spreads are derived and a sensitivity analysis is performed. To provide a justification for these models, an empirical analysis is carried out, which employs two different datasets: Constant Maturity Yields and Moody's Long-Term for the period December 1992–November 2003 Corporate Bond Yield Averages and Lehman Brothers Eurodollar Indices covering the period June 1996–July 2006. Long memory properties of Treasury and corporate bond yields as well as credit spreads are thus investigated.

KEY WORDS Credit Risk; Structural Models; Credit Spreads; Semiparametric Methods; Fractional Integration

1 Introduction

In the academic literature models for the pricing of risky debt can be subdivided into two classes: firm's value and reduced form models.

The philosophy underlying firm's value models is to assume there is a fundamental process usually interpreted as the total value of the assets of the firm that has issued the bonds we are interested in. The value of the firm is assumed to move around stochastically and hence a stochastic process for the evolution of the firms' underlying assets is assumed. This is the driving force behind the dynamics of the prices of all securities issued by the firm. The well-known structural approach due to the seminal paper of Merton (1974) assumes that the company has issued only shares and a zero-coupon bond. The firm defaults if the value of its assets is lower than the promised debt payment at maturity. As a consequence all claims on the firm's value are modeled as derivative securities with the firm's value as underlying. Merton's model has been extended for instance by Black and Cox (1976), Geske (1977), Shimko et al. (1993) and Leland (1994) to allow for more realistic assumptions, such as the

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possibility of default before maturity, coupon payments, stochastic interest rates. The most common alternative to structural models is given by the reduced-form approach, which directly models the default process of risky debt. In combination with assumptions on the evolution of the risk-free rate and the recovery rate in the event of default, this is used to value risky debt. See, for example, Jarrow and Turnbull (1995), Duffie and Singleton (1997) and Madan and Unal (1998).

It is commonly agreed that structural models of credit risk have a poor performance in predicting corporate bonds prices. One of the main critiques to the classical Merton model and structural models in general, is that they predict credit spreads that are lower than the ones observed in the market. This is due to the fact that the assumptions behind the model are far for realistic. In particular the assumption of a geometric Brownian motion (gBm) for the firms' underlying assets proposed in the original paper by Merton could be the explanation for this poor performance. Many empirical evidence have shown that, when a gBm is used to describe log-returns, such a specification cannot describe the behaviour of financial assets mainly because the actual returns look to show some form of dependence.

The contribution of this paper to the literature is twofold. First of all, we investigate the empirical properties of credit spreads, with specific reference to their long memory features. Although a number of empirical studies on credit spread dynamics have been carried out (see Pedrosa and Roll, 1998; Prigent et al., 2001; Kiesel et al., 2001), no research has been performed to date to investigate the long memory properties of credit spread. Applied analysis on credit spreads so far has been carried out within the classical $I(0)$ vs $I(1)$ framework, i.e. by testing for stationarity vs nonstationarity of spreads only. It is well known that the distinction between $I(0)$ and $I(1)$ processes can be too restrictive. In contrast to $I(0)$ time series in which shocks die out at an exponential rate, or an $I(1)$ series with an infinite persistence (no mean reversion), Adenstedt (1974), Granger (1980) and Granger and Joyeux (1980) proposed an $I(d)$ time series with $0 < d < 1$ in which shocks dissipate at a slow hyperbolic rate. We show that credit spreads are likely to be long memory nonstationary processes, i.e. $I(d)$ processes with $d > 0.5$.

Secondly, we propose the fractional version of two structural credit risk models. The presence of long memory in credit spreads time series would provide a justification for the theoretical models proposed, that, in turn, would be able to explain realized credit spreads better than traditional credit risk structural models. A sensitivity analysis on bond prices and credit spreads predicted by such models shows that indeed the predicted credit spreads are closer to the real ones than those predicted by the original Merton model.

The rest of the paper is organized as follows. Section 2 reports the results of an empirical application on credit spreads which applies the econometric methodologies reported in Appendix A. Section 3 presents the fractional version of the Merton and Black and Cox models. Section 4 concludes.

2 Empirical Evidence Using Credit Spreads Data

In what follows we will investigate whether credit spreads can be described by long range dependent series. This will provide a justification for the models presented in section 3. We consider two different datasets. The first one is given by the Moody's Long-Term Corporate Bond Yield Averages. The second one consists of Lehman Brothers Eurodollar Indices.

2.1 Dataset 1

First we employ the following data: 30-year Historical US Treasury Constant Maturity Yields and Moody's Aaa, Aa, A and Baa Long-Term Corporate Bond Yield Averages. The data covers the period from December 1992 to November 2003, for $n = 2703$ observations. Spreads are calculated as the difference between corporate yields and Treasury yields, as well as between different corporate yields. Thus we have 15 series: Treasury yields (denoted by T), corporate yields (Aaa, Aa, A, and Baa), spreads over Tresasury (sTAaa, sTAa, sTA, and sTBaa), spreads between corporate yields (sAaaAa, sAaaA, sAaaBaa, sAaA, sAaBaa, and sABaa). These series are plotted in Figure 1–Figure 3.

[Figures 1–3 about here.]

Table 1 and Table 2 report summary statistics and normality tests for the series involved and for their first differences respectively.

[Tables 1–2 about here.]

Tests for normality show that distributions of yields and spreads, and their first differences, are highly non-normal. Differences in credit spreads, with the exceptions of dsTreasAaa, dsAaBaa and dsABaa are positively skewed. This implies that the probability of a loss from the return is bigger than the probability of a loss from return from a normal distribution. Moreover, differences are also leptokurtic.
Table 3 and Table 4 report unit root and stationarity tests for yields and spreads and their first differences.

[Tables 3–4 about here.]

Yields and spreads seem to be non-stationary. We have ambiguous results only for sAaaAa and sABaa. In both cases the KPSS test reject the null of stationarity, but for the former the null of a unit root is rejected by the PP test at a 1% level of significance, whereas for the latter the null of a unit root is rejected by the ADF with lag length 4 at a 5% level of significance and by the PP test at a 1% level of significance. However first differences of the series are clearly stationary, as shown by Table 4. As a consequence, we use the differenced series to estimate the difference parameter.
First we estimate the parameter d for yields and spreads using different values of m and J. In particular, when $J = 1$ (no pooling), $m = [n^\alpha]$ with $\alpha \in \{0.4, 0.5, 0.6, 0.7, 0.8\}$. As pointed out in Diebold and Inoue (2001) the

choice of m is very important because even though a large value for m, would result in reducing standard error, this would induce bias in the estimator. This is because eq. (10), on which the log-periodogram estimate is based, is valid only for frequencies close to zero. Even though a popular choice is $m = \sqrt{n}$, many authors suggest $m = n^{4/5}$ (see for instance Hassler et al., 2006, pag. 189), which is the mean-square optimal choice. Table 5 and Table 6 report the d estimates for the yields and the spreads when $l = 0$.

[Tables 5–6 about here.]

Table 7 and Table 8 report the d estimates for the yields and the spreads when $l = 1$.

[Tables 7–8 about here.]

Table 9 and Table 10 report the d estimates for the yields and the spreads when $l = 0$ and the data is tapered.

[Tables 9–10 about here.]

Table 11 and Table 12 report the d estimates for the yields and the spreads when $l = 1$ and the data is tapered.

[Tables 11–12 about here.]

From the analysis of Table 5 through Table 12 it is clear that yields and spreads are likely to be long memory and nonstationary processes. In each table we report also the test statistic for the null hypothesis of $d = 1$. First of all, the null is never rejected for the yields series, no matter if we consider tapered or non-tapered data, $l = 0$ or $l = 1$, $J = 1$, $J = 2$ or $J = 3$. As far as spreads are concerned, the number of rejections of the null increases when moving from non-tapered to tapered data. On the other hand, this number decreases when moving from $l = 0$ to $l = 1$ and as J increases. Moreover the rejections seem to be concentrated to the highest values of m, in particular to $m = \left[n^{0.7}\right]$. However, even when the null is rejected, it is quite clear that the fractional difference parameter belong to the interval $\left(\frac{1}{2}, 1\right)$, which, again, entails long memory and non-stationarity.

Table 13 and Table 14 report the local Whittle estimates for yields and spreads respectively.

[Tables 13–14 about here.]

Basically, the local Whittle estimations and the related tests to verify whether the fractional difference parameter is different from 1, confirm the results we got from the log-periodogram techniques.

Table 15 reports the result of Nielsen (2005) LM test for yields when setting $d = 1$ in eq. (18) or $\mathbf{d} = \iota$ in eq. (21) for the multivariate case. Panel A reports univariate tests whereas Panel B reports multivariate tests. Table 16 reports the result of Nielsen (2005) LM test for spreads when setting $d = 1$ in eq. (18) or $\mathbf{d} = \iota$ in eq. (21) for the multivariate case. Panel A reports univariate tests whereas Panel B reports multivariate tests.

[Tables 15–16 about here.]

The tables show that every time we do not allow for short run dynamics (i.e. $p = 0$), and we consider the univariate procedure, first differencing is sufficient to achieve stationarity for yields, but not for spreads. The null of stationarity after first differencing is rejected for the series sBaa when $p = 1$ and $p = 2$ and for the series sAaA when $p = 1$. Also in the multivariate case for yields and for spreads over treasury when $p = 0$ the null hypothesis of stationarity after first differencing is rejected. This is not the case when we allow for for short run dynamics ($p > 0$).

Table 17 reports the results of some unit root and stationarity tests for the residuals from the cointegrating regression for all possible bivariate systems of yields.

[Table 17 about here.]

Table 17 shows that there is no evidence of cointegration between Treasury and corporate yields. The only exception is the system Aaa–Aa for which the DF and the KPSS tests lead to different conclusions. This surprising result could be explained by the fact that the usual concept of cointegration may be too restrictive. Treasury and corporate yields may in fact be fractionally cointegrated.

Table 18 through Table 21 report the results of the three step procedure of Dittmann (2004) when residuals from the narrow band FDLS estimation are used. Table 18 and Table 19 report the results for the residuals, whereas Table 20 and Table 21 report the results for the differenced residuals.

[Tables 18–21 about here.]

Table 22 through Table 25 report the results of the three step procedure of Dittmann (2004) when tapered residuals from the narrow band FDLS estimation are used. Table 22 and Table 23 report the results for the residuals, whereas Table 24 and Table 25 report the results for the differenced residuals.

[Tables 22–25 about here.]

From the analysis of Table 18 through Table 25, it is clear that the fractional differencing parameter is always statistically different (and less) than 1 only for the systems Aaa-Aa and Aa-A. From Table 18 (non-tapered data, $l = 0$ and $J = 1$), it seems that the some fractional cointegration is present also in the system Aaa-A. When tapered data is used, some evidence of fractional cointegration in the system T-Aaa, can be found as well.

These results are different from the ones in Della Ratta and Urga (2005) in which the authors find fractional cointegration in all the bivariate systems except T-Baa and Aaa-Baa. This discrepancy is due to the fact that they use $m = [n^{0.9}]$.

On the other hand, looking at the Dittmann (2004) estimation procedure, the residuals from the cointegrating relation seem to be non-stationary even after the fractional differention, according to the KPSS test, only for the system T-Aaa. For each other bivariate system the null hypothesis of stationarity cannot be reject by using the KPSS test.

2.2 Dataset 2

Our second dataset consists of Lehman Brothers Eurodollar Aaa, Aa, A and Baa Indices and U.S. Global Treasury Index. The indices include primarily corporates bonds (even though they can include government-related and securitized bonds). The data covers the period from June 1996 to July 2006, for $n = 2613$ observations. Spreads are calculated as the difference between corporate yields and Treasury yields, as well as between different corporate yields. Thus we have 15 series: Lehman Brothers U.S. Global Treasury Index (denoted by T), corporate yields (Aaa, Aa, A, and Baa), spreads over Tresasury (sTAaa, sTAa, sTA, and sTBaa), spreads between corporate yields (sAaaAa, sAaaA, sAaaBaa, sAaA, sAaBaa, and sABaa). These series are plotted in Figure 4–Figure 6.

[Figures 4–6 about here.]

Table 26 and Table 27 report summary statistics and normality tests for the series involved and for their first differences respectively. All the series are highly non-normal.

[Tables 26–27 about here.]

Table 28 and Table 29 report unit root and stationarity tests for yields and spreads and their first differences.

[Tables 28–29 about here.]

The series are clearly non-stationary. As far as first differences are concerned, only for the series ΔT the KPSS tests reject the null of stationarity (at a 5% level of significance). Thus we can conclude that first differences of the series are stationary.

Table 30 and Table 31 report the d estimates for the yields and the spreads when $l = 0$.

[Tables 30–31 about here.]

Table 32 and Table 33 report the d estimates for the yields and the spreads when $l = 1$.

[Tables 32–33 about here.]

Table 34 and Table 35 report the d estimates for the yields and the spreads when $l = 0$ and the data is tapered.

[Tables 34–35 about here.]

Table 36 and Table 37 report the d estimates for the yields and the spreads when $l = 1$ and the data is tapered.

[Tables 36–37 about here.]

Table 38 and Table 39 report the local Whittle estimates for yields and spreads respectively.

[Tables 38–39 about here.]

Table 40 reports the result of Nielsen (2005) LM test for yields when setting $d = 1$ in eq. (18) or $\mathbf{d} = \iota$ in eq. (21) for the multivariate case. Panel A reports univariate tests whereas Panel B reports multivariate tests. Table 41 reports the result of Nielsen (2005) LM test for spreads when setting $d = 1$ in eq. (18) or $\mathbf{d} = \iota$ in eq. (21) for the multivariate case. Panel A reports univariate tests whereas Panel B reports multivariate tests.

[Tables 40–41 about here.]

From the tables is clear that for $p = 0$ all the first differenced credit spreads series are not stationary except sBaa, sAaabaa, sAaA and sAaBaa. Further, for the series sAaaAa the null of stationarity after first differencing is rejected for the series also when $p = 1$. Among the yields, the null is rejected also for the time series T, result confirmed also by the fact that the fractional differencing parameter appears to be bigger than one from the GPH estimation.

Table 42 reports the results of some unit root and stationarity tests for the residuals from the cointegrating regression for all possible bivariate systems of yields.

[Table 42 about here.]

Table 42 shows that there is no evidence of cointegration between Treasury and corporate yields. Ambiguous results are found for the systems Aa–Baa and A–Baa and, to some extend, for the systems T–Baa and T–Baa. The explanation could be the presence of fractional cointegration.

3 Merton and Black and Cox Fractional Models

In the structural approach to credit risk the firm liabilities are thought as contingent claims issued against the firm underlying assets. A stochastic process for the evolution of the firm underlying assets, V, and the conditions under which a default is triggered as well as the payoff of the risky debt in the event of default are specified. Merton (1974) assumes that the firm has only issued zero coupon bonds with maturity T and total face value D, that default may happen only at maturity. Denote by $\overline{B}(t, T)$ and S_t the prices in t of a defaultable zero coupon bond and the equity respectively. Both $\overline{B}$ and S are function of V and more generally all claims on the firm's value are evaluated as derivative securities with the firm's value as underlying. The term structure of interest rate is assumed to be deterministic and flat and the firm pays no dividend over the life of the debt. In case of default bondholders are assumed to have absolute priority, i.e. bond value at time T is $\overline{B}(T, T) = \min(D, V_T)$

and the equity is simply a call option, $S_T = \max(V_T - D, 0)$. Whereas the original model assumes a Geometric Brownian motion for the firm value, in this paper we consider the following dynamics for V:

$$\mathrm{d}V_t = \mu V_t \mathrm{d}t + \sigma V_t \mathrm{d}B_H(t) \tag{1}$$

where $B_H(t)$ denotes a fractional Brownian motion and $H \in \left(\frac{1}{2}, 1\right)$ is the Hurst parameter. The fractional Brownian motion is a Gaussian process with zero mean, stationary increments, variance

$$\mathbb{E}\left[B_H^2(t)\right] = t^{2H}$$

and covariance

$$\mathbb{E}\left[B_H(t)B_H(s)\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Usually it is assumed that $B_H(0) = 0$. For any $H \in (0,1)$ the process $B_H(t)$ is self-similar in the sense that

$$B_H(t) \stackrel{d}{=} a^{-H} B_H(at).$$

The parameter H negotiates whether the fractional Brownian motion has independent increments ($H = 1/2$), positive covariance between two increments over non-overlapping time intervals ($1/2 < H \leq 1$), or negative covariance between increments ($0 < H < 1/2$). If $1/2 < H \leq 1$ we say that $B_H(t)$ has a long range dependence, since

$$\sum_{n=-\infty}^{\infty} \gamma(n) = \infty, \tag{2}$$

where

$$\gamma(n) = \mathrm{Cov}\left[B_H(1), B_H(n+1) - B_H(n)\right] = \frac{1}{2}\left[|n+1|^{2H} + |n-1|^{2H} - 2|n|^{2H}\right].$$

When $H = 1/2$, $\gamma(n) = 0$ for all $n \neq 0$ whereas for $1/2 < H \leq 1$

$$\gamma(n) \sim H(2H-1)|n|^{2H-2}, \quad \text{as} \quad |n| \to \infty, \tag{3}$$

where "$\sim$" means that the ratio of the left and right hand sides tends to one. Therefore, when $H = 1/2$, (1) implies that the $\log V_t$ are independent normal random variables. However, in recent years, many empirical evidence have shown that this assumption cannot be used to describe the behaviour of financial assets because: 1) the empirical distributions of the log-price variations are far from being normal; 2) the actual returns look to show some form of dependence. In this paper we will address only the second issue and use fractional Brownian motion to model long-range dependence.

In what follows we denote by $N(\cdot)$ the cumulative probability distribution function of a standard normal random variable:

$$N(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} \mathrm{e}^{-\frac{1}{2}u^2} \mathrm{d}u,$$

and by $n(\cdot) = N'(\cdot)$ the density function. The next result follows directly from the fractional Black-Scholes formula derived in Hu and Øksendal (2003) using the Wick–Itô–Skorohod calculus:

Theorem 3.1 *Assuming absolute priority for the bondholders, a geometric fractional Brownian motion* (1) *with* $H \in \left(\frac{1}{2}, 1\right)$ *for the firm asset, and that the firm has only issued zero coupon bonds with maturity* T *and total face value* D*, when the risk-free rate is constant and equal to* r*, the value of the equity at time* 0 *is*

$$S_0 = V_0 N(d_1) - D\mathrm{e}^{-rT} N(d_2) \tag{4}$$

where

$$d_1 = \sigma^{-1} T^{-H} \left(\log(V_0/D) + rT\right) + \frac{1}{2}\sigma T^H$$
$$d_2 = d_1 - \sigma T^H$$

and $N(\cdot)$ *denotes the standard Normal cumulative distribution function. The price of the bond is*

$$\overline{B}(0,T) = V_0 - S_0. \tag{5}$$

The spread is

$$s(0,T) = -\frac{1}{T} \log\left(\frac{\overline{B}(0,T)}{D} \mathrm{e}^{rT}\right) = -\frac{1}{T} \log\left(\frac{\overline{B}(0,T)}{\ell V_0}\right), \tag{6}$$

where $\ell = \frac{D\mathrm{e}^{-rT}}{V_0}$ *is the firm leverage.*

Clearly for $H = \frac{1}{2}$ we get the classical Merton model and (4) reduces to the Black-Scholes formula for a call option.

Corollary 3.2 *The price of the equity at time* t *is given by*

$$\begin{aligned} S_t = V_t N &\left(\frac{\log(V_t/D) + r(T-t) + \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}} \right) \\ &- D\mathrm{e}^{-r(T-t)} N \left(\frac{\log(V_t/D) + r(T-t) - \frac{1}{2}\sigma^2\left(T^{2H} - t^{2H}\right)}{\sigma\sqrt{T^{2H} - t^{2H}}} \right) \end{aligned}$$

option prices evaluated at different times t *but same time to maturity* $T-t$ *will give different results with fractional Black-Scholes formula, whereas the classical formula will give identical prices, i.e. if* $T_2 - t_2 = T_1 - t_1$, *then, denoting by $S(t,T)$ price at time t with maturity T and Hurst exponent H,*

$$S^H(t_1,T_1) \neq S^H(t_2,T_2)$$

unless $H = \frac{1}{2}$. This is due to the non-Markovian of fBM which forces S_t^H to be dependent on the underlying process evolution during the increment between different t's.

Table 43 reports the results of the sensitivity analysis for the fractional Merton model.

[Table 43 about here.]

Some observations are in order. First of all the fractional Merton model shares some features of the original model. Note that the corporate bond is equivalent to a long position in a default free bond with nominal D and a short position in a put option written on V and with strike D or, alternatively, to a long position in the asset value and a short position in a call written on V and with strike D. Therefore, when the value of the firm is much bigger that the debt D, then the put option is deep out-of-the-money, the probability of default, $N(-d_2)$ is low and corporate debt trades as if it is default-free. Conversely, when $V_t \ll D$ then the call component is small and the bond is approximately equal to the value of the equity. As a consequence the volatility of the corporate debt depends on the volatility of the underlying asset when the the put option trades deep in-the-money. It is clear from Table 43 that the credit spread is a decreasing function of the risk-free rate. The intuition is as follows: an increase in the default-free spot interest rate, keeping the value of the firm constant, makes the probability of default decline. This, in turn, makes the corporate bond price increase and, consequently, the spread decline. Moreover credit spreads are an increasing function of the firm's leverage. The explanation is straightforward: the bigger the leverage, the bigger the probability of default. Clearly, an increase in the probability of default results in a decrease of the corporate bond price and therefore in an increase in the credit spread. Of course spreads are increasing functions of the firm volatility because equity investors do always benefit from an increase in asset volatility which makes the equity price increase. On the other hand this results in a decrease in the corporate bond price and, thus, in an increase in the spread. Another implication of the model is that the credit spreads tend to zero as the maturity of the zero-coupon bond tends to zero when $V > D$ and explode when $V \leq D$.
In Figure 7–Figure 9 we plotted the value of the spreads as a function of time to maturity for three values of firm leverage ($\ell \in \{0.8, 1, 1.2\}$) and three values of the parameter H ($H \in \{0.55, 0.8, 0.95\}$). The risk-free is $r = 0$, so that $\ell = D/V$. In every graph, part (a) plots the credit spreads against time for $T \in [0, 30]$ and part (b) for $T \in [0, 1]$, to show the different behaviour for time to maturity bigger and smaller than one.

[Figures 7–9 about here.]

It is clear that when H is close to $\frac{1}{2}$, i.e. under the classical Merton model, credit spreads are hump shaped with respect to maturity (rising at first and then falling), for values of leverage below unity and are decreasing for values of leverage bigger than one. For values of the memory parameter H bigger than $\frac{1}{2}$, spreads are increasing ($\ell = 80\%$) and first decreasing and then increasing ($\ell = 100\%$ and $\ell = 120\%$).

However one of main critiques to the classical Merton model is that it predicts credit spreads which are lower than the ones observed in the market. However, in the fractional Brownian motion framework, when $T > 1$ spreads are increasing function of the long memory parameter H. This implies that the theoretical spreads in the fractional framework are bigger that the spreads predicted by the classical Merton model ($H = \frac{1}{2}$). The fractional model entails much more realistic spreads because it takes into account the dependency structure of financial returns. For $T = 1$ the spreads are a constant function of H and for $T < 1$ they are a decreasing function of H, meaning that for short maturities credit spreads are underestimated.

A more realistic assumption can be made about the conditions under which default is triggered. The company issuing debt, for instance, can be assumed to default not only at maturity T but the first time its assets fall below some default barrier, say L. To be more precise, if the assets hit the barrier L before time T, the option ceases to exist and the bondholders receive the assets value or some recovered portion of it. If the barrier is not hit before T, then the equity at maturity is, as in Merton model, the payoff on a European call option. This is the well known first passage time model proposed by Black and Cox (1976). A straightforward implication of the model is that the equity can be viewed as a down-and-out call option. To be more precise the time of default is given by the stopping time $\tau = \min\{k \geq t : V_k < L\}$. Denoting by $\mathbb{E}_t^Q[\cdot]$ the expectation under the risk neutral probability Q conditional to the information available up to time t and by I the indicator function, the equity value at time t is given by

$$S_t = \mathbb{E}_t^Q \left[\mathrm{e}^{-\int_t^T r(u)\mathrm{d}u} (V_T - D)^+ I_{\{\tau > T\}} \right].$$

It is well known that this expression leads to a nice closed-form solution when the interest rate is constant and V follows a geometric Brownian motion:

$$S_t = \left[V_t N(x_1) - D\mathrm{e}^{-r(T-t)} N(x_1 - \sigma\sqrt{T-t}) \right]$$
$$- \left[V_t \left(\frac{L}{V_t} \right)^{2\theta} N(y_1) - D\mathrm{e}^{-r(T-t)} \left(\frac{L}{V_t} \right)^{2\theta - 2} N(y_1 - \sigma\sqrt{T-t}) \right],$$

where

$$\theta = \frac{r + \frac{1}{2}\sigma^2}{\sigma^2}$$

$$x_1 = \begin{cases} \frac{\log(V_t/D)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} & \text{if } D \geq L \\ \frac{\log(V_t/L)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} & \text{if } D < L \end{cases},$$

$$y_1 = \begin{cases} \frac{\log(L^2/(V_t D))+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} & \text{if } D \geq L \\ \frac{\log(L/V_t)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} & \text{if } D < L \end{cases}.$$

Denoting by $\bar{r} = r - \frac{1}{2}\sigma^2$, a closed-form solution is available for the risk neutral probability of default:

$$\begin{aligned} &1 - Q(\tau > T, V_T > D) \\ = N\left(\frac{\log(D/V_t) - \bar{r}(T-t)}{\sigma\sqrt{T-t}}\right) &+ \left(\frac{L}{V_t}\right)^{\frac{2\bar{r}}{\sigma^2}} N\left(\frac{\log(L^2/(DV_t)) + \bar{r}(T-t)}{\sigma\sqrt{T-t}}\right). \end{aligned}$$

Intuitively, this default probability is higher than the corresponding probability in the classical approach, which is obtained as the special case where $L = 0$. The corresponding payoff to bond investors at maturity is

$$\begin{aligned} &\min(V_T, D)I_{\{\tau > T\}} + V_T I_{\{\tau \leq T\}} \\ &= D - (D - V_T)^+ + (V_T - D)^+ I_{\{\tau \leq T\}}. \end{aligned}$$

This position is equivalent to a portfolio composed of a risk-free loan with face value D maturing at T, a short European put on the firm with strike D and maturity T and a long European down-and-in call on the firm with strike D and maturity T. The price of the bond can be derived using the corresponding Black-Scholes formulae. In order to derive a fractional version of this model, we need some assumptions. First of all we assume that $r = 0$ and that both the amount of the debt and the value of the firm are bigger than the threshold L. While the former assumption may be restrictive, the latter is very realistic. Now we can use the result in Carr et al. (1998): under the previous assumptions in an arbitrage-free economy a position in a down-and-out call option is equivalent to being long a call struck at D and short D/L puts struck at L^2/D:

Theorem 3.3 *Assuming a geometric fractional Brownian motion with $H \in \left(\frac{1}{2}, 1\right)$ for the firm asset, that the firm has only issued zero coupon bonds with maturity T and total face value D and that it could default not only at maturity but also at the first time its assets fall below the barrier $L < V_0$, when the risk-free rate is $r = 0$, the value of the equity at time 0 for $D > L$ is*

$$S_0 = V_0 N(d_1) - D\mathrm{e}^{-rT} N(d_2) - \left[LN(y_1) - \frac{D}{L} V_0 N(y_2)\right] \tag{7}$$

where

$$y_1 = \sigma^{-1} T^{-H} \log\left(\frac{L^2}{V_0 D}\right) + \frac{1}{2}\sigma T^H$$
$$y_2 = y_1 - \sigma T^H.$$

The price of the bond is

$$\overline{B}(0,T) = V_0 - S_0. \tag{8}$$

The spread is

$$s(0,T) = -\frac{1}{T}\log\left(\frac{\overline{B}(0,T)}{D}\right) = -\frac{1}{T}\log\left(\frac{\overline{B}(0,T)}{\ell V_0}\right), \tag{9}$$

where $\ell = \frac{D}{V_0}$ *is the firm leverage.*

Table 44 reports the results of the sensitivity analysis for the fractional Black and Cox model.

[Table 44 about here.]

Note that, denoting by s_{M} and s_{BC} the spreads resulting from the fractional Merton and Black and Cox model respectively, we have

$$\frac{\partial}{\partial x} s_{\mathrm{BC}} = \left(\frac{\partial}{\partial x} s_{\mathrm{M}}\right)\left[n(d_2) - \frac{V_0}{L} n(y_2)\right], \quad x \in \{\sigma, H\}$$

Simple algebra yields

$$n(d_2) - \frac{V_0}{L} n(y_2) = n(d_2)\left\{1 - \exp\left[-\frac{2}{\sigma^2 T^{2H}} \log(L/V_0)\log(L/D)\right]\right\} > 0$$

since $L < D$ and $L < V_0$. This means that, as in the fractional Merton model, in the fractional Black and Cox model spreads are increasing functions of both the volatility and the memory parameter.

In Figure 10–Figure 12 we plotted the value of the spreads as a function of time to maturity for three values of firm leverage ($\ell \in \{0.8, 1, 1.2\}$) and three values of the parameter H ($H \in \{0.55, 0.8, 0.95\}$). In particular we chose $V = 100$, $L = 70$ and $D \in \{80, 100, 120\}$. In every graph, part (a) plots the credit spreads against time for $T \in [0, 30]$ and part (b) for $T \in [0, 1]$, to show the different behaviour for time to maturity bigger and smaller than one. The spreads predicted by the Black and Cox model are below those predicted by Merton fractional model because $S_{\mathrm{BC}} \leq S_{\mathrm{M}}$, which implies $\overline{B}_{\mathrm{BC}} \geq \overline{B}_{\mathrm{M}}$ and therefore $s_{\mathrm{BC}} \leq s_{\mathrm{M}}$.
Clearly, for $\ell = 80\%$ spreads keep the usual hump shape, whereas for $\ell \geq 1$ spreads behave like a decreasing function of T, even for values of H close to unity.

[Figures 10–12 about here.]

4 Conclusions and Future Work

This paper has two main purposes. First, it proposes a continuous-time credit risk model that is consistent with the observed behaviour of credit spread and can be used for pricing credit sensitive instruments, including credit derivatives. In particular the fractional versions of well known structural credit risk models (Merton and Black and Cox models) have been investigated. A sensitivity analysis has been performed in order to understand the properties of the model. Second, we have investigated the empirical properties of credit spreads, with specific reference to their long memory characteristics. Using semiparametric estimators of the fractional difference parameter d, we have shown that yields and spreads are generally long memory nonstationary processes. This is substantially confirmed by the recent LM test proposed by Nielsen (2005).

However, as pointed out by Dolado et al. (2005) and by Mikosch and Stărică (2004) for the case of multiple breaks, the long memory feature could be a spurious effect caused by the presence of one or more structural breaks. In particular the former presents a simple example to illustrate the source of confusion between a long memory process and a short memory one subject to structural breaks. Consider the sequence $\{y_t\}_{t=1,\ldots,T}$ with the following data generating process

$$y_t = \alpha_1 + (\alpha_2 - \alpha_1)DU_t(\lambda) + u_t = \begin{cases} \alpha_1 + u_t & t \leq T_B \\ \alpha_2 + u_t & t > T_B \end{cases}$$

where u_t is a zero-mean $I(0)$ process with autocovariances $\gamma_u(j)$ and $DU_t(\lambda) = I_{(t>T_B)} \equiv I_{(t>\lambda T)}$ with λ the fraction of the sample where the break occurs. A straightforward application of the ergodic theorem allows us to derive the asymptotic behavior of the sample autocovariance of $\{y_t\}$

$$\hat{\gamma}_y(j) = \frac{1}{T}\sum_{t=j+1}^{T}(y_t - y_{t-j}) - (\bar{y}_T)^2 \to \gamma_u(j) + \lambda(1-\lambda)(\alpha_2 - \alpha_1)^2 \quad \text{a.s.}$$

and thus even though $\gamma_u(j)$ approaches 0 as $j \uparrow \infty$ because u_t is $I(0)$,

$$\lim_{j\uparrow\infty} = \lambda(1-\lambda)(\alpha_2 - \alpha_1)^2.$$

As long as $\alpha_2 \neq \alpha_1$, the limit is not zero and thus the autocovariance function of the process mimics a slow hyperbolic decay characteristic of a long memory process. To assess whether credit spreads are genuine long memory processes we could implement the time-domain test of a process being $I(d)$, $0 < d \leq 1$, under the null, against the alternative of being $I(0)$ with deterministic components subject to structural breaks, proposed in this same paper. Another possibility is to apply a jump detection method based on bi-power variation (see Barndorff-Nielsen and Shephard, 2004; Tauchen and Zhou, 2006; Barndorff-Nielsen and Shephard, 2006; Andersen et al., 2006) filter out jumps from the time series of interest and analyze the memory properties of the filtered series.

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Appendix

A Econometric Methodology

A.1 Fractional Integration

Consider a stationary sequence X_t. Assume that the spectral density of the sequence, i.e. $f(\lambda)$, $-\pi < \lambda \leq \pi$ such that $\gamma(s) = \int_{-\pi}^{\pi} f(\lambda)\cos(s\lambda)\mathrm{d}s$, $s = 0, \pm 1, \ldots$, satisfies for $0 < G < \infty$

$$f(\lambda) \sim G\lambda^{-2d} \quad \text{as} \quad \lambda \to 0^+, \tag{10}$$

where $d \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ is the fractional differencing parameter which can be related to H via

$$d = H - \frac{1}{2}.$$

The main idea behind a fractional differentiation is that for many time series the distinction between $I(0)$ and $I(1)$ processes can be too restrictive. Even though many series show autocorrelation up to very long lags as in (2), taking the first differences may be excessive and lead to overdifferencing. Thus, the main idea of some authors (see Granger, 1980; Granger and Joyeux, 1980, among the others) was to introduce an $I(d)$ time series with $0 < d < 1$ in which shocks dissipate at a slow hyperbolic rate, in contrast to $I(0)$ time series in which shocks die out at an exponential rate, or an $I(1)$ series with an infinite persistence (no mean reversion) If the series X_t is $I(d)$, it becomes stationary after d-th differencing, i.e.

$$\Delta^d X_t = e_t,$$

where e_t is white noise, $\Delta = 1 - L$ and L denotes the lag operator, $L^k X_t = X_{t-k}$, and $(1-L)^d$ has the binomial expansion

$$(1-L)^d = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(d+1)}{\Gamma(d-k+1)k!} L^k,$$

where $\Gamma(\cdot)$ is the Gamma function.

Note that the non-summability condition (2) for $1/2 < H \leq 1$ implies that the spectral density is unbounded at zero frequency. The spectral density (10) has, in fact, a pole when $0 < d < 1/2$ and a zero when $-1/2 < d < 0$ at $\lambda = 0$.

One of the most common procedure to estimate the parameter d in the frequency domain with stationary data, $-1/2 < d < 1/2$, is the log-periodogram regression proposed by Geweke and Porter-Hudak (1983), GPH henceforth. This is semiparametric method in that no assumptions on the behaviour of the spectral density apart from the origin is made (in other words, avoid parameterization of the short run component is avoided). Suppose that model (10) is approximately valid for the spectral density of a stationary long memory process

when the first m Fourier frequencies, λ_j, $j = 1, \ldots, m$, are considered. The reason why only the first frequencies are considered is that long range dependency can be captured by studying the behaviour of big lags in the autocorrelation function or, equivalently low frequencies in the spectral density.
Define the discrete Fourier transform of X_t for n observations and for $\lambda_j = 2\pi j/n$:

$$w(\lambda_j) = \frac{1}{\sqrt{2\pi n}} \sum_{t=1}^{n} X_t \mathrm{e}^{\mathrm{i}\lambda_j t}$$

and the periodogram

$$I(\lambda_j) = |w(\lambda_j)|^2 .$$

Taking the logarithm of both sides of (10) yields

$$\log f(\lambda_j) \approx \log G - 2d \log \lambda_j \quad j = 1, \ldots, m$$

or equivalently

$$\log I(\lambda_j) \approx c + da_j + \epsilon_j \quad j = 1, \ldots, m. \tag{11}$$

with $\epsilon_j = \log \frac{I(\lambda_j)}{f(\lambda_j)}$, $c = \log G$ and $a_j = -2 \log \lambda_j$.
Since the random variables $I(\lambda_j)/f(\lambda_j)$ are, at least for $d = 0$, asymptotically i.i.d. distributed, eq. (11) represents a linear regression model, with regressand $\log I(\lambda_j)$, regressors $-2 \log \lambda_j$ and slope d, which can be estimated by ordinary least squares (OLS). This is exactly the GPH estimator, except that here the quantity $2\sin(\lambda_j/2)$ is replaced by its first order Taylor expansion λ_j.
Robinson (1995b) proposed to trim the very low l frequencies and to consider the logs of a pooled periodogram of J periodogram values as the dependent variable in the log-periodogram regression. To be more precise, let J and $(m - l)/J$ be integers and

$$Y_h^{(J)} = \log \left(\sum_{j=1}^{J} I(\lambda_{h+j-J}) \right), \quad h = l + J, l + 2J, \ldots, m.$$

The parameter d is now estimated by OLS from the regression

$$Y_h^{(J)} = c^{(J)} + da_h + \epsilon_h^{(J)}, \quad h = l + J, l + 2J, \ldots, m$$

and therefore we have

$$\widehat{d} = \left(\sum_{h=l(J)}^{m} W_h^2 \right)^{-1} \sum_{h=l(J)}^{m} W_h Y_h^{(J)}, \tag{12}$$

where

$$W_h = a_h - \bar{a}, \quad \bar{a} = \frac{J}{m-l} \sum_{h=l(J)}^{m} a_h.$$

When both l and m tend to infinity with the sample size n but more slowly and under the assumption of Gaussianity, Robinson (1995b) has derived the limiting distribution for the GPH estimator $\widehat{d}$:

$$\sqrt{m}\left(\widehat{d} - d\right) \rightarrow_d N\left(0, \frac{\pi^2}{24}\right). \tag{13}$$

If in the log-periodogram regression we use $A_j = -\log\left[4\sin^2(\lambda_j/2)\right]$ as regressors,

$$\left(\widehat{d} - d\right) \rightarrow_d N\left(0, \sigma^2_{\text{GPH}}\right),$$

with

$$\sigma_{\text{GPH}} = \sqrt{\frac{\pi^2}{6\sum_{j=l+1}^{m}(A_j - \bar{A})^2}}$$

where $\bar{A} = \frac{1}{m-l}\sum_{j=l+1}^{m} A_j$. This suggest how to construct a test statistic for

$$\text{H}_0 : \widehat{d} = d.$$

The test statistic is

$$t_d = \frac{\widehat{d}(X) - d}{\sigma_{\text{GPH}}} \tag{14}$$

or

$$\tau_d = \frac{\widehat{d-1}(\Delta X) + 1 - d}{\sigma_{\text{GPH}}} \tag{15}$$

and it has to be compared with standard normal percentiles.

This model easily extends to the multivariate case. Suppose that now X_t is a K-dimensional vector, whose k-th component is X_{kt}. We assume that for $k = 1, \ldots, K$

$$f_{kk}(\lambda) \sim G_k \lambda^{-2d_k} \quad \text{as} \quad \lambda \to 0^+,$$

where $-1/2 < d_k < 1/2$ and $f_{kk}(\cdot)$ denotes the spectral density of the k-th series. Now we have K regressions

$$Y_{kh}^{(J)} = c_k^{(J)} + d_k a_h + \epsilon_{kh}^{(J)},$$
$$h = l+J, l+2J, \ldots, m \quad k = 1, \ldots, K,$$

where

$$Y_{kh}^{(J)} = \log\left(\sum_{j=1}^{J} I_{kk}(\lambda_{h+j-J})\right), \quad h = l+J, l+2J, \ldots, m \quad k = 1, \ldots, K,$$

and

$$I_{kk}(\lambda) = \frac{1}{2\pi n}\left|\sum_{t=1}^{n} X_{kt} \mathrm{e}^{\mathrm{i}\lambda t}\right|^2 \quad k = 1, \ldots, K. \tag{16}$$

Denoting by $\widehat{\mathbf{d}}$ and $\widehat{\mathbf{c}}$ the estimates for $\mathbf{d} = (d_1, \ldots, d_K)'$ and $\mathbf{c} = \left(c_1^{(J)}, \ldots, c_K^{(J)}\right)'$ respectively, it follows

$$\begin{pmatrix} \widehat{\mathbf{c}} \\ \widehat{\mathbf{d}} \end{pmatrix} = \operatorname{vec}\left(Y^{(J)'} S(S'S)^{-1}\right) \tag{17}$$

where $S = (S_{l+J}, S_{l+2J}, \ldots, S_m)'$and $Y^{(J)} = (Y_1^{(J)}, \ldots, Y_K^{(J)})$, with $S_h = (1, a_h)'$ and $Y_k^{(J)} = \left(Y_{k,l+J}^{(J)}, Y_{k,l+2J}^{(J)}, \ldots, Y_{k,m}^{(J)}\right)'$, $k = 1, \ldots, K$.

Once again, for Gaussian stationary and invertible time series the estimator $\widehat{\mathbf{d}}$ is consistent and asymptotically multivariate normal. Thus it becomes straightforward to test the homogeneous restriction

$$\mathrm{H}_0 : P\mathbf{d} = \mathbf{0}$$

where P is a $A \times K$ matrix of rank $A < K$. The test statistic is

$$\widehat{\mathbf{d}}' P' \left[[0, P]\{(S'S) \otimes \widehat{\Omega}^{-1}\}(0, P)'\right]^{-1} P\widehat{\mathbf{d}},$$

where $\widehat{\Omega} = \frac{J}{m-l}\sum_{h=l(J)}^{m} \widehat{\epsilon_h^{(J)}}\left(\widehat{\epsilon_h^{(J)}}\right)'$, $\widehat{\epsilon_h^{(J)}} = \left(\widehat{\epsilon_{1h}^{(J)}}, \ldots, \widehat{\epsilon_{Kh}^{(J)}}\right)'$ are the residuals of the log-periodogram regression and $\otimes$ denotes the Kronecker product. The test statistics has asymptotic χ_A^2 under H_0. For instance it could be useful to test if the difference parameter is common to every series. In this case P is the $(K-1) \times (K+1)$ matrix

$$P = \begin{pmatrix} 1 & -1 & 0 & \cdots & \cdots & \cdots \\ 0 & 1 & -1 & 0 & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & \cdots & 0 & 1 & -1 \end{pmatrix}.$$

Basically the same hypothesis is tested in Nielsen (2005). He proposed a Lagrange multiplier (LM) test for the K-dimensional series $\{X_t, t = 1, \ldots, n\}$ generated by

$$\Delta^{d+\theta} X_t = e_t \mathbb{I}\,(t \geq 1) \equiv e_t^{\#} \quad t = 0, \pm 1, \pm 2, \ldots, \tag{18}$$

where $\mathbb{I}(\cdot)$ denotes the indicator function. This definition (type II $I(d+\theta)$ process) entails that the results that follow are valid for all d and θ, whereas for a type I process they would be valid only for $d+\theta \in (-1/2, 1/2)$. The difference between the two types of processes is discussed in Robinson (2005).

The hypothesis to be tested is

$$\mathrm{H}_0 : \theta = 0. \tag{19}$$

From (18), the Gaussian log-likelihood function is

$$L(\theta, \Sigma) = -\frac{n}{2} \log (2\pi|\Sigma|) - \frac{1}{2} \sum_{t=1}^{n} \Delta^{d+\theta} X_t' \Sigma^{-1} \Delta^{d+\theta} X_t.$$

Denoting by $\eta = (\mathrm{vec}(\Sigma)', \theta')'$, the multivariate LM test statistic for testing H_0 is

$$LM = \left. \frac{\partial L(\eta)}{\partial \eta'} \right|_{\theta=0, \Sigma=\widehat{\Sigma}} \left[- \left. \frac{\partial^2 L(\eta)}{\partial \eta \partial \eta'} \right|_{\theta=0, \Sigma=\widehat{\Sigma}} \right]^{-1} \left. \frac{\partial L(\eta)}{\partial \eta} \right|_{\theta=0, \Sigma=\widehat{\Sigma}}.$$

Since it can be shown that

$$\left. \frac{\partial L(\theta, \Sigma)}{\partial \theta} \right|_{\theta=0, \Sigma=\widehat{\Sigma}} = \mathrm{tr} \left(\widehat{\Sigma}^{-1} S_{10} \right)$$

and that the relevant block in the Hessian matrix is

$$- \left. \frac{\partial^2 L(\theta, \Sigma)}{\partial \theta^2} \right|_{\theta=0, \Sigma=\widehat{\Sigma}} = \mathrm{tr} \left(\widehat{\Sigma}^{-1} M_{11} \right)$$

it follows that

$$LM = \frac{\mathrm{tr} \left(\widehat{\Sigma}^{-1} S_{10} \right)^2}{\mathrm{tr} \left(\widehat{\Sigma}^{-1} M_{11} \right)}. \tag{20}$$

Consider the processes $Z_t = \Delta^d X_t$, $Z_{t-1}^* = \sum_{j=1}^{t-1} j^{-1} Z_{t-j}$ and $Z_{t-2}^{**} = \sum_{j=1}^{t-2} j^{-1} Z_{t-j-1}^*$ and define $Z = (Z_1, \ldots, Z_n)'$, $Z^* = \left(Z_1^*, \ldots, Z_{n-1}^*\right)'$, $Z^{**} = \left(Z_1^{**}, \ldots, Z_{n-2}^{**}\right)'$, $\underline{Z} = (Z_2, \ldots, Z_n)'$ and $\underline{\underline{Z}} = (Z_3, \ldots, Z_n)'$. The matrices used in (20) are defined as follows: $\widehat{\Sigma} = \frac{1}{n} Z'Z$ is a consistent estimate of the covariance matrix Σ, $S_{10} = \sum_{t=2}^{n} Z_{t-1}^* Z_t' = Z^{*\prime} \underline{Z}$, $M_{11} = S_{11} + \frac{1}{2}(S_{20} + S_{20}')$, with $S_{11} = Z^{*\prime} Z^*$ and $S_{20} = \sum_{t=3}^{n} Z_{t-2}^{**} Z_t' = Z^{**\prime} \underline{\underline{Z}}$. If the assumption of Gaussianity is made, under the null the test statistic (20) is asymptotically χ_1^2.

The test can be readily extended to the case of different θ for each variable. Suppose now that

$$\Delta^{d_k + \theta_k} X_{kt} = e_{kt}^{\#}, \quad k = 1, \ldots, K, \quad t = 0, \pm 1, \pm 2, \ldots. \tag{21}$$

Now $\theta = (\theta_1, \ldots, \theta_K)'$ is a K-vector. Denote by $L_K(\theta, \Sigma)$ the log-likelihood for Gaussian data generated by (21). It can be shown that

$$\left. \frac{\partial L_K(\theta, \Sigma)}{\partial \theta} \right|_{\theta=0, \Sigma=\widehat{\Sigma}} = J_K' \, \mathrm{vec} \left(\widehat{\Sigma}^{-1} S_{10}' \right)$$

and

$$-\left.\frac{\partial^2 L_K(\theta,\Sigma)}{\partial\theta\partial\theta'}\right|_{\theta=0,\Sigma=\widehat{\Sigma}} = S_{11}\odot\widehat{\Sigma}^{-1}+\left(\widehat{\Sigma}^{-1}S'_{20}\right)\odot I_K$$

where $\odot$ denotes the Hadamard product, I_K is the K-dimensional identity matrix and J_K is the $K^2\times K$ $J_K=(\mathrm{vec}\,E_{11},\ldots,\mathrm{vec}\,E_{KK})$, $E_{ii}=e_ie_i'$, where e_i is the i-th unit K-vector. Nielsen (2005) shows that, if the assumption of Gaussianity is made, under the null (19), the test statistic

$$LM_K=\mathrm{vec}\left(\widehat{\Sigma}^{-1}S'_{10}\right)' J_K\left(S_{11}\odot\widehat{\Sigma}^{-1}+\left(\widehat{\Sigma}^{-1}S'_{20}\right)\odot I_K\right)^{-1} J_K'\,\mathrm{vec}\left(\widehat{\Sigma}^{-1}S'_{10}\right)$$

is asymptotically χ^2_K. This due to the fact that now the number of restriction tested is K. The test can accommodate also deterministic trends. Suppose that we observe instead of X_t the series

$$X_t^0=\beta z_t+X_t$$

where z_t is a vector of deterministic components, for instance $z_t=1$ or $z_t=(1,t)'$. Assuming that the matrix $\sum_{t=1}^n\tilde{z}_t\tilde{z}_t'$ is positive definite for n sufficiently large, where $\tilde{z}_t=\Delta^d z_t$, we can estimate β by OLS regressing Z_t on $\tilde{z}_t$. The test statistic is then based on the residuals $\tilde{X}_t=X_t^0-\hat{\beta}z_t$.

If one wants to allow for short run dynamics, the following vector autoregressive (VAR) could be considered:

$$\Phi(L)e_t=\varepsilon_t \quad t=0,\pm1,\pm2,\ldots$$

where ε_t is $I(0)$, $\Phi(z)$ is a matrix polynomial of order p such that $\Phi(1)$ has full rank and e_t, the process in (18), is stationary. Now the test is based on the residuals from the regression

$$Z_t=\hat{\Phi}_1Z_{t-1}+\ldots+\hat{\Phi}_pZ_{t-p}+\hat{\varepsilon}_t \quad t=p+1,\ldots,n.$$

Let $\hat{\varepsilon}^*_{t-1}=\sum_{j=1}^{t-1}j^{-1}\hat{\varepsilon}_{t-j}$, $\hat{\varepsilon}^{**}_{t-2}=\sum_{j=1}^{t-2}j^{-1}\hat{\varepsilon}^*_{t-j-1}$, $\hat{\varepsilon}=(\hat{\varepsilon}_{p+1},\ldots,\hat{\varepsilon}_n)'$, $\hat{\varepsilon}^*=\left(\hat{\varepsilon}^*_{p+1},\ldots,\hat{\varepsilon}^*_{n-1}\right)'$ and $\hat{\varepsilon}^{**}=\left(\hat{\varepsilon}^{**}_{p+1},\ldots,\hat{\varepsilon}^{**}_{n-2}\right)'$. Define also $\underline{\hat{\varepsilon}}=(\hat{\varepsilon}_{p+2},\ldots,\hat{\varepsilon}_n)'$ and $\underline{\underline{\hat{\varepsilon}}}=(\hat{\varepsilon}_{p+3},\ldots,\hat{\varepsilon}_n)'$ and consider for $t=p+1,\ldots,n$ the pK-vector $\tilde{Z}_{t-1}=\left(Z'_{t-1},\ldots,Z'_{t-p}\right)'$ and the $pK\times(n-p)$ matrix $\tilde{Z}=\left(\tilde{Z}_{p+1},\ldots,\tilde{Z}_n\right)'$. Now the test statistics are

$$LM(p)=\frac{\mathrm{tr}\left(\widehat{\Sigma}^{-1}\hat{S}_{10}\right)^2}{\mathrm{tr}\left(\widehat{\Sigma}^{-1}M_{11}-\hat{S}'_{Z1}S_{ZZ}^{-1}\hat{S}_{Z1}\right)} \tag{22}$$

$$LM_K(p)=\mathrm{vec}\left(\widehat{\Sigma}^{-1}\hat{S}'_{10}\right)' J_K\left(\hat{S}_{11}\odot\widehat{\Sigma}^{-1}+\left(\widehat{\Sigma}^{-1}\hat{S}'_{20}\right)\odot I_K-\left(\hat{S}'_{Z1}S_{ZZ}^{-1}\hat{S}_{Z1}\right)\odot\widehat{\Sigma}^{-1}\right)^{-1}$$

$$\times J_K'\,\mathrm{vec}\left(\widehat{\Sigma}^{-1}\hat{S}'_{10}\right) \tag{23}$$

with $\widehat{\Sigma} = \frac{1}{n-p}\hat{\varepsilon}'\hat{\varepsilon}$, $\hat{S}_{10} = \hat{\varepsilon}^{*\prime}\underline{\hat{\varepsilon}}$, $\hat{S}_{11} = \hat{\varepsilon}^{*}\hat{\varepsilon}^{*\prime}$, $\hat{S}_{20} = \hat{\varepsilon}^{**\prime}\underline{\hat{\varepsilon}}$, $\hat{M}_{11} = \hat{S}_{11} + \frac{1}{2}(\hat{S}_{20} + \hat{S}_{20}')$, $S_{ZZ} = \tilde{Z}\tilde{Z}'$ and $\hat{S}_{Z1} = \sum_{t=p+2}^{n-1} \tilde{Z}_{t-1}\hat{\varepsilon}_{t-1}^{*} = \underline{\tilde{Z}}\hat{\varepsilon}^{*\prime}$ where in the last equality we defined $\underline{\tilde{Z}} = \left(\tilde{Z}_{p+1}, \ldots, \tilde{Z}_{n-1}\right)'$.

As discussed in Velasco (2000), for $0 < d < 1/2$ the log-periodogram regression estimator is still consistent even when the data is not Gaussian. Instead of Gaussianity a fourth order stationary linear process condition is required. To achieve asymptotic normality also tapering is needed. Define the tapered discrete Fourier transform which uses the full cosine bell as

$$w_k^g(\lambda) = \left(2\pi \sum_{t=1}^{n} g_t^2\right)^{-1/2} \sum_{t=1}^{n} g_t X_{kt} \mathrm{e}^{\mathrm{i}\lambda t} \quad k = 1, \ldots, K,$$

where

$$g_t = \frac{1}{2}\left\{1 - \cos\left(\frac{2\pi t}{n}\right)\right\}.$$

Note that $\sum_{t=1}^{n} g_t = \frac{1}{2}$ and $\sum_{t=1}^{n} g_t^2 = \frac{3}{8}n$. Next, the vector of difference parameter is estimated as in (17), except that the periodogram in (16) is replaced by

$$I_{kk}^g(\lambda) = \frac{1}{2\pi n}\left|w_k^g(\lambda)\right|^2 \quad k = 1, \ldots, K. \tag{24}$$

The use of (24) reduces the bias of the periodogram on the tails, because tapering downweights the observations at both extremes of the sample and does not change the central part.

The asymptotic theory developed by Robinson (1995b) for stationary $I(d)$ processes has been extended to non-stationary $(d \geq 1/2)$ and non-invertible $(d \leq -1/2)$ series by Velasco (1999a,b). Consider $X_t \sim I(d)$ with $1/2 + \gamma \leq d < 3/2 + \gamma$, with γ integer. Then we will apply the log-periodogram regression to the series $W_t = \Delta^\gamma X_t$. Denote by $\widehat{d}(W)$ the corresponding estimate. The estimate for the original series will be

$$\widehat{d}(X) = \widehat{d}(W) + \gamma.$$

In other words, we first differenciate the original series γ times, apply the log-periodogram regression to the differencied series and finally adjust the resulting estimate with the number of differences taken. Basically the same idea applies to the case of non-invertible series, for which the data is first integrated and then the number of integrations taken is subtracted by the resulting estimate.

To capture also short range dependencies in the process X_t, Autoregressive Fractionally Integrated Moving Average (ARFIMA) processes have been proposed defined as

$$\phi(L) X_t \Delta^d = \theta(L) e_t, \tag{25}$$

where $\phi(L)$ and $\theta(L)$ involve autoregressive and moving average coefficients of order p and q respectively:

$$\phi(L) = 1 - \sum_{i=1}^{p} \phi_i L^i$$
$$\theta(L) = 1 + \sum_{i=1}^{q} \theta_i L^i,$$

with roots lying outside the unit circle. The requirement on $\phi(\cdot)$ implies stationarity and the one on $\theta(\cdot)$ implies invertibility. Using a, The spectral density is

$$f(\lambda;\zeta) = \frac{\sigma^2}{2\pi} \left|1 - \mathrm{e}^{\mathrm{i}\lambda}\right|^{-2d} \left|\frac{\theta(\mathrm{e}^{\mathrm{i}\lambda})}{\phi(\mathrm{e}^{\mathrm{i}\lambda})}\right|^2, \quad -\pi < \lambda \leq \pi,$$

where $\zeta = (d, \phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q, \sigma^2)'$. The vector of parameter ζ can be estimated by maximizing an approximate form of frequency-domain Gaussian likelihood. This amounts to minimize the Whittle objective function

$$L(\zeta) = \sum_{j=1}^{n-1} \left\{ \log f(\lambda_j;\zeta) + \frac{I(\lambda_j)}{f(\lambda_j;\zeta)} \right\}.$$

A problem with this approach is to choose the autoregressive and moving average orders p and q. Under- or over-specifying p and q could, indeed, lead to bias in the estimates of d. To overcome these difficulties we could use the local Whittle estimator, $\tilde{d}$ proposed by Künsch (1987), which is based on (10) and maximizes the frequency-domain Gaussian likelihood only for frequencies in the neighborhood of zero. The Whittle log-likelihood is $-m/2$ times

$$Q(G,d) = \frac{1}{m} \sum_{j=1}^{m} \left\{ \log\left[G\lambda_j^{-2d}\right] + \frac{\lambda_j^{2d}}{G} I(\lambda_j) \right\}.$$

Concentrating $Q(G,d)$ with respect to G entails the minimization of the function $Q(\hat{G},d)$, with

$$\hat{G} = \arg \min_{0<G<+\infty} Q(G,d) = \frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j).$$

The corresponding estimator is:

$$\tilde{d} = \arg \min_{-\frac{1}{2}<d<\frac{1}{2}} \left\{ \log\left[\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j)\right] - 2\frac{d}{m} \sum_{j=1}^{m} \log \lambda_j \right\}.$$

Robinson (1995a) shows that, if m goes to infinity but slower that n,

$$\sqrt{m}\left(\tilde{d} - d\right) \rightarrow_d N\left(0, \frac{1}{4}\right).$$

Thus, the test statistic for for

$$\mathrm{H}_0 : \tilde{d} = d$$

is

$$t_d = 2\sqrt{m}\left(\tilde{d}(X) - d\right) \tag{26}$$

or, for first differenced data,

$$\tau_d = 2\sqrt{m}\left(\widetilde{d-1}(\Delta X) + 1 - d\right) \tag{27}$$

and it has to be compared with standard normal percentiles.

Looking at (13), it is clear that the local Whittle estimator is asymptotically more efficient than the GPH estimator for the same choice of m.

A.2 Fractional Cointegration

Consider again the time series X_t and suppose that it is $I(d)$. Suppose that there exist α such that $\alpha' X_t$ is $I(\gamma)$ with $d \geq b$ positive real numbers. In this case we say that X_t is fractionally cointegrated and denoted by $CI(d, d-\gamma)$. Suppose that $X_t = (y_t, x_t')'$ is a $(K+1)$-vector with $x_t = (x_{1t}, \ldots, x_{Kt})'$.

It is common in the literature to compute first the residuals from an equilibrium relation between non-stationary series integrated of order $d \in (1/2, 3/2)$, and then to apply the methodologies proposed in subsection A.1, to the residuals (or to the differenced residuals). Let us start with the case $K = 1$. Suppose $x_t \equiv x_{1t} \sim I(d)$ and $y_t \sim I(d)$ are observable with $d \in (1/2, 3/2)$, and that $u_t \sim I(\delta)$, where

$$y_t = \beta x_t + u_t, \quad t = 1, \ldots, n$$

and $0 \leq \delta < d$. Consider a consistent estimator for β and denote by $\hat{u}_t$ the observed residuals. If the the equilibrium errors are stationary, $\delta \leq 1/2$, we can estimate δ as follows:

$$\widehat{\delta}(\hat{u}) = \left(\sum_{h=l+1}^{m} W_h^2\right)^{-1} \sum_{h=l+1}^{m} W_h \log I_{\hat{u}\hat{u}}(\lambda_h).$$

This is (12) with $J = 1$. For non-stationary residuls we have

$$\widehat{\delta}(\Delta\hat{u}) = 1 + \left(\sum_{h=l+1}^{m} W_h^2\right)^{-1} \sum_{h=l+1}^{m} W_h \log I_{\Delta\hat{u}\Delta\hat{u}}(\lambda_h).$$

Hassler et al. (2006) derive the condition under which these estimator are $\log n$-consistent and asymptotically normal, provided that $\hat{\beta}$ converges fast enough. In particular they assume that

$$\begin{cases} \hat{\beta} - \beta = \mathrm{O_P}(n^{\delta - d}) & \text{if } \delta + d \geq 1 \\ \hat{\beta} - \beta = \mathrm{O_P}(n^{1-2d}) & \text{if } \delta + d < 1 \end{cases}, \tag{28}$$

and

$$m \sim An^a, \quad l \sim Bn^b, \quad 0 < b < a < 1, \quad 0 < A, B < \infty. \tag{29}$$

Assumption (28) requires different rate of convergence for the β-estimator, depending on the overall memory of regressors and errors, $\delta + d$. In particular, when $\delta + d < 1$, a slower rate of convercence is required. Assumption (29) restricts the bandwidth numbers l and m to a power of n. Usually only small values are chosen for l ($l = 0$ or $l = 1$.)

They show[†] that for Gaussian u_t and x_t, under (28) and (29) as $n \to \infty$

$$\begin{cases} \log n \left(\widehat{\delta}(\hat{u}) - \delta \right) \to_p 0 & \text{if } 0 \leq \delta < 1/2 \text{ and } \delta < d - 1/2 < 1 \\ \log n \left(\widehat{\delta}(\Delta\hat{u}) - \delta \right) \to_p 0 & \text{if } 1/2 < \delta < d - 1/2 < 1 \end{cases}.$$

Moreover asymptotic Normality as in (13) for both $\widehat{\delta}(\hat{u})$ and $\widehat{\delta}(\Delta\hat{u})$ is showed and, most important, $\log n$-consistency even for non-Gaussian data when the pooled-tapered estimator is used:

$$\begin{cases} \log n \left(\widehat{\delta}^{(J)}(\hat{u}) - \delta \right) \to_p 0 & \text{if } d + \delta < 1 \text{ and } 0 \leq \delta < d < 3/2 \\ \log n \left(\widehat{\delta}^{(J)}(\Delta\hat{u}) - \delta \right) \to_p 0 & \text{if } 1/2 < \delta < d < 3/2 \end{cases}, \quad \text{as } n \to \infty.$$

Note that $\widehat{\delta}(\hat{u})$ is consistent only if $0 \leq \delta < 1/2$ with $\delta < d - 1/2 < 1$ as for differenced residuals, when l and m are chosen appropriately. Note that assumption (28) is fulfilled when $\hat{\beta}$ is the OLS estimates for $\delta \in [0, 3/2) - \{1/2\}$. Eq. (28) is satisfied also the narrow band frequency domain LS (FDLS) estimator proposed in Robinson and Marinucci (2001) when a bandwidth is chosen appropriately. This improves the asymptotic and finite sample properties of OLS estimates. However it is important to notice that they define say that the process a_t, is $I(d)$ if there exist a zero mean scalar $I(0)$ process, ξ_t, $t \in \mathbb{Z}$, and a scalar μ such that

$$a_t = \mu + \Delta^{-d} \xi_t^{\#} \quad t \in \mathbb{Z}, \quad d > 0. \tag{30}$$

Let us present their estimator in our setting assuming now that $K \geq 1$. Suppose that

$$y_t = \beta' x_t + u_t = \sum_{i=1}^{K} \beta_i x_{it} + u_t, \quad t = 1, \ldots, n \tag{31}$$

for $x_{it} \sim I(d_i)$, $d_i \in (1/2, 3/2)$, $y_t \sim I(d_{\max})$ and $u_t \sim I(\delta)$, $0 \leq \delta < d_{\min}$ where $d_{\min} = \min_i d_i$ and $d_{\max} = \max_i d_i$. Note that the formulation (30) does not need the requirements on d_i because of the truncation. Thus

[†]Actually some more technical conditions than the ones presented here are required. (See Hassler et al., 2006, for the whole set of assumptions.)

following Robinson and Marinucci (2001) we can estimate β by the frequency domain least square statistic

$$\hat{\beta}_m = \hat{F}_{xx}(m)\hat{F}_{xy}(m), \quad 1 \leq m \leq n/2, \tag{32}$$

where for the column vector or scalar sequence b_t, $t = 1, \ldots, n$, possibly identical to a_t, $\hat{F}_{ab}(m)$ denotes the averaged cross-periodogram:

$$\hat{F}_{ab}(m) = 2\Re\left\{\frac{2\pi}{n}\sum_{j=1}^{m} I_{ab}(\lambda_j)\right\} - \frac{2\pi}{n} I_{ab}(\pi)\mathbb{I}\left(m = \frac{n}{2}\right),$$

and

$$I_{ab}(\lambda) = w_a(\lambda)w_b'(-\lambda)$$

is the cross-periodogram. Note that, denoting by $[\cdot]$ the integer part and by $\bar{a} = n^{-1}\sum_{t=1}^{n} a_t$ we have

$$\hat{F}_{ab}\left(\left[\frac{n}{2}\right]\right) = \frac{1}{n}\sum_{t=1}^{n}(a_t - \bar{a})(b_t - \bar{b})', \tag{33}$$

and thus

$$\hat{\beta}_{[n/2]} = \left(\sum_{t=1}^{n}(x_t - \bar{x})(x_t - \bar{x})'\right)^{-1}\sum_{t=1}^{n}(x_t - \bar{x})(y_t - \bar{y})'$$

is the OLS estimate with intercept. Moreover $\hat{F}_{ab}(m)$ can be looked at as the contribution from the first m frequencies to the mean-corrected sample covariance (33). If

$$\frac{1}{m} + \frac{m}{n} \to 0 \quad \text{as } n \to \infty \tag{34}$$

holds, the authors show that, denoting by $\hat{\beta}_{im}$ the i-th element of $\hat{\beta}_m$ one has for $d_i > 1/2$, $\delta \geq 0$, $d_i + \delta < 1$

$$\begin{cases} \hat{\beta}_{im} - \beta_i = \mathrm{O_P}(n^{\delta - d_i} m^{1 - d_{\min} - \delta}) \\ \hat{\beta}_{i[n/2]} - \beta_i = \mathrm{O_P}(n^{1 - d_{\min} - d_i}) \end{cases}, \quad i = 1, \ldots, K$$

which indicates that the FDLS estimator converges faster than OLS. Moreover, under (34) and when $d_1 = \cdots = d_K > 1/2$, $delta > 0$ and $d_i + \delta > 1$

$$\begin{cases} \hat{\beta}_{im} - \beta_i = \mathrm{O_P}(n^{\delta - d_i}) \\ \hat{\beta}_{i[n/2]} - \beta_i = \mathrm{O_P}(n^{\delta - d_i}) \end{cases}, \quad i = 1, \ldots, K$$

meaning that in this case, as long as at least an arbitrarily slowly increasing number m of frequencies is included, omission of higher frequencies does not change the speed of convergence.

When $-1/2 < \delta < 1/2$, to estimate the vector $(\delta, d_1 - 1, \ldots, d_K - 1)'$, derive the residuals from (31) and simply use (17) and thus solve $K + 1$ regressions with dependent variables $(\log I_{\hat{u}\hat{u}}(\lambda_j), \log I_{\Delta x_1 \Delta x_1}(\lambda_j), \ldots, \log I_{\Delta x_K \Delta x_K}(\lambda_j))$. When the residuals are likely to be non-stationary, estimate the vector $(\delta - 1, d_1 - 1, \ldots, d_K - 1)'$ by (17) and thus solve $K + 1$ regressions with dependent variables $(\log I_{\Delta\hat{u}\Delta\hat{u}}(\lambda_j), \log I_{\Delta x_1 \Delta x_1}(\lambda_j), \ldots, \log I_{\Delta x_K \Delta x_K}(\lambda_j))$. Define by $\widehat{\mathbf{d}}(\hat{u})$ and $\widehat{\mathbf{d}}(\Delta\hat{u})$ the estimates of the vector $(\delta, d_1, \ldots, d_K)'$ based on original and differenced residuals respectively. Assume that

$$\begin{cases} \hat{\beta}_i - \beta_i = \mathrm{O_P}(n^{\delta - d_i}) & \text{if } \delta + d_i \geq 1 \\ \hat{\beta}_i - \beta_i = \mathrm{O_P}(n^{1 - d_{\min} - d_i}) & \text{if } \delta + d_i < 1 \end{cases}, \quad i = 1, \ldots, K. \tag{35}$$

Clearly under assumptions (29)–(35), $\widehat{\delta}$ is still $\log n$-consistent and under the additional assumption of Normality

$$\begin{cases} 2\sqrt{m}\left(\widehat{\mathbf{d}}(\hat{u}) - \mathbf{d}\right) \to_d N(0, \Omega) & \text{if } 0 \leq \delta < 1/2 \\ 2\sqrt{m}\left(\widehat{\mathbf{d}}(\Delta\hat{u}) - \mathbf{d}\right) \to_d N(0, \Omega) & \text{if } 1/2 < \delta < 1 \end{cases}.$$

To test the null of cointegration in fractionally cointegrated model we could use the Lagrange multiplier test of Nielsen (2004). Suppose that $X_t = (y_t, x_t')'$, with $y_t \sim I(d)$, is generated by the fractionally cointegrated system

$$y_t = \beta' x_t + z_t \quad t = 1, 2, \ldots \tag{36}$$

$$\Delta^{\delta+\theta} z_t = u_{1t}^{\#} \quad t = 1, 2, \ldots \tag{37}$$

$$\Delta^{d} x_t = u_{2t}^{\#} \quad t = 1, 2, \ldots \tag{38}$$

where $\delta = d - b$ with $d \geq b \geq 3/4 + \epsilon$, for some $\epsilon > 0$ and $u_t = (u_{1t}, u_{2t}')'$ is an error component. Under the null

$$\mathrm{H}_0 : \theta = 0$$

X_t is $CI(d, b)$. Moreover we assume that

$$\phi(L) u_{1t} = e_{1t} \quad t = 1, 2, \ldots$$

$$\Phi(L) u_{2t} = e_{2t} \quad t = 1, 2, \ldots$$

where $\phi(z)$ and $\Phi(z)$ are polynomials of order p with coefficient gathered in $\gamma = (\gamma_1', \gamma_2')'$, $\Phi(1)$ has full rank, meaning that there is no cointegration among the components of x_t and $e_t = (e_{1t}, e_{2t}')' \sim \text{iid}(0, \Sigma)$ with

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & \sigma_{21}' \\ \sigma_{21} & \Sigma_{22} \end{bmatrix}.$$

Assuming Gaussianity of the errors the log-likelihood function is

$$L(\theta, \beta, \Sigma, \gamma) = -\frac{n}{2} \log |\Sigma| - \frac{1}{2} \sum_{t=1}^{n} \begin{pmatrix} \phi(L)\Delta^{\gamma+\theta} z_t \\ \Phi(L)\Delta^d x_t \end{pmatrix}' \Sigma^{-1} \begin{pmatrix} \phi(L)\Delta^{\gamma+\theta} z_t \\ \Phi(L)\Delta^d x_t \end{pmatrix}$$

Next define

$$e_{1.2t} = e_{1t} - \sigma_{21}' \Sigma_{22}^{-1} e_{2t}$$

which is e_{1t} centered about its mean conditional on e_{2t}, and the corresponding variance

$$\sigma_{1.2}^2 = \sigma_{11}^2 - \sigma_{21}' \Sigma_{22}^{-1} \sigma_{21}.$$

It can be shown that the MLE of β under the null can be obtained as the non-linear least squares estimator in the augmented regression

$$\Delta^\delta y_t = \beta' \Delta^\delta x_t + (1 - \phi(L))\Delta^\delta (y_t - \beta' x_t) + c' \Phi(L) \Delta^d x_t + e_{1.2t} \tag{39}$$

Thus the estimate we get is not the one we would have got estimating (36) by OLS. The two estimator coincide only if $\sigma_{21} = 0$ and $\phi(L) = 1$. When $\sigma_{21} \neq 0$ or $\phi(L) \neq 1$ the OLS estimator is biased because of endogeneity and serial correlation. When there is no autoregressive term in the equilibrium errors, i.e. when $\phi(L) = 1$, (39) reduces to

$$\Delta^\delta y_t = \beta' \Delta^\delta x_t + \sum_{k=0}^{p} c_k' \Delta^d x_{t-k} + e_{1.2t}. \tag{40}$$

The normalized score statistic under the null

$$S_n = \frac{1}{\sqrt{n}} \frac{\partial L(\theta, \beta, \Sigma, \gamma)}{\partial \theta} \bigg|_{\theta=0, \beta=\hat{\beta}, \Sigma=\widehat{\Sigma}, \gamma=\hat{\gamma}}$$

can be shown to be

$$S_n = \frac{1}{\sqrt{n}\hat{\sigma}_{1.2}^2} \sum_{t=1}^{n} \sum_{j=1}^{t-1} j^{-1} \hat{e}_{1,t-j} \hat{e}_{1.2t}$$

with

$$\hat{e}_{1t} = \hat{\phi}(L)\Delta^{\delta}(y_t - \beta' x_t)$$
$$\hat{e}_{1.2t} = \hat{\phi}(L)\Delta^{\delta}(y_t - \beta' x_t) - c'\hat{\Phi}(L)\Delta^{d} x_t$$
$$\hat{\sigma}_{1.2} = \sqrt{\frac{1}{n}\sum_{t=1}^{n} \hat{e}'_{1.2t}\hat{e}_{1.2t}}$$

A numerical approximation to the one sided test H_0 against $\mathrm{H}_1 : \theta > 0$ is

$$\widehat{LM} = \frac{\sqrt{n}\sum_{t=1}^{n}\sum_{j=1}^{t-1} j^{-1}\hat{e}_{1,t-j}\hat{e}_{1.2t}}{\sqrt{\sum_{t=1}^{n}\left(\sum_{j=1}^{t-1} j^{-1}\hat{e}_{1,t-j}\right)^2 \sum_{t=1}^{n}\hat{e}^2_{1.2t}}} \tag{41}$$

which is to be compared with the quantiles of the standard normal distribution.

A fractionally cointegrated system can be estimated using the results of Dittmann (2004), which proposes an alternative to Granger (1986) error correction model for fractionally integrated systems. We can use a three steps procedure. Basically the first two step coincide with the procedure presented before: the residuals from (31) are calculated and the difference parameter $\widehat{\delta}(\hat{u})$ estimated via the GPH method, for instance. In the third step one computes

$$\varpi_t \equiv \Delta^{\widehat{\delta}(\hat{u})}\hat{u}_t \quad t = 1, \ldots, n \tag{42}$$

and verifies that ϖ_t is a stationary process, using for instance the KPSS test.

Figure 1: Treasury, Aaa, Aa, A and Baa yields.

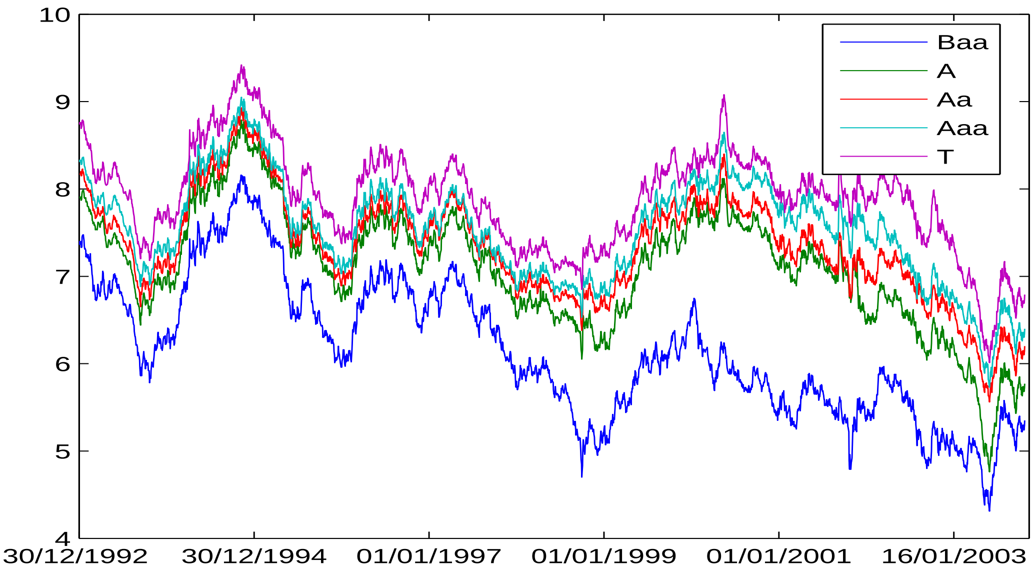


Figure 2: Aaa, Aa, A and Baa spreads over Treasury yields.

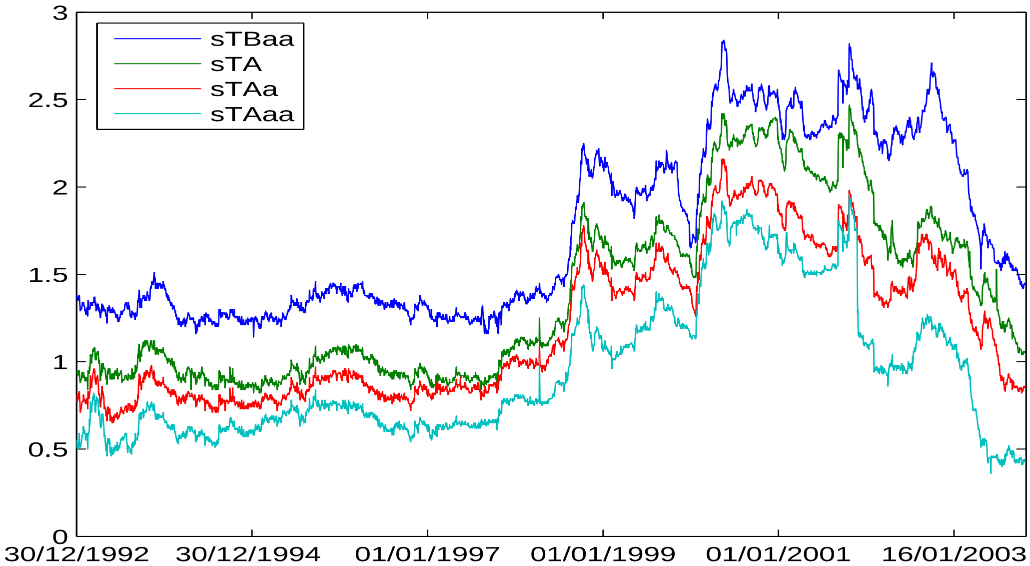


Figure 3: Spreads between corporate yields.

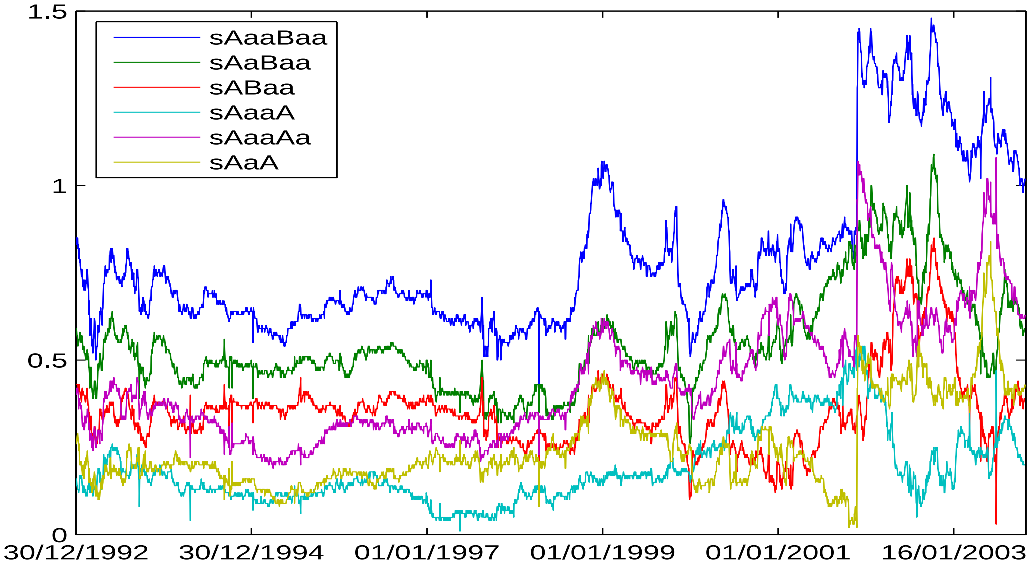


Figure 4: Treasury, Aaa, Aa, A and Baa yields.

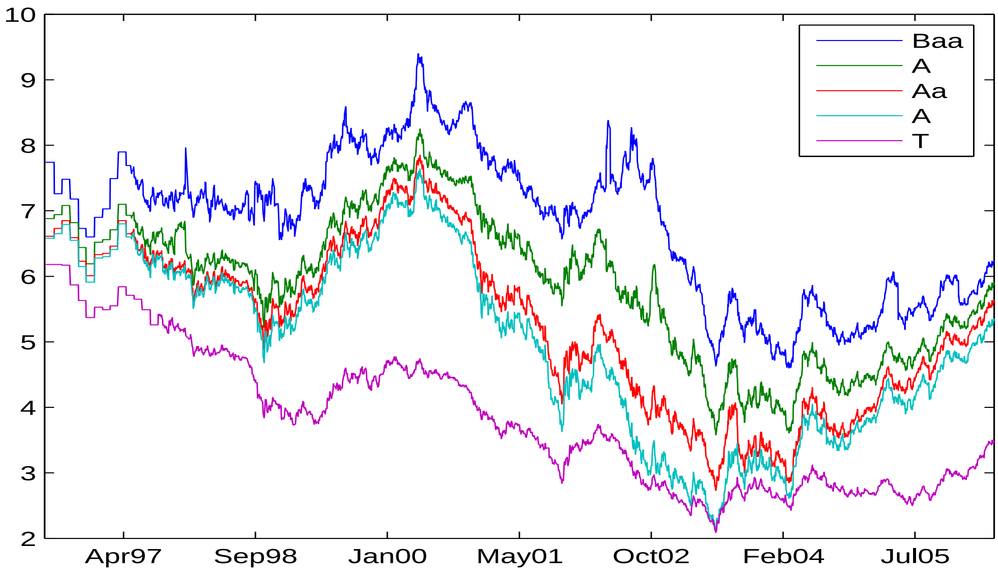


Figure 5: Aaa, Aa, A and Baa spreads over Treasury yields.

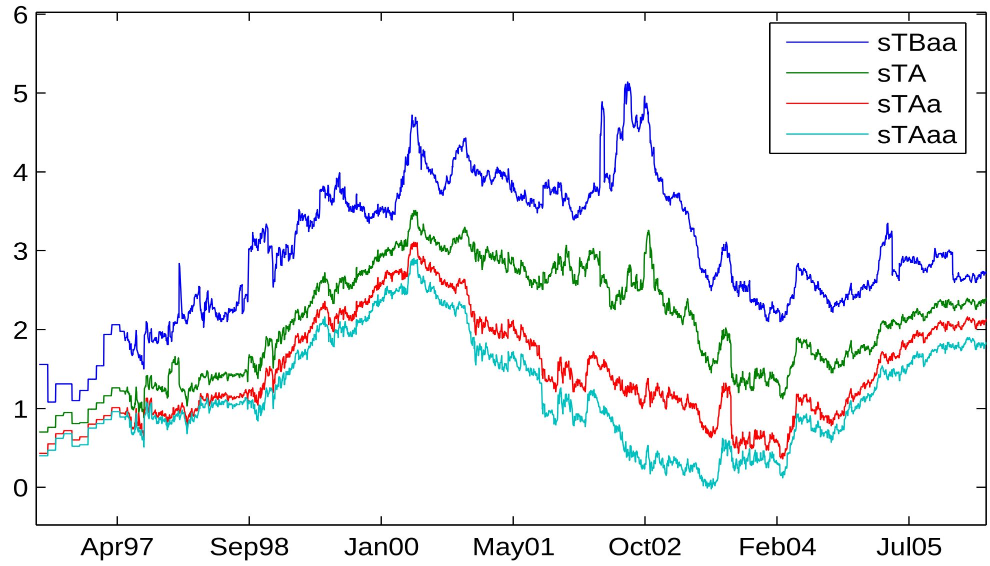


Figure 6: Spreads between corporate yields.

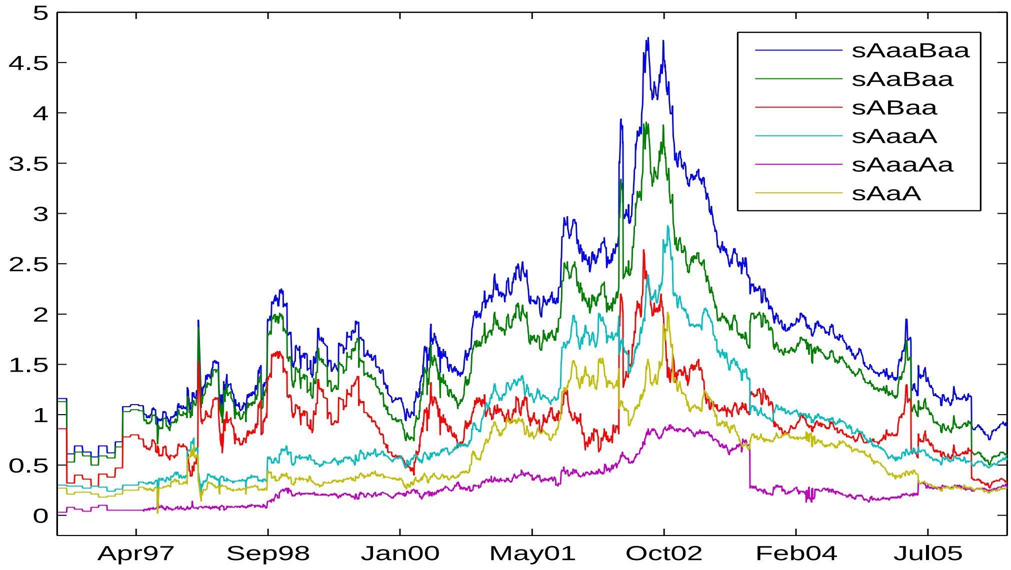
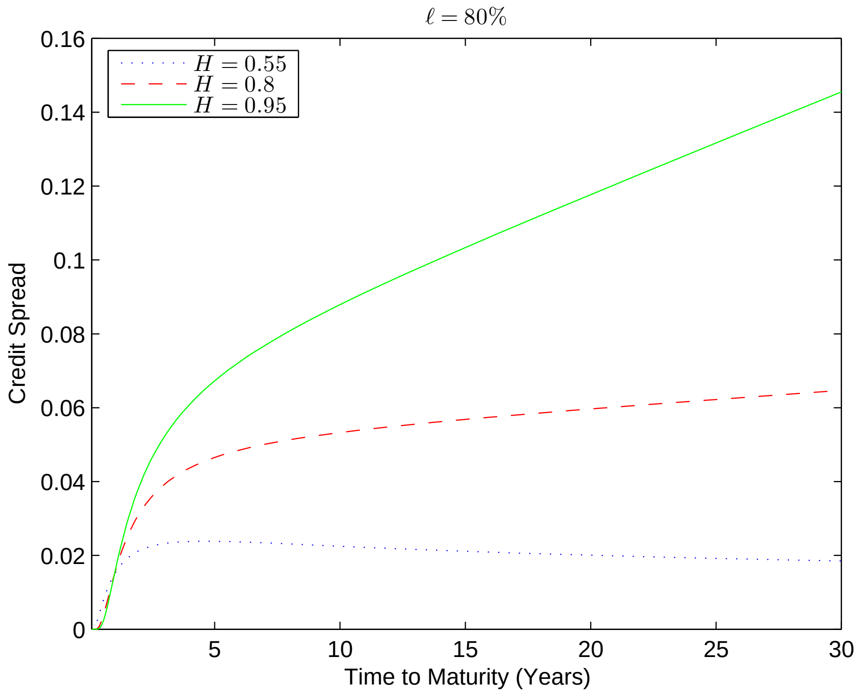
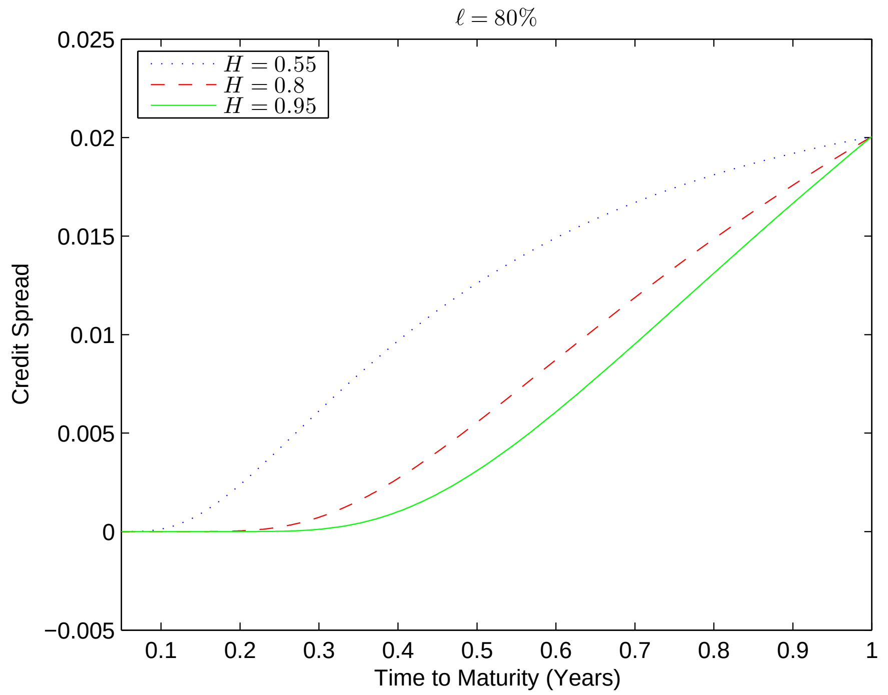


Figure 7: Credit spreads resulting in the fractional Merton model against maturity when $\sigma = 0.2$, $D/V = 0.8$, $r = 0$.

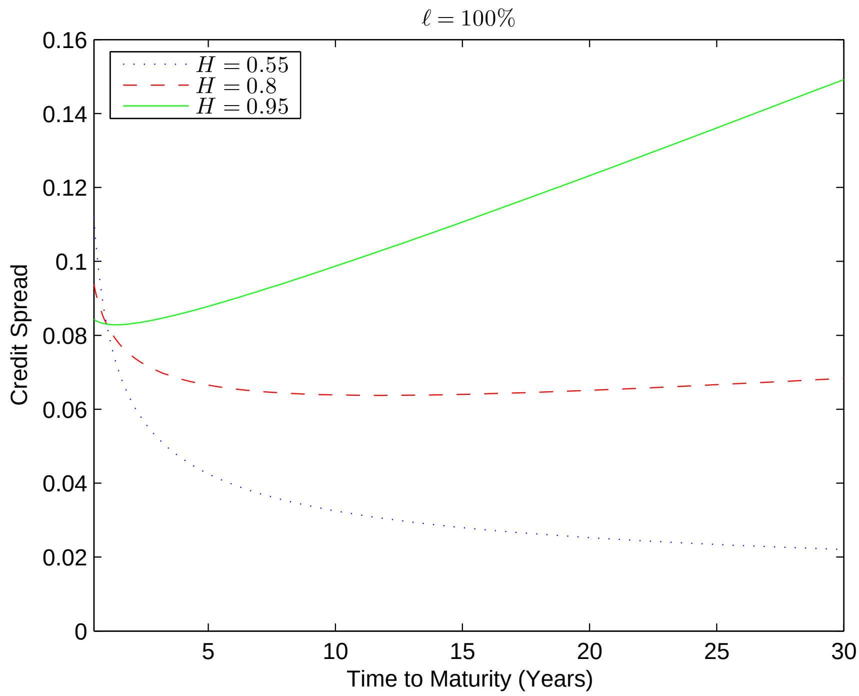


(a) $0 < T < 30$

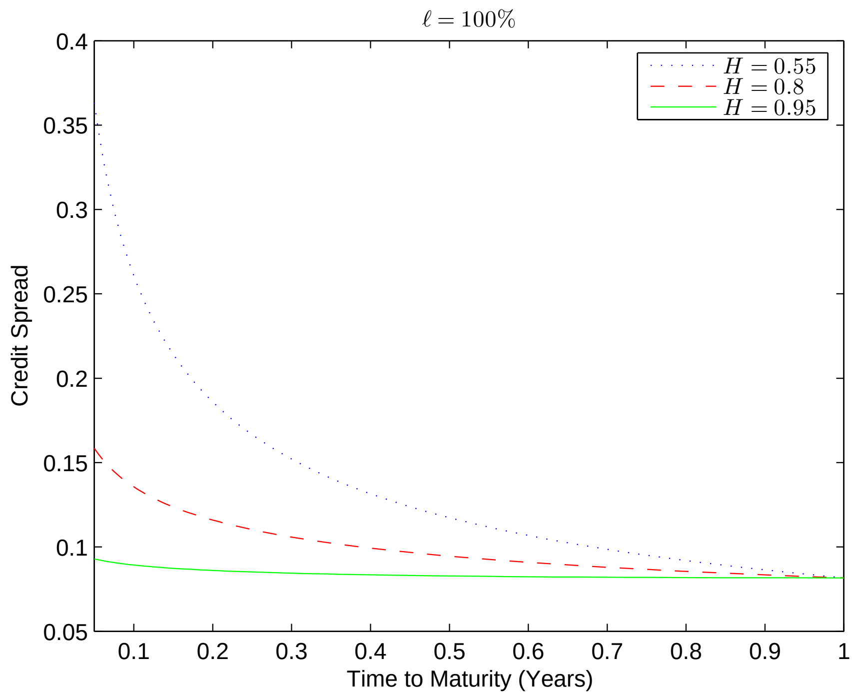


(b) $0 < T < 1$.

Figure 8: Credit spreads resulting in the fractional Merton model against maturity when $\sigma = 0.2$, $D/V = 1$, $r = 0$.

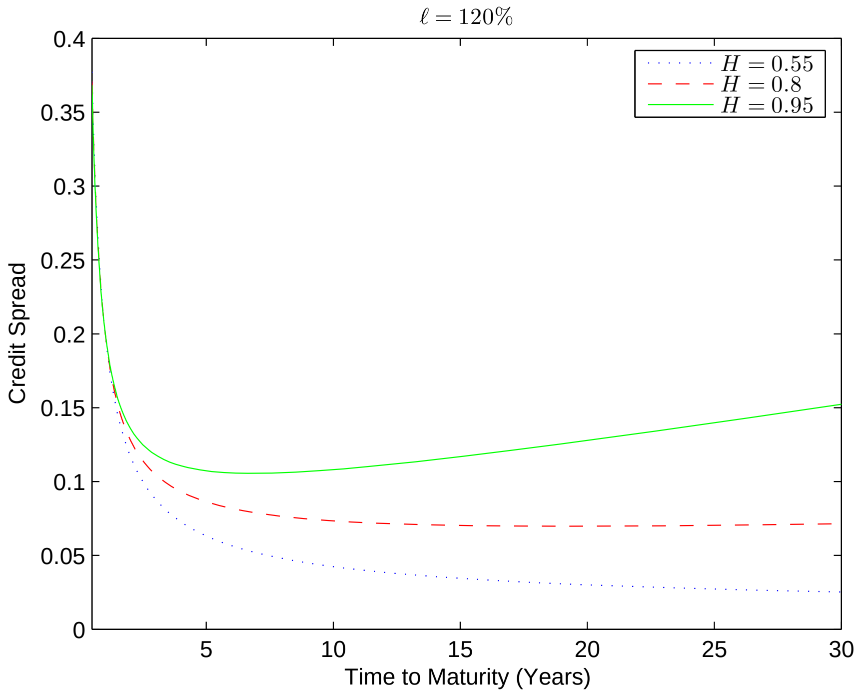


(a) $0 < T < 30$

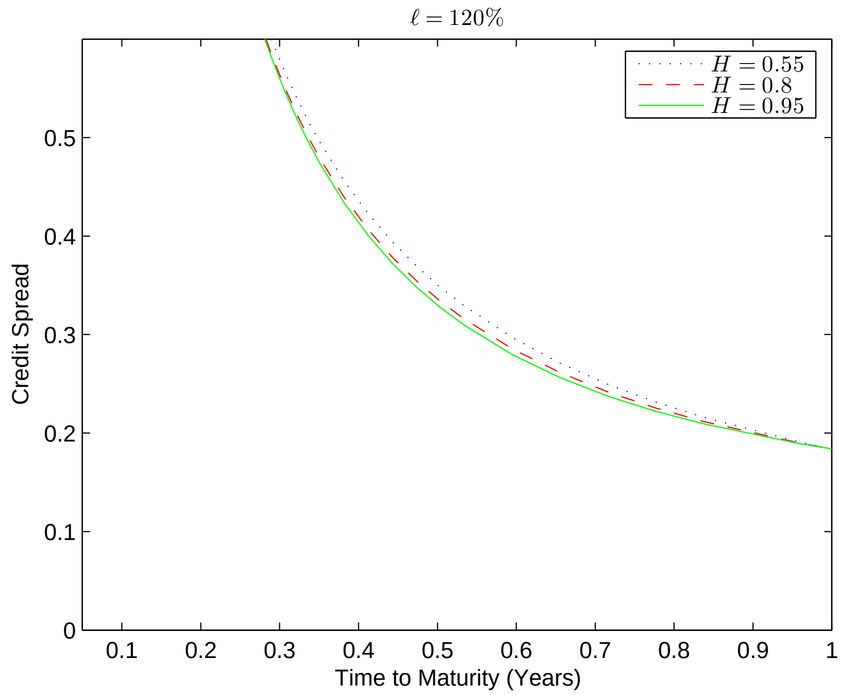


(b) $0 < T < 1$.

Figure 9: Credit spreads resulting in the fractional Merton model against maturity when $\sigma = 0.2$, $D/V = 1.2$, $r = 0$.

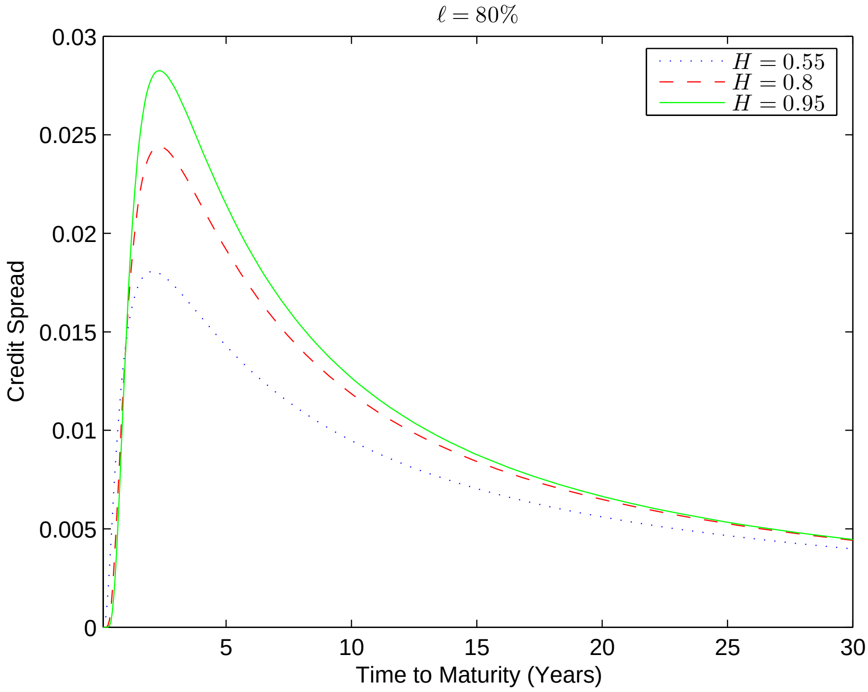


(a) $0 < T < 30$

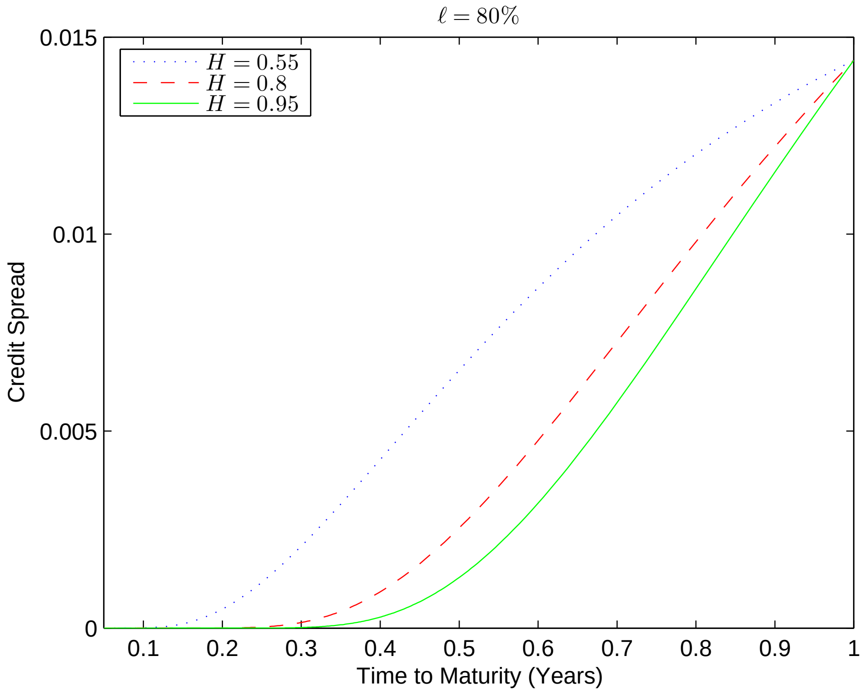


(b) $0 < T < 1$.

Figure 10: Credit spreads resulting in the fractional Black and Cox model against maturity when $\sigma = 0.2$, $V = 100$, $D = 80$ and $L = 70$.

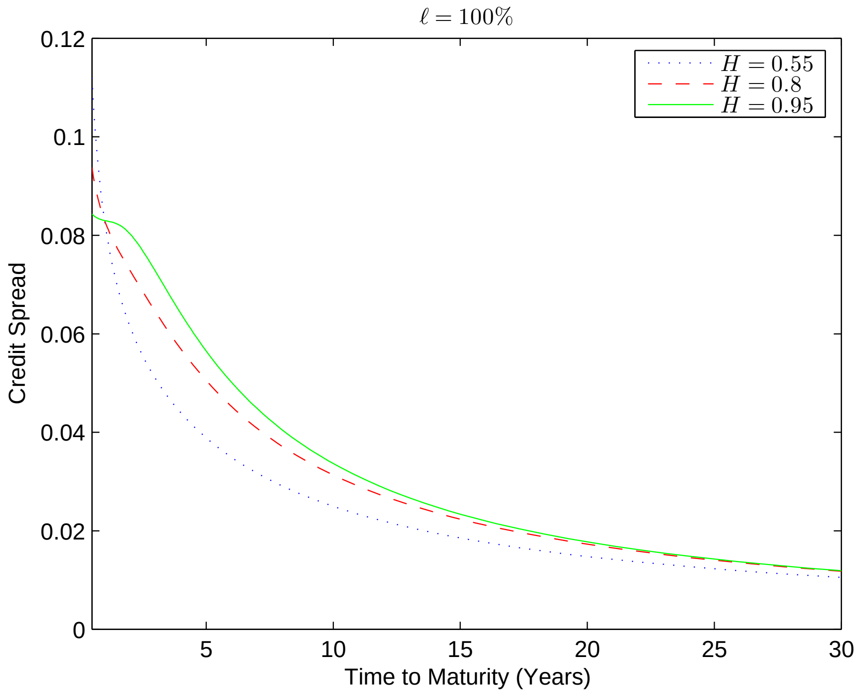


(a) $0 < T < 30$

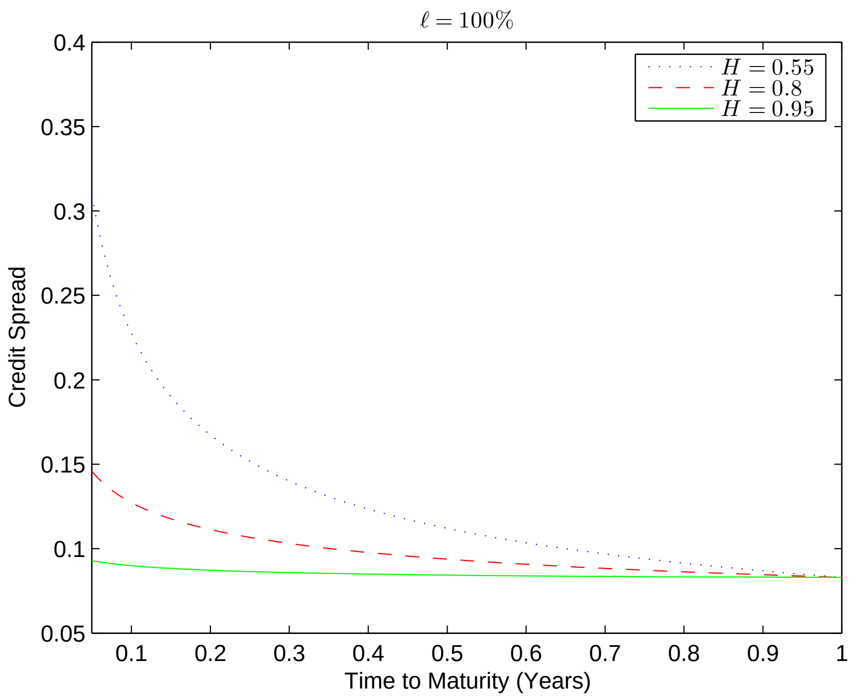


(b) $0 < T < 1$.

Figure 11: Credit spreads resulting in the fractional Black and Cox model against maturity when $\sigma = 0.2$, $V = 100$, $D = 100$ and $L = 70$.

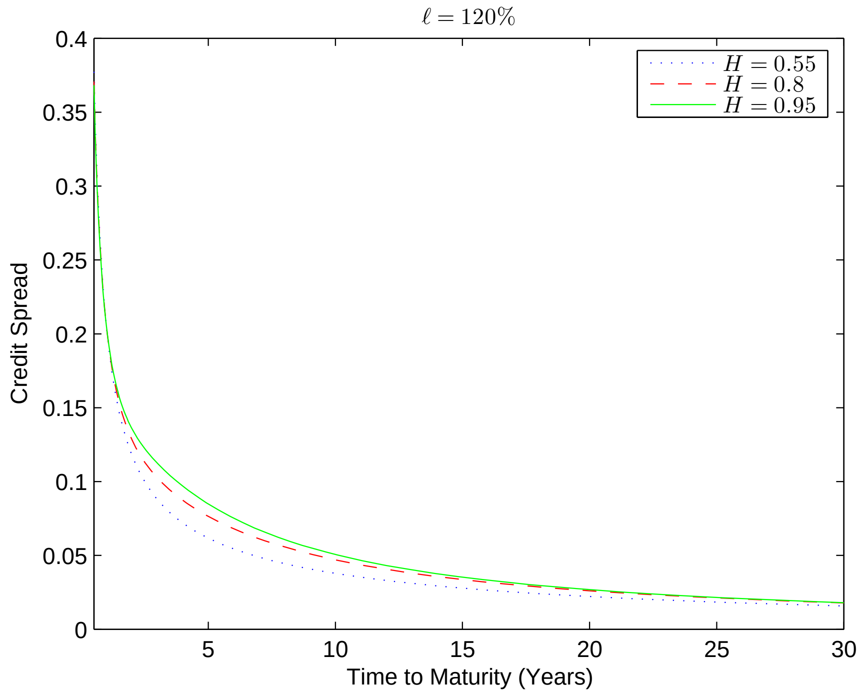


(a) $0 < T < 30$

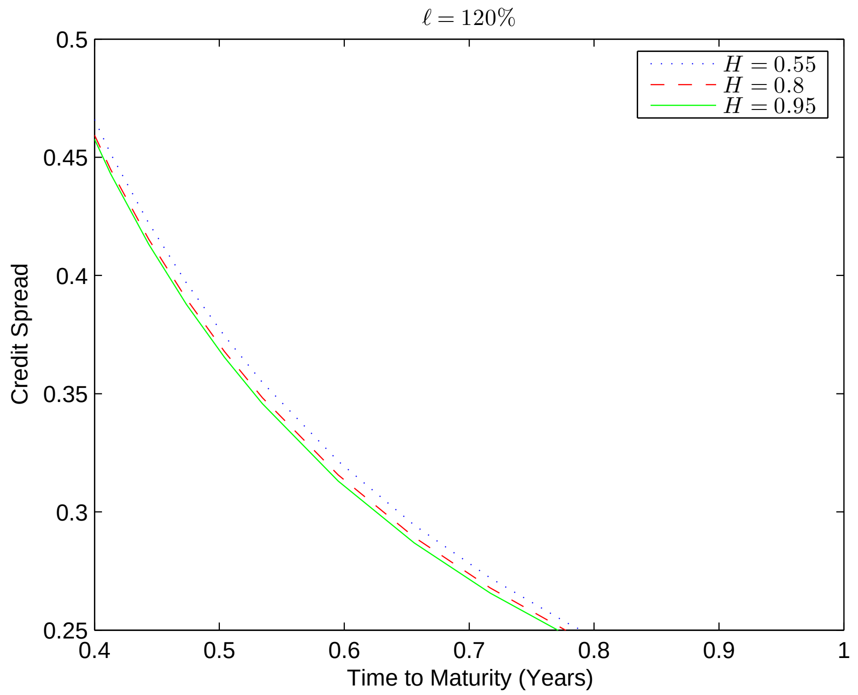


(b) $0 < T < 1$.

Figure 12: Credit spreads resulting in the fractional Black and Cox model against maturity when $\sigma = 0.2$, $V = 100$, $D = 120$ and $L = 70$.



(a) $0 < T < 30$



(b) $0 < T < 1$.

Table 1: Summary Statistics and Normality Tests for yields and spreads. Both the Jarque–Bera test and the Normality test proposed by Doornik and Hansen (1994) are computed. In both cases the null hypothesis is that the series is normally distributed and the test statistic is χ^2_2. The p-value is in square bracket. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	mean	std	skewness	ex. Kurtosis	JB test	DH test
T	6.1536	0.78947	0.30498	-0.57126	78.655 [0.0000]**	140.41 [0.0000]**
Aaa	7.0975	0.68853	-0.40296	0.37445	88.943 [0.0000]**	69.261 [0.0000]**
Aa	7.3463	0.60072	-0.12787	0.079963	8.0862 [0.0175]*	7.5408 [0.0230]*
A	7.5368	0.58879	-0.1924	-0.11221	18.095 [0.0001]**	21.806 [0.0000]**
Baa	7.8947	0.57677	-0.23461	0.29	34.269 [0.0000]**	26.234 [0.0000]**
sTAaa	0.94396	0.40039	0.85386	-0.47124	353.46 [0.0000]**	1291.5 [0.0000]**
sTAa	1.1927	0.41573	0.54433	-1.1281	276.81 [0.0000]**	934.56 [0.0000]**
sTA	1.3833	0.49232	0.61294	-1.0284	288.36 [0.0000]**	1033.2 [0.0000]**
sTBaa	1.7412	0.49927	0.51375	-1.3408	321.38 [0.0000]**	1197.4 [0.0000]**
sAaaAa	0.24873	0.12367	1.3572	2.0211	1289.9 [0.0000]**	1343 [0.0000]**
sAaaA	0.43932	0.17651	1.0234	0.68153	524.14 [0.0000]**	901.43 [0.0000]**
sAaaBaa	0.79721	0.2279	1.2453	0.5105	727.98 [0.0000]**	2312.4 [0.0000]**
sAaA	0.19058	0.10543	1.0911	0.49992	564.49 [0.0000]**	1330.5 [0.0000]**
sAaBaa	0.54848	0.14479	1.1672	1.0723	743.29 [0.0000]**	1182.2 [0.0000]**
sABaa	0.35789	0.11427	1.5693	3.7498	2693.1 [0.0000]**	1260.6 [0.0000]**

Table 2: Summary Statistics and Normality Tests for first differences of yields and spreads. Both the Jarque–Bera test and the Normality test proposed by Doornik and Hansen (1994) are computed. In both cases the null hypothesis is that the series is normally distributed and the test statistic is χ^2_2. The p-value is in square bracket. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	mean	std	skewness	ex. Kurtosis	JB test	DH test
ΔT	-0.00074	0.051925	0.30619	2.0323	507.20	254.79
					[0.0000]**	[0.0000]**
ΔAaa	-0.00079	0.048771	0.31655	2.6307	824.26	382.40
					[0.0000]**	[0.0000]**
ΔAa	-0.00073	0.046392	0.43888	1.9246	503.78	210.14
					[0.0000]**	[0.0000]**
ΔA	-0.00071	0.04719	0.43378	2.0412	553.84	230.94
					[0.0000]**	[0.0000]**
ΔBaa	-0.00072	0.047793	0.42962	1.9843	526.41	221.83
					[0.0000]**	[0.0000]**
ΔsTAaa	-4.81E-05	0.024443	-2.0015	55.643	350377.5	12448
					[0.0000]**	[0.0000]**
ΔsTAa	1.48E-05	0.022006	0.45617	8.3799	7999.7	1970.7
					[0.0000]**	[0.0000]**
ΔsTA	3.70E-05	0.023	0.57711	14.692	24451.1	3923.6
					[0.0000]**	[0.0000]**
ΔsTBaa	2.22E-05	0.022466	0.5992	9.0243	9330.2	2012.1
					[0.0000]**	[0.0000]**
ΔsAaaAa	6.29E-05	0.01623	5.8072	144.87	2.3780E+06	6833
					[0.0000]**	[0.0000]**
ΔsAaaA	8.51E-05	0.017467	4.9598	130.89	1.9397E+06	10617
					[0.0000]**	[0.0000]**
ΔsAaaBaa	7.03E-05	0.018208	4.3241	112.28	1.4277E+06	11489
					[0.0000]**	[0.0000]**
ΔsAaA	2.22E-05	0.013233	0.66513	52.071	3.0545E+05	16289
					[0.0000]**	[0.0000]**
ΔsAaBaa	7.40E-06	0.014959	-0.19375	10.042	11371	2719.2
					[0.0000]**	[0.0000]**
ΔsABaa	-1.48E-05	0.01553	-0.10372	28.283	90065	8920.3
					[0.0000]**	[0.0000]**

Table 3: Unit root and stationarity tests for yields and spreads. The Dickey-Fuller (DF) or augmented Dickey-Fuller with the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out. n is the lag length in the ADF and it is chosen by the AIC. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	DF-ADF	n	KPSS	KPSS HFO	PP
T	-1.8998	0	18.272**	4.8192**	-1.8811
Aaa	-1.6007	1	9.8415**	2.6134**	-1.5274
Aa	-1.8819	1	8.5263**	2.2743**	-1.771
A	-1.9308	1	6.0752**	1.623**	-1.8066
Baa	-2.0792	1	5.3553**	1.4406**	-2.0162
sTAaa	-1.4212	39	12.07**	3.16**	-1.2784
sTAa	-1.415	96	17.553**	4.5677**	-1.253
sTA	-1.4234	73	17.974**	4.6607**	-1.1279
sTBaa	-1.219	4	19.013**	4.9377**	-1.1825
sAaaAa	-2.37	41	11.285**	3.0905**	-2.8852**
sAaaA	-1.9799	1	18.582**	4.9404**	-2.0816
sAaaBaa	-1.6043	57	15.306**	4.0569**	-1.983
sAaA	-1.9006	10	12.082**	3.1866**	-2.2683
sAaBaa	-2.292	1	11.63**	3.121**	-2.4266
sABaa	-3.0041*	4	2.3387**	0.63969**	-3.1083**

Table 4: Unit root and stationarity tests for the first differences of yields and spreads. The Dickey-Fuller (DF) or augmented Dickey-Fuller with the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out. n is the lag length in the ADF and it is chosen by the AIC. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	DF-ADF	n	KPSS	KPSS HFO	PP
ΔT	-50.796**	0	0.038103	0.037796	-50.788**
ΔAaa	-50.254**	0	0.065246	0.06474	-50.225**
ΔAa	-49.933**	0	0.05892	0.05892	-49.894**
ΔA	-50.361**	0	0.066975	0.065249	-50.345**
ΔBaa	-50.116**	0	0.061583	0.059524	-50.087**
ΔsTAaa	-7.5675**	38	0.30846	0.3141	-63.294**
ΔsTAa	-5.8175**	95	0.2924	0.2924	-62.675**
ΔsTA	-5.3436**	72	0.34606	0.35011	-64.169**
ΔsTBaa	-23.651**	3	0.26461	0.24729	-59.851**
ΔsAaaAa	-9.4063**	40	0.035604	0.035017	-62.101**
ΔsAaaA	-63.273**	0	0.038956	0.036094	-63.162**
ΔsAaaBaa	-8.3088**	56	0.059622	0.059054	-58.503**
ΔsAaA	-18.297**	9	0.047834	0.051928	-73.938**
ΔsAaBaa	-60.905**	0	0.043784	0.044609	-60.648**
ΔsABaa	-64.597**	0	0.035307	0.032132	-64.219**

Table 5: d estimates for the yields with $l = 0$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	0.80087	0.97492	0.88708	0.97457	0.91151
		$\tau_{d=1}$	-1.2036	-0.15158	-0.68253	-0.15374	-0.53491
	$m = 51$	$\hat{d}(\Delta X)$	0.84594	0.94272	0.87103	0.91001	0.93016
		$\tau_{d=1}$	-1.5125	-0.56232	-1.2661	-0.88346	-0.68566
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	0.96895	1.0195	0.92161	0.97034	0.97393
		$\tau_{d=1}$	-0.48028	0.30185	-1.2127	-0.45883	-0.40322
	$m = 252$	$\hat{d}(\Delta X)$	0.9419	0.98311	0.92178	0.94766	0.99618
		$\tau_{d=1}$	-1.3749	-0.39957	-1.851	-1.2385	-0.090393
	$m = 556$	$\hat{d}(\Delta X)$	0.97927	0.99219	0.97855	0.98767	1.0323
		$\tau_{d=1}$	-0.7327	-0.27612	-0.75813	-0.43585	1.141
	$m = 22$	$\hat{d}(\Delta X)$	1.0896	1.1732	1.2483	1.2048	1.1473
		$\tau_{d=1}$	0.3291	0.63644	0.91224	0.75256	0.54123
	$m = 50$	$\hat{d}(\Delta X)$	0.91791	1.0427	1.0301	1.031	1.019
		$\tau_{d=1}$	-0.52307	0.27201	0.19174	0.19757	0.12082
$J = 2$	$m = 114$	$\hat{d}(\Delta X)$	1.0018	1.0379	1.0097	1.0277	1.0257
		$\tau_{d=1}$	0.01843	0.39641	0.10166	0.28978	0.2693
	$m = 252$	$\hat{d}(\Delta X)$	0.97895	1.0072	0.98284	0.99102	1.0219
		$\tau_{d=1}$	-0.343	0.11798	-0.27969	-0.14631	0.35682
	$m = 556$	$\hat{d}(\Delta X)$	0.99833	1.0078	1.0004	1.0049	1.0229
		$\tau_{d=1}$	-0.041077	0.19234	0.010702	0.12011	0.56376
	$m = 21$	$\hat{d}(\Delta X)$	0.87718	1.1111	1.0974	1.124	1.0247
		$\tau_{d=1}$	-0.32127	0.29064	0.25485	0.32441	0.064606
	$m = 51$	$\hat{d}(\Delta X)$	0.83253	0.99082	0.96514	0.97111	0.95098
		$\tau_{d=1}$	-0.83107	-0.04555	-0.17298	-0.14337	-0.24328
$J = 3$	$m = 114$	$\hat{d}(\Delta X)$	0.98173	1.0239	0.99281	1.0057	1.015
		$\tau_{d=1}$	-0.15055	0.19682	-0.059257	0.046875	0.12323
	$m = 252$	$\hat{d}(\Delta X)$	0.96576	0.99235	0.97424	0.97644	1.0134
		$\tau_{d=1}$	-0.44573	-0.099633	-0.33529	-0.30662	0.1748
	$m = 555$	$\hat{d}(\Delta X)$	0.97733	0.99088	0.98522	0.9814	1.0062
		$\tau_{d=1}$	-0.44888	-0.18057	-0.29255	-0.36831	0.12348

Table 6: d estimates for the spreads with $l = 0$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	0.92097	1.05	1.027	0.99384	0.67217	0.65317	0.90996	0.98424	0.87955	1.0693
		$\tau_{d=1}$	-0.47769	0.30223	0.16305	-0.037205	-1.9816*	-2.0965*	-0.54425	-0.095236	-0.7281	0.4191
	$m = 51$	$\hat{d}(\Delta X)$	1.0332	1.0306	1.072	1.1047	0.80634	0.82961	0.95578	0.994	0.81231	0.91661
		$\tau_{d=1}$	0.32557	0.30043	0.70673	1.0283	-1.9013	-1.6728	-0.43409	-0.058928	-1.8426	-0.81866
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.0458	0.99239	1.0342	1.0863	0.87859	0.91356	0.97636	0.99051	0.97782	0.88735
		$\tau_{d=1}$	0.70796	-0.11771	0.52927	1.3347	-1.8782	-1.3371	-0.36577	-0.14687	-0.34314	-1.7426
	$m = 252$	$\hat{d}(\Delta X)$	1.0549	1.0704	1.0868	1.1103	0.98959	1.0024	1.0507	0.96664	1.0448	0.97755
		$\tau_{d=1}$	1.2982	1.6663	2.0547*	2.6091**	-0.24635	0.057549	1.1992	-0.78952	1.0607	-0.53117
	$m = 556$	$\hat{d}(\Delta X)$	1.0348	0.99189	0.9903	1.0914	0.97802	0.96553	1.0733	0.90041	0.98604	0.98274
		$\tau_{d=1}$	1.2285	-0.28653	-0.34269	3.2291**	-0.77674	-1.2181	2.59**	-3.5191**	-0.49324	-0.61002
	$m = 22$	$\hat{d}(\Delta X)$	0.84579	0.83857	0.98118	0.92743	0.69372	0.60359	0.79276	0.98821	0.71874	0.85606
		$\tau_{d=1}$	-0.56658	-0.59311	-0.069149	-0.26662	-1.1253	-1.4565	-0.76143	-0.043302	-1.0334	-0.52884
	$m = 50$	$\hat{d}(\Delta X)$	1.0075	1.0331	1.1084	1.1012	0.80097	0.84875	0.95023	0.98799	0.77459	0.93089
		$\tau_{d=1}$	0.047495	0.21117	0.69051	0.6447	-1.2683	-0.96381	-0.31713	-0.07656	-1.4364	-0.44038
$J = 2$	$m = 114$	$\hat{d}(\Delta X)$	1.0607	1.0256	1.0642	1.0635	0.89574	0.9342	0.93122	0.9979	0.9928	0.9255
		$\tau_{d=1}$	0.63486	0.26751	0.67237	0.66457	-1.0912	-0.68868	-0.71985	-0.022027	-0.075375	-0.77971
	$m = 252$	$\hat{d}(\Delta X)$	1.0906	1.0847	1.1043	1.1113	0.99528	1.0063	1.0284	0.95773	1.0317	1.0113
		$\tau_{d=1}$	1.4769	1.3808	1.6996	1.8135	-0.076846	0.10267	0.46259	-0.6889	0.51733	0.18443
	$m = 556$	$\hat{d}(\Delta X)$	1.0334	0.99621	1.0075	1.0822	0.98171	0.96581	1.0503	0.88844	0.98291	0.98403
		$\tau_{d=1}$	0.82167	-0.093305	0.18488	2.0223*	-0.44978	-0.84092	1.2378	-2.7438**	-0.42024	-0.39273
	$m = 21$	$\hat{d}(\Delta X)$	1.0932	1.0585	1.2068	1.0247	0.68511	0.53466	0.83354	0.94063	0.72614	0.93701
		$\tau_{d=1}$	0.24371	0.15295	0.54093	0.064676	-0.82366	-1.2172	-0.43542	-0.1553	-0.71632	-0.16477
	$m = 51$	$\hat{d}(\Delta X)$	1.0793	1.0999	1.1471	1.1204	0.80996	0.82075	0.96278	0.98229	0.75197	0.94084
		$\tau_{d=1}$	0.39336	0.49595	0.72994	0.59768	-0.94307	-0.88952	-0.1847	-0.087908	-1.2309	-0.29361
$J = 3$	$m = 114$	$\hat{d}(\Delta X)$	1.0777	1.0638	1.0863	1.085	0.87889	0.92741	0.91604	0.97628	0.98823	0.92951
		$\tau_{d=1}$	0.64038	0.52583	0.71137	0.7004	-0.99806	-0.59817	-0.69193	-0.1955	-0.096991	-0.5809
	$m = 252$	$\hat{d}(\Delta X)$	1.0853	1.0891	1.0949	1.1083	0.99974	1.0001	1.0329	0.94609	1.0055	1.0203
		$\tau_{d=1}$	1.1101	1.1597	1.2348	1.4097	-0.0034405	0.00074993	0.42821	-0.70174	0.072236	0.26444
	$m = 555$	$\hat{d}(\Delta X)$	1.0304	1.0037	0.99824	1.0812	0.98445	0.95898	1.0474	0.87589	0.9791	0.98987
		$\tau_{d=1}$	0.60192	0.072986	-0.03487	1.6083	-0.30782	-0.81216	0.93849	-2.457*	-0.4138	-0.20047

Table 7: d estimates for the yields with $l = 1$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.0353	1.0736	0.89097	0.99382	0.93387
		$\tau_{d=1}$	0.17203	0.35853	-0.53124	-0.030098	-0.32219
	$m = 51$	$\hat{d}(\Delta X)$	0.94903	0.97784	0.8668	0.90459	0.94177
		$\tau_{d=1}$	-0.43916	-0.19088	-1.1476	-0.82201	-0.50171
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.0294	1.0439	0.92612	0.97535	0.98441
		$\tau_{d=1}$	0.42055	0.62844	-1.0566	-0.35257	-0.22294
	$m = 252$	$\hat{d}(\Delta X)$	0.96888	0.99244	0.92397	0.94852	1.0026
		$\tau_{d=1}$	-0.70267	-0.17059	-1.7165	-1.1622	0.058915
	$m = 556$	$\hat{d}(\Delta X)$	0.99411	0.99716	0.98197	0.98975	1.037
		$\tau_{d=1}$	-0.20233	-0.097479	-0.61933	-0.35224	1.2698
	$m = 23$	$\hat{d}(\Delta X)$	1.0026	1.1408	1.1049	1.0806	1.025
		$\tau_{d=1}$	0.008386	0.44602	0.33236	0.25526	0.079348
	$m = 51$	$\hat{d}(\Delta X)$	0.92601	1.0564	1.0201	0.98881	0.99223
		$\tau_{d=1}$	-0.42809	0.32632	0.11613	-0.064734	-0.044986
$J = 2$	$m = 115$	$\hat{d}(\Delta X)$	1.0316	1.0563	1.0043	1.0119	1.0314
		$\tau_{d=1}$	0.3109	0.55491	0.041876	0.11683	0.30894
	$m = 253$	$\hat{d}(\Delta X)$	0.98455	1.0027	0.98385	0.97935	1.0147
		$\tau_{d=1}$	-0.24245	0.041635	-0.25345	-0.3241	0.23073
	$m = 557$	$\hat{d}(\Delta X)$	0.98428	0.98638	0.98776	0.97939	1.0106
		$\tau_{d=1}$	-0.37764	-0.32737	-0.29403	-0.49522	0.25581
	$m = 22$	$\hat{d}(\Delta X)$	1.3385	1.2626	1.3282	1.2539	1.2117
		$\tau_{d=1}$	0.7788	0.60426	0.75521	0.5842	0.48704
	$m = 52$	$\hat{d}(\Delta X)$	0.9806	1.0903	1.0449	1.0139	1.012
		$\tau_{d=1}$	-0.08864	0.41274	0.20518	0.063583	0.054933
$J = 3$	$m = 115$	$\hat{d}(\Delta X)$	1.0333	1.0352	1.0093	1.0102	1.0356
		$\tau_{d=1}$	0.26047	0.27487	0.072759	0.079342	0.27822
	$m = 253$	$\hat{d}(\Delta X)$	0.99747	1.0067	0.99208	0.99224	1.0216
		$\tau_{d=1}$	-0.031876	0.084883	-0.099657	-0.097712	0.272
	$m = 556$	$\hat{d}(\Delta X)$	1.0008	1.0079	1.0011	0.99927	1.0173
		$\tau_{d=1}$	0.01551	0.15227	0.022004	-0.014192	0.3351

Table 8: d estimates for the spreads with $l = 1$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	0.74987	0.90468	0.85302	0.87147	0.80844	0.52991	0.91703	0.99587	0.83407	1.3064
		$\tau_{d=1}$	-1.2187	-0.46443	-0.71613	-0.62622	-0.93333	-2.2904*	-0.40423	-0.020122	-0.80843	1.4926
	$m = 51$	$\hat{d}(\Delta X)$	0.98108	0.96415	1.0047	1.0691	0.88827	0.80681	0.96923	1.0014	0.78319	0.98951
		$\tau_{d=1}$	-0.16299	-0.30889	0.040815	0.59539	-0.96263	-1.6645	-0.2651	0.011927	-1.868	-0.090371
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.024	0.95789	0.99952	1.0671	0.92361	0.91376	0.98493	0.99356	0.9812	0.91611
		$\tau_{d=1}$	0.34372	-0.60219	-0.006876	0.96001	-1.0925	-1.2334	-0.21554	-0.092087	-0.26894	-1.1997
	$m = 252$	$\hat{d}(\Delta X)$	1.0452	1.059	1.0735	1.1025	1.0187	1.0086	1.0599	0.96614	1.0507	0.99742
		$\tau_{d=1}$	1.0194	1.3323	1.66	2.315*	0.42313	0.19378	1.3516	-0.76431	1.1452	-0.058244
	$m = 556$	$\hat{d}(\Delta X)$	1.0292	0.98286	0.97963	1.0867	0.99183	0.967	1.0788	0.89724	0.98653	0.99265
		$\tau_{d=1}$	1.003	-0.58871	-0.69992	2.9775**	-0.28077	-1.1336	2.708**	-3.5303**	-0.46291	-0.25249
	$m = 23$	$\hat{d}(\Delta X)$	0.93695	0.93291	1.0259	0.88609	0.70816	0.55502	0.84467	1.0097	0.71308	1.0857
		$\tau_{d=1}$	-0.19975	-0.21254	0.081996	-0.36087	-0.92458	-1.4097	-0.4921	0.030827	-0.90899	0.27151
	$m = 51$	$\hat{d}(\Delta X)$	1.0691	1.0495	1.1185	1.096	0.86277	0.83437	0.97586	1.0125	0.79056	0.99666
		$\tau_{d=1}$	0.3999	0.28669	0.68558	0.55523	-0.79403	-0.95836	-0.13968	0.072256	-1.2118	-0.019327
$J = 2$	$m = 115$	$\hat{d}(\Delta X)$	1.0495	1.0361	1.0838	1.0714	0.93674	0.95939	0.95783	0.96523	0.97625	0.94174
		$\tau_{d=1}$	0.48755	0.35526	0.82554	0.70339	-0.62323	-0.40004	-0.41539	-0.34249	-0.23392	-0.57398
	$m = 253$	$\hat{d}(\Delta X)$	1.0602	1.0759	1.0841	1.0892	1.0275	1.0134	1.0508	0.94644	1.0174	1.0334
		$\tau_{d=1}$	0.94483	1.1915	1.3194	1.399	0.43186	0.21027	0.79669	-0.84045	0.2729	0.52394
	$m = 557$	$\hat{d}(\Delta X)$	1.0254	0.9916	0.98942	1.0748	0.98586	0.95837	1.0537	0.87611	0.9766	0.99081
		$\tau_{d=1}$	0.61105	-0.20193	-0.2543	1.7968	-0.33985	-1.0004	1.2893	-2.9768**	-0.56217	-0.22084
	$m = 22$	$\hat{d}(\Delta X)$	0.82371	0.73205	0.89577	0.76287	0.62827	0.57538	0.80618	1.0203	0.61229	0.97652
		$\tau_{d=1}$	-0.40559	-0.6165	-0.2398	-0.54557	-0.85525	-0.97695	-0.44592	0.046625	-0.89202	-0.054012
	$m = 52$	$\hat{d}(\Delta X)$	1.0593	1.0357	1.1252	1.0662	0.82906	0.86221	0.98179	0.99795	0.75392	0.96371
		$\tau_{d=1}$	0.27088	0.16321	0.57235	0.30249	-0.78121	-0.62973	-0.083205	-0.0093506	-1.1247	-0.16586
$J = 3$	$m = 115$	$\hat{d}(\Delta X)$	1.0742	1.0027	1.0552	1.0367	0.93158	0.95869	0.96859	0.98545	0.97857	0.95627
		$\tau_{d=1}$	0.58004	0.021021	0.4311	0.2871	-0.53473	-0.32284	-0.24551	-0.11371	-0.1675	-0.34179
	$m = 253$	$\hat{d}(\Delta X)$	1.0805	1.0819	1.0811	1.0863	1.0135	1.0144	1.0398	0.9437	1.0301	1.0493
		$\tau_{d=1}$	1.0135	1.0316	1.0212	1.0866	0.16971	0.18157	0.50066	-0.70873	0.3787	0.62016
	$m = 555$	$\hat{d}(\Delta X)$	1.0246	0.99001	0.99484	1.071	0.97738	0.9567	1.0557	0.86794	0.99075	0.99221
		$\tau_{d=1}$	0.47749	-0.19368	-0.10012	1.3757	-0.43851	-0.83947	1.0799	-2.5603*	-0.17931	-0.15106

Table 9: d estimates for the yields with $l = 0$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.0794	1.1038	1.0122	1.0886	1.0329
		$\tau_{d=1}$	0.47993	0.62732	0.073489	0.53561	0.19873
	$m = 51$	$\hat{d}(\Delta X)$	0.9769	0.99086	0.94898	0.97784	0.97821
		$\tau_{d=1}$	-0.22673	-0.089711	-0.50083	-0.21758	-0.21394
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.0054	0.9199	0.95393	0.97763	0.97777
		$\tau_{d=1}$	0.084006	-1.2391	-0.71274	-0.34599	-0.34382
	$m = 252$	$\hat{d}(\Delta X)$	0.98594	0.94146	0.97549	0.96459	0.99369
		$\tau_{d=1}$	-0.3326	-1.3853	-0.58009	-0.83783	-0.14941
	$m = 556$	$\hat{d}(\Delta X)$	0.9836	0.95627	0.99043	0.9805	1.0066
		$\tau_{d=1}$	-0.57937	-1.5454	-0.33825	-0.689	0.23245
	$m = 22$	$\hat{d}(\Delta X)$	1.0903	1.1548	1.0586	1.1207	1.0611
		$\tau_{d=1}$	0.33182	0.56892	0.21532	0.44332	0.2246
	$m = 50$	$\hat{d}(\Delta X)$	0.99765	1.0371	0.98705	0.99847	0.99164
		$\tau_{d=1}$	-0.015005	0.23667	-0.082552	-0.009731	-0.053293
$J = 2$	$m = 114$	$\hat{d}(\Delta X)$	1.0041	0.93739	0.95375	0.95836	0.97683
		$\tau_{d=1}$	0.042722	-0.65527	-0.4841	-0.43581	-0.24247
	$m = 252$	$\hat{d}(\Delta X)$	0.98426	0.95082	0.9613	0.95778	0.96824
		$\tau_{d=1}$	-0.25658	-0.80147	-0.63075	-0.68805	-0.5176
	$m = 556$	$\hat{d}(\Delta X)$	0.98484	0.96438	0.9765	0.97148	0.99067
		$\tau_{d=1}$	-0.37275	-0.87604	-0.57799	-0.70142	-0.22945
	$m = 21$	$\hat{d}(\Delta X)$	1.1436	1.2969	1.1541	1.1949	1.0546
		$\tau_{d=1}$	0.3755	0.77655	0.40303	0.50977	0.14291
	$m = 51$	$\hat{d}(\Delta X)$	0.95943	0.9916	0.97575	0.97473	0.96302
		$\tau_{d=1}$	-0.20131	-0.041709	-0.12033	-0.12542	-0.18352
$J = 3$	$m = 114$	$\hat{d}(\Delta X)$	1.0032	0.92731	0.96023	0.96149	0.98751
		$\tau_{d=1}$	0.026266	-0.599	-0.32773	-0.31739	-0.10294
	$m = 252$	$\hat{d}(\Delta X)$	0.99589	0.95255	0.96919	0.96247	0.97723
		$\tau_{d=1}$	-0.053537	-0.61759	-0.401	-0.48846	-0.29638
	$m = 555$	$\hat{d}(\Delta X)$	0.97881	0.96188	0.97832	0.96462	0.98357
		$\tau_{d=1}$	-0.41944	-0.75476	-0.42926	-0.70044	-0.32534

Table 10: d estimates for the spreads with $l = 0$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	0.97058	0.96698	1.0416	0.8911	0.69883	0.8081	0.60979	1.1058	0.77158	0.59618
		$\tau_{d=1}$	-0.17786	-0.1996	0.2512	-0.65824	-1.8204	-1.16	-2.3586*	0.63965	-1.3807	-2.4409*
	$m = 51$	$\hat{d}(\Delta X)$	1.0374	1.065	1.1789	1.1409	0.89207	1.0805	1.0218	0.88402	0.88381	0.88352
		$\tau_{d=1}$	0.36733	0.63831	1.7565	1.3829	-1.0596	0.79042	0.21421	-1.1386	-1.1407	-1.1435
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.1106	1.071	1.1172	1.128	0.93064	0.97942	1.0287	0.97902	0.96461	0.85941
		$\tau_{d=1}$	1.7107	1.0986	1.8124	1.9799*	-1.073	-0.31839	0.44416	-0.32454	-0.5474	-2.1749*
	$m = 252$	$\hat{d}(\Delta X)$	1.132	1.1836	1.186	1.1225	1.0037	0.99484	1.0374	0.93534	0.95569	0.92352
		$\tau_{d=1}$	3.1225**	4.3453**	4.4004**	2.8991**	0.087883	-0.12205	0.88423	-1.5301	-1.0485	-1.8097
	$m = 556$	$\hat{d}(\Delta X)$	1.02	1.0401	1.063	1.0987	0.91555	0.94069	0.97827	0.83367	0.96585	0.95882
		$\tau_{d=1}$	0.70726	1.4167	2.2276*	3.4893**	-2.9841**	-2.0958*	-0.76778	-5.8775**	-1.2067	-1.4552
	$m = 22$	$\hat{d}(\Delta X)$	0.96895	0.98127	1.0732	0.86534	0.77152	0.7885	0.56508	1.2826	0.73123	0.55281
		$\tau_{d=1}$	-0.11409	-0.06881	0.26884	-0.49475	-0.83946	-0.77708	-1.598	1.0383	-0.9875	-1.6431
	$m = 50$	$\hat{d}(\Delta X)$	1.0326	1.0731	1.1363	1.1284	0.94519	1.1126	1.0066	0.97601	0.87644	0.8499
		$\tau_{d=1}$	0.20771	0.46576	0.86871	0.81818	-0.34925	0.71751	0.041853	-0.15285	-0.78738	-0.95646
$J = 2$	$m = 114$	$\hat{d}(\Delta X)$	1.0805	1.0439	1.1015	1.0938	0.96059	1.0102	0.99244	1.0037	0.92699	0.84934
		$\tau_{d=1}$	0.84255	0.45924	1.0624	0.9813	-0.41247	0.10692	-0.079132	0.038292	-0.76416	-1.5769
	$m = 252$	$\hat{d}(\Delta X)$	1.1098	1.1472	1.1692	1.1222	0.99875	1.011	1.0186	0.93698	0.942	0.90653
		$\tau_{d=1}$	1.7894	2.3985*	2.7579**	1.9915*	-0.020376	0.17942	0.30339	-1.0269	-0.94523	-1.5233
	$m = 556$	$\hat{d}(\Delta X)$	1.0183	1.0255	1.0658	1.0917	0.90921	0.94521	0.99247	0.84796	0.9632	0.94889
		$\tau_{d=1}$	0.45093	0.62662	1.6177	2.2556*	-2.233*	-1.3476	-0.1852	-3.7395**	-0.90508	-1.257
	$m = 21$	$\hat{d}(\Delta X)$	0.95851	0.96511	1.0426	0.84411	0.7998	0.78809	0.65575	1.2381	0.67676	0.61542
		$\tau_{d=1}$	-0.10852	-0.091265	0.1114	-0.40777	-0.52367	-0.5543	-0.90044	0.6229	-0.8455	-1.006
	$m = 51$	$\hat{d}(\Delta X)$	1.0536	1.1105	1.17	1.1285	0.92703	1.0581	1.0939	0.93573	0.86702	0.90717
		$\tau_{d=1}$	0.26585	0.54859	0.84367	0.63778	-0.36215	0.28831	0.46586	-0.31894	-0.65993	-0.46067
$J = 3$	$m = 114$	$\hat{d}(\Delta X)$	1.0889	1.0992	1.144	1.1274	0.92305	0.98988	0.99682	1.0123	0.92951	0.85291
		$\tau_{d=1}$	0.7324	0.8172	1.1863	1.0497	-0.63411	-0.083381	-0.02624	0.10151	-0.58086	-1.2122
	$m = 252$	$\hat{d}(\Delta X)$	1.1289	1.1832	1.1992	1.1519	0.99718	1.004	1.018	0.93562	0.92674	0.91203
		$\tau_{d=1}$	1.678	2.3846*	2.5927*	1.9766*	-0.036721	0.052639	0.23472	-0.83792	-0.95354	-1.145
	$m = 555$	$\hat{d}(\Delta X)$	1.027	1.0473	1.076	1.1006	0.90895	0.93513	0.99848	0.83203	0.96065	0.95801
		$\tau_{d=1}$	0.53412	0.93722	1.5036	1.9924*	-1.8026	-1.2843	-0.030047	-3.3255**	-0.77899	-0.83126

Table 11: d estimates for the yields with $l = 1$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.1884	1.2771	1.155	1.2036	1.1253
		$\tau_{d=1}$	0.9178	1.3499	0.75544	0.99195	0.61038
	$m = 51$	$\hat{d}(\Delta X)$	1.0009	1.0365	0.99467	1.0026	1.0038
		$\tau_{d=1}$	0.0079118	0.31429	-0.045894	0.022692	0.032615
$J = 1$	$m = 114$	$\hat{d}(\Delta X)$	1.0197	0.9328	0.97565	0.98891	0.98967
		$\tau_{d=1}$	0.28106	-0.96113	-0.34828	-0.15855	-0.14768
	$m = 252$	$\hat{d}(\Delta X)$	0.9912	0.94918	0.98725	0.969	1.0003
		$\tau_{d=1}$	-0.19874	-1.1472	-0.28793	-0.69992	0.0075866
	$m = 556$	$\hat{d}(\Delta X)$	0.98626	0.96081	0.99699	0.98349	1.0105
		$\tau_{d=1}$	-0.47201	-1.3464	-0.10337	-0.56734	0.36221
	$m = 23$	$\hat{d}(\Delta X)$	1.1827	1.2709	1.1755	1.2025	1.1098
		$\tau_{d=1}$	0.57871	0.85821	0.5561	0.6416	0.3478
	$m = 51$	$\hat{d}(\Delta X)$	0.98636	1.0224	0.99647	0.98728	0.97068
		$\tau_{d=1}$	-0.078932	0.12977	-0.020433	-0.073616	-0.16963
$J = 2$	$m = 115$	$\hat{d}(\Delta X)$	1.0152	0.92745	0.95922	0.96206	0.99032
		$\tau_{d=1}$	0.15	-0.71469	-0.40176	-0.37374	-0.095404
	$m = 253$	$\hat{d}(\Delta X)$	1.0219	0.9704	0.99306	0.98199	0.99199
		$\tau_{d=1}$	0.34416	-0.4645	-0.10899	-0.28258	-0.12572
	$m = 557$	$\hat{d}(\Delta X)$	0.9963	0.96762	0.98787	0.97465	0.99454
		$\tau_{d=1}$	-0.088829	-0.77797	-0.29139	-0.60916	-0.13124
	$m = 22$	$\hat{d}(\Delta X)$	1.3521	1.2921	1.196	1.2095	1.1056
		$\tau_{d=1}$	0.81	0.67203	0.45085	0.48195	0.24288
	$m = 52$	$\hat{d}(\Delta X)$	1.0366	1.0708	1.0233	1.0017	0.98548
		$\tau_{d=1}$	0.16712	0.3234	0.10638	0.0078825	-0.066381
$J = 3$	$m = 115$	$\hat{d}(\Delta X)$	1.0083	0.92826	0.94957	0.93996	0.97668
		$\tau_{d=1}$	0.064737	-0.56064	-0.39417	-0.46927	-0.18223
	$m = 253$	$\hat{d}(\Delta X)$	1.0106	0.96208	0.98062	0.97014	0.98304
		$\tau_{d=1}$	0.13345	-0.47735	-0.24392	-0.37584	-0.21347
	$m = 556$	$\hat{d}(\Delta X)$	0.98997	0.9647	0.98136	0.97179	0.98751
		$\tau_{d=1}$	-0.19448	-0.6843	-0.36142	-0.54701	-0.24205

Table 12: d estimates for the spreads with $l = 1$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
$J=1$	$m=23$	$\hat{d}(\Delta X)$	0.86505	0.8581	0.94091	0.79418	0.83822	0.75347	0.62092	0.99287	0.69988	0.56592
		$\tau_{d=1}$	-0.65751	-0.69136	-0.28792	-1.0028	-0.78825	-1.2012	-1.847	-0.034751	-1.4622	-2.1149*
	$m=51$	$\hat{d}(\Delta X)$	1.0027	1.033	1.1588	1.1389	0.98358	1.103	1.0998	0.7973	0.87553	0.92263
		$\tau_{d=1}$	0.023083	0.28432	1.3682	1.197	-0.14149	0.88733	0.86003	-1.7464	-1.0724	-0.66664
	$m=114$	$\hat{d}(\Delta X)$	1.1025	1.0576	1.1019	1.1246	0.97604	0.9788	1.0628	0.95275	0.96755	0.87341
		$\tau_{d=1}$	1.4664	0.82384	1.457	1.7813	-0.34274	-0.30318	0.89768	-0.67571	-0.46409	-1.8105
	$m=252$	$\hat{d}(\Delta X)$	1.1291	1.1848	1.1835	1.1201	1.0308	0.99568	1.0537	0.91961	0.95622	0.93445
		$\tau_{d=1}$	2.9156**	4.1726**	4.143**	2.7123**	0.69581	-0.097571	1.2127	-1.8148	-0.98834	-1.4798
	$m=556$	$\hat{d}(\Delta X)$	1.0139	1.0345	1.0567	1.0966	0.92509	0.93878	0.98391	0.82166	0.96636	0.96557
		$\tau_{d=1}$	0.47584	1.1835	1.9465	3.3193**	-2.5735*	-2.103*	-0.55262	-6.1267**	-1.1558	-1.1829
$J=2$	$m=23$	$\hat{d}(\Delta X)$	0.89271	0.86956	0.94986	0.77017	0.81107	0.80397	0.7709	1.2047	0.70425	0.67184
		$\tau_{d=1}$	-0.3399	-0.41325	-0.15884	-0.72813	-0.59854	-0.62103	-0.72581	0.64859	-0.93696	-1.0396
	$m=51$	$\hat{d}(\Delta X)$	1.042	1.0632	1.1423	1.1319	0.94575	1.0717	1.1563	0.85854	0.90909	0.97809
		$\tau_{d=1}$	0.24298	0.36558	0.8236	0.76345	-0.31391	0.41514	0.90425	-0.81851	-0.52602	-0.12678
	$m=115$	$\hat{d}(\Delta X)$	1.0948	1.0903	1.1329	1.1433	0.94662	1.0034	1.052	0.98472	0.95561	0.88498
		$\tau_{d=1}$	0.93353	0.88919	1.3089	1.4114	-0.52588	0.033477	0.51238	-0.15049	-0.43726	-1.1331
	$m=253$	$\hat{d}(\Delta X)$	1.1244	1.1939	1.2087	1.1395	1.0068	1.0072	1.0359	0.94463	0.94727	0.92958
		$\tau_{d=1}$	1.9528	3.0428**	3.2749**	2.1893*	0.10712	0.11225	0.56383	-0.86897	-0.82741	-1.1051
	$m=557$	$\hat{d}(\Delta X)$	1.0082	1.0404	1.0672	1.1002	0.90628	0.93846	0.99459	0.82947	0.9682	0.96534
		$\tau_{d=1}$	0.19659	0.97044	1.6136	2.4077*	-2.2519*	-1.4787	-0.13008	-4.0975**	-0.7641	-0.83286
$J=3$	$m=22$	$\hat{d}(\Delta X)$	0.9163	0.86516	0.95769	0.75386	0.81551	0.8946	0.80897	1.307	0.69999	0.65095
		$\tau_{d=1}$	-0.19256	-0.31024	-0.097341	-0.5663	-0.42447	-0.2425	-0.43951	0.7063	-0.69025	-0.80307
	$m=52$	$\hat{d}(\Delta X)$	1.0687	1.0744	1.1692	1.1628	0.97588	1.1609	1.1934	0.84064	0.94133	1.004
		$\tau_{d=1}$	0.31377	0.34001	0.77312	0.74403	-0.11023	0.73544	0.88385	-0.72831	-0.26812	0.018248
	$m=115$	$\hat{d}(\Delta X)$	1.0883	1.0526	1.1059	1.122	0.97594	1.0441	1.0511	0.95795	0.95823	0.89021
		$\tau_{d=1}$	0.68975	0.41134	0.82757	0.95377	-0.18802	0.34434	0.39943	-0.32866	-0.32645	-0.85802
	$m=253$	$\hat{d}(\Delta X)$	1.0954	1.1608	1.1712	1.121	1.0127	1.0202	1.0389	0.92055	0.94951	0.93231
		$\tau_{d=1}$	1.201	2.0244*	2.155*	1.5237	0.15936	0.25447	0.48964	-1.0002	-0.63562	-0.85205
	$m=555$	$\hat{d}(\Delta X)$	0.99814	1.0306	1.0598	1.0936	0.90888	0.94024	1.0034	0.8293	0.97406	0.96729
		$\tau_{d=1}$	-0.035999	0.59346	1.1587	1.8141	-1.7666	-1.1586	0.066679	-3.3095**	-0.50301	-0.63409

Table 13: Local Whittle estimates for the yields. For every series X, this table report the estimates $\tilde{d}(X)$ and $\tilde{d}(\Delta X) \equiv \widetilde{d-1}(\Delta X) + 1$ along with the test statistics (26) and (27) respectively. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

		T	Aaa	Aa	A	Baa
$m = 23$	$\tilde{d}(X)$	1.1577	1.167	1.1097	1.1454	1.1219
	$t_{d=1}$	1.5123	1.6017	1.0521	1.3948	1.1689
$m = 51$	$\tilde{d}(X)$	0.91182	0.94809	0.92025	0.90521	0.88131
	$t_{d=1}$	-1.2594	-0.74138	-1.1391	-1.3539	-1.6952
$m = 114$	$\tilde{d}(X)$	0.99674	0.99071	0.96684	0.96289	0.96243
	$t_{d=1}$	-0.0697	-0.19833	-0.708	-0.7925	-0.80225
$m = 252$	$\tilde{d}(X)$	0.9941	0.99666	0.98005	0.97504	0.99569
	$t_{d=1}$	-0.18737	-0.10596	-0.63354	-0.79245	-0.13684
$m = 556$	$\tilde{d}(X)$	0.98095	0.97878	0.97273	0.96478	0.97887
	$t_{d=1}$	-0.89842	-1.0006	-1.2861	-1.6611	-0.99643
$m = 23$	$\tilde{d}(\Delta X)$	0.96063	1.0084	1.0473	1.026	0.99513
	$\tau_{d=1}$	-0.37759	0.08021	0.45354	0.24948	-0.04672
$m = 51$	$\tilde{d}(\Delta X)$	0.93782	1.0165	1.0079	0.9944	0.97732
	$\tau_{d=1}$	-0.88811	0.2356	0.11253	-0.08003	-0.32394
$m = 114$	$\tilde{d}(\Delta X)$	1.0372	1.0134	1.0083	1.0101	1.0269
	$\tau_{d=1}$	0.79448	0.28525	0.17759	0.21594	0.57464
$m = 252$	$\tilde{d}(\Delta X)$	0.98669	0.98386	0.972	0.97201	1.0024
	$\tau_{d=1}$	-0.42253	-0.51249	-0.889	-0.88861	0.077778
$m = 556$	$\tilde{d}(\Delta X)$	0.98806	0.99025	0.97874	0.97453	0.99326
	$\tau_{d=1}$	-0.56288	-0.45975	-1.0024	-1.2014	-0.3177

Table 14: Local Whittle estimates for the spreads. For every series X, this table report the estimates $\tilde{d}(X)$ and $\tilde{d}(\Delta X) \equiv \widetilde{d-1}(\Delta X)+1$ along with the test statistics (26) and (27) respectively. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

		sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
$m = 23$	$\tilde{d}(X)$	0.93646	0.98644	1.0337	0.96975	0.79523	0.84721	0.87801	0.9524	0.78504	0.89519
	$t_{d=1}$	-0.60941	-0.1301	0.3232	-0.29017	-1.9641	-1.4655	-1.1701	-0.45658	-2.0619*	-1.0053
$m = 51$	$\tilde{d}(X)$	1.0659	1.0772	1.1409	1.104	0.88337	0.96783	0.98093	0.9698	0.79327	0.88544
	$t_{d=1}$	0.94154	1.1032	2.0125*	1.4849	-1.6658	-0.45951	-0.27232	-0.43128	-2.9527**	-1.6362
$m = 114$	$\tilde{d}(X)$	1.0828	1.0495	1.0834	1.0701	0.91027	0.99811	0.94407	0.9712	0.98375	0.90814
	$t_{d=1}$	1.7686	1.0575	1.7815	1.4974	-1.9161	-0.04037	-1.1943	-0.61498	-0.3471	-1.9617*
$m = 252$	$\tilde{d}(X)$	1.0842	1.0728	1.0877	1.0965	1.0034	1.0272	1.0454	0.9167	1.0119	1.0119
	$t_{d=1}$	2.6731**	2.312*	2.783**	3.0635**	0.10737	0.86367	1.4401	-2.6448**	0.37758	0.37762
$m = 556$	$\tilde{d}(X)$	1.0083	0.99229	1.0046	1.0537	0.94755	0.9577	1.0333	0.85168	0.9668	0.96479
	$t_{d=1}$	0.38937	-0.36375	0.219	2.5311*	-2.4737*	-1.995	1.5712	-6.9946**	-1.5658	-1.6603
$m = 23$	$\tilde{d}(\Delta X)$	0.95017	0.97652	1.0213	0.96277	0.62929	0.7002	0.82383	0.99918	0.78182	0.89659
	$\tau_{d=1}$	-0.47793	-0.22524	0.20405	-0.35709	-3.5558**	-2.8756**	-1.6898	-0.00782	-2.0927*	-0.99187
$m = 51$	$\tilde{d}(\Delta X)$	1.0623	1.0774	1.1407	1.1004	0.80978	0.88246	0.91923	0.95135	0.7861	0.89481
	$\tau_{d=1}$	0.88943	1.1058	2.01*	1.4336	-2.7169**	-1.6788	-1.1536	-0.69483	-3.055**	-1.5023
$m = 114$	$\tilde{d}(\Delta X)$	1.0776	1.05	1.0822	1.0677	0.86685	0.94205	0.90191	0.96217	0.97921	0.91494
	$\tau_{d=1}$	1.6573	1.0669	1.7545	1.4466	-2.8433**	-1.2375	-2.0946*	-0.80773	-0.44394	-1.8163
$m = 252$	$\tilde{d}(\Delta X)$	1.0873	1.0787	1.0903	1.1	0.99018	1.0055	1.0362	0.913	1.0149	1.0148
	$\tau_{d=1}$	2.7718**	2.4972*	2.8678**	3.1736**	-0.31183	0.17487	1.1506	-2.7621**	0.47373	0.46939
$m = 556$	$\tilde{d}(\Delta X)$	1.0273	1.008	1.0162	1.0703	0.96682	0.96454	1.054	0.85775	0.98494	0.97835
	$\tau_{d=1}$	1.2862	0.37586	0.76442	3.3159**	-1.5646	-1.6724	2.545*	-6.7085**	-0.71027	-1.0208

Table 15: Nielsen (2005) LM test for yields. For the univariate case, we set $d = 1$ in eq. (18). For the multivariate case, we set $\mathbf{d} = \iota$ in eq. (21). Panel A reports univariate tests whereas Panel B reports multivariate tests.

Panel A

		T	Aaa	Aa	A	Baa
$p = 0$	LM	0.24403	0.98816	0.74011	0.25188	1.4597
	pval	0.62131	0.32019	0.38963	0.61575	0.22698
$p = 1$	LM	0.2163	0.16096	0.68366	0.7104	0.076076
	pval	0.64187	0.68828	0.40833	0.39931	0.78269
$p = 2$	LM	0.16922	0.37611	0.5547	0.66573	0.12288
	pval	0.68081	0.53969	0.4564	0.41454	0.72593
$p = 3$	LM	0.10598	0.23807	0.30477	0.38305	0.04553
	pval	0.74476	0.62561	0.58091	0.53598	0.83103
$p = 4$	LM	0.003461	0.020401	0.057072	0.054574	0.013225
	pval	0.95308	0.88642	0.81119	0.81529	0.90845

Panel B

$p = 0$	LM	167.6335	LMK	169.8412
	pval	0	pval	0
$p = 1$	LM	0.050609	LMK	3.7343
	pval	0.822	pval	0.5883
$p = 2$	LM	0.005161	LMK	3.9713
	pval	0.9427	pval	0.5536
$p = 3$	LM	0.003228	LMK	2.2841
	pval	0.9547	pval	0.8086
$p = 4$	LM	0.099125	LMK	1.9812
	pval	0.7529	pval	0.8517

Table 16: Nielsen (2005) LM test for spreads. For the univariate case, we set $d = 1$ in eq. (18). For the multivariate case, we set $\mathbf{d} = \iota$ in eq. (21). Panel A reports univariate tests whereas Panel B reports multivariate tests for spreads over Treasury only.

Panel A

	sAaa	sAa	sA	sBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
$p = 0$	43.789	44.47	53.892	10.476	55.159	68.764	8.1254	254.31	39.43	79.58
	3.66E-11	2.58E-11	2.12E-13	0.00121	1.11E-13	1.11E-16	0.004365	0	3.40E-10	0.00E+00
$p = 1$	3.0048	1.177	1.6468	8.4342	0.14955	0.015179	7.6248	26.407	0.092168	0.00071
	0.083017	0.27797	0.19939	0.003682	0.69897	0.90195	0.005757	2.77E-07	0.76144	0.97875
$p = 2$	3.2169	2.8481	4.0746	6.6772	0.36829	0.074718	0.087621	5.5076	0.096976	0.23447
	0.072881	0.09148	0.043533	0.009765	0.54394	0.78459	0.76722	0.018934	0.75549	0.62823
$p = 3$	1.047	1.5122	2.4191	1.6061	0.05593	0.039755	0.053543	1.8383	0.092241	0.15987
	0.30619	0.2188	0.11986	0.20505	0.81305	0.84196	0.81701	0.17516	0.76135	0.68927
$p = 4$	0.51029	0.74678	0.92112	0.375	0.13419	0.020081	0.2416	0.79341	0.000393	0.095139
	0.47501	0.3875	0.33718	0.54029	0.71412	0.88731	0.62305	0.37307	0.98419	0.75774

Panel B

$p = 0$	LM	232.1004	LMK	282.6042
	pval	0	pval	0
$p = 1$	LM	0.004572	LMK	10.178
	pval	0.94609	pval	0.037538
$p = 2$	LM	0.067499	LMK	4.669
	pval	0.79501	pval	0.32297
$p = 3$	LM	0.038253	LMK	1.4772
	pval	0.84494	pval	0.83067
$p = 4$	LM	0.03184	LMK	1.0435
	pval	0.85838	pval	0.90313

Table 17: Cointegration analysis: unit root and stationarity tests for bivariate systems. Cointegration analysis is performed for all possible bivariate systems $X - Y$. For each pair of variables X and Y, the Dickey-Fuller without the constant (DF), augmented Dickey-Fuller without the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out on the estimated OLS residuals of the regression of Y on X and a constant. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered.

	T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
DF	-1.1126	-1.2185	-1.2286	-1.3477	-5.1035**	-3.3781*	-2.7935	-3.3225*	-2.8467	-3.642**
ADF1	-0.76146	-0.98468	-1.0283	-1.2659	-4.3831**	-2.8898*	-2.6082	-2.4953	-2.4872	-2.9817*
ADF2	-0.81788	-0.98766	-1.0388	-1.3245	-4.5047**	-2.875*	-2.7763	-2.2745	-2.4988	-2.9505*
ADF3	-0.90426	-0.99256	-1.0319	-1.374	-4.3949**	-2.8312	-2.8307	-2.1635	-2.5038	-2.9765*
ADF4	-0.91223	-0.98571	-1.0051	-1.3784	-4.4816**	-2.7972	-2.8929	-2.0932	-2.5522	-3.1397*
ADF5	-0.90747	-0.97528	-0.93669	-1.3485	-4.5053**	-2.7663	-2.8436	-2.0015	-2.5385	-3.1477*
PP	-0.9226	-1.036	-1.042	-1.407	-4.681**	-2.926*	-2.869*	-2.304	-2.656	-3.249*
KPSS	4.009**	4.8108**	4.9315**	5.0898**	2.6821**	10.335**	7.4391**	9.3078**	7.949**	1.5756**
KPSS HFO	1.0565**	1.2671**	1.2948**	1.3483**	0.80581**	2.8043**	2.0253**	2.459**	2.1548**	0.43417*

Table 18: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the narrow band FDLS residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $l = 0$ and $J = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\hat{u})$	0.98741	0.98462	0.98743	1.0212	0.92114	0.92985	1.0344	0.8899	0.97552	0.96388
	$t_{d=1}$	-0.44504	-0.54344	-0.44409	0.74761	-2.7866**	-2.4789*	1.2171	-3.8907**	-0.86504	-1.2765
	KPSS	0.445	0.39127	0.37659	0.29511	0.31628	0.33678	0.20062	0.27833	0.23695	0.18659
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\hat{u})$	0.98909	0.98569	0.98748	1.0202	0.92156	0.92646	1.0352	0.88891	0.97519	0.96386
	$t_{d=1}$	-0.38541	-0.5055	-0.44253	0.71274	-2.7719**	-2.5987**	1.2445	-3.9257**	-0.8766	-1.2769
	KPSS	0.44318	0.39026	0.37776	0.29763	0.31383	0.33882	0.19815	0.27005	0.23158	0.17879
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\hat{u})$	0.98961	0.98609	0.98749	1.02	0.92185	0.92528	1.0354	0.88881	0.97508	0.96389
	$t_{d=1}$	-0.36731	-0.49162	-0.44197	0.70594	-2.7616**	-2.6405**	1.2524	-3.9291**	-0.88061	-1.276
	KPSS	0.44265	0.38987	0.3781	0.29819	0.31238	0.33967	0.19752	0.26918	0.23071	0.17757
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\hat{u})$	0.98987	0.98629	0.9875	1.0199	0.92201	0.92465	1.0355	0.88877	0.97504	0.9639
	$t_{d=1}$	-0.35801	-0.4844	-0.44166	0.70353	-2.7561**	-2.6627**	1.2561	-3.9307**	-0.8819	-1.2758
	KPSS	0.44238	0.38966	0.37828	0.29842	0.31166	0.34014	0.19724	0.26876	0.23045	0.17726
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\hat{u})$	0.99	0.98639	0.98751	1.0199	0.92208	0.9243	1.0356	0.88872	0.97501	0.9639
	$t_{d=1}$	-0.35355	-0.48089	-0.44151	0.70252	-2.7536**	-2.675**	1.2584	-3.9323**	-0.88304	-1.2756
	KPSS	0.44225	0.38956	0.37837	0.29854	0.31135	0.3404	0.19706	0.26834	0.23022	0.17701
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\hat{u})$	0.99005	0.98644	0.98751	1.0199	0.92211	0.92416	1.0356	0.88872	0.975	0.9639
	$t_{d=1}$	-0.35159	-0.47914	-0.44143	0.7021	-2.7525**	-2.6799**	1.2595	-3.9322**	-0.8835	-1.2755
	KPSS	0.44219	0.38951	0.37841	0.29859	0.31121	0.34051	0.19698	0.26835	0.23013	0.17694

Table 19: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the narrow band FDLS residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $J = l = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\hat{\delta}(\hat{u})$	0.98774	0.98584	0.98878	1.0257	0.93441	0.92952	1.0409	0.88566	0.97728	0.97376
	$t_{d=1}$	-0.42108	-0.48645	-0.38547	0.88216	-2.2531*	-2.4212	1.4056	-3.9282**	-0.78036	-0.90133
	KPSS	0.44426	0.38887	0.37405	0.28797	0.29696	0.33725	0.19358	0.2817	0.23506	0.17716
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\hat{\delta}(\hat{u})$	0.98928	0.9866	0.98837	1.0241	0.93471	0.92582	1.0415	0.88455	0.97678	0.97373
	$t_{d=1}$	-0.36827	-0.46051	-0.39939	0.82715	-2.2431*	-2.5484	1.4254	-3.9661**	-0.79767	-0.90255
	KPSS	0.44277	0.3885	0.37606	0.29144	0.29485	0.33974	0.19142	0.27332	0.22993	0.16981
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\hat{\delta}(\hat{u})$	0.98974	0.98688	0.98826	1.0237	0.93493	0.92453	1.0417	0.88444	0.97663	0.97375
	$t_{d=1}$	-0.35233	-0.45066	-0.40338	0.81505	-2.2353*	-2.5926	1.4316	-3.9698**	-0.80268	-0.90179
	KPSS	0.44235	0.38832	0.37666	0.29225	0.29359	0.34073	0.19087	0.27243	0.2291	0.16866
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\hat{\delta}(\hat{u})$	0.98998	0.98703	0.9882	1.0236	0.93506	0.92385	1.0418	0.88439	0.97659	0.97376
	$t_{d=1}$	-0.34415	-0.44546	-0.40541	0.81013	-2.231*	-2.616	1.4346	-3.9715**	-0.80427	-0.90158
	KPSS	0.44213	0.38821	0.37696	0.2926	0.29295	0.34128	0.19061	0.27201	0.22885	0.16837
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\hat{\delta}(\hat{u})$	0.9901	0.98711	0.98817	1.0235	0.93511	0.92348	1.0418	0.88434	0.97655	0.97376
	$t_{d=1}$	-0.34023	-0.44291	-0.40643	0.80778	-2.2291*	-2.6289	1.4364	-3.9733**	-0.80567	-0.90141
	KPSS	0.44203	0.38816	0.37712	0.29278	0.29268	0.34158	0.19046	0.27159	0.22864	0.16814
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\hat{\delta}(\hat{u})$	0.99015	0.98714	0.98815	1.0235	0.93514	0.92332	1.0418	0.88434	0.97653	0.97376
	$t_{d=1}$	-0.33851	-0.44163	-0.40693	0.80668	-2.2282*	-2.6341	1.4374	-3.9732**	-0.80624	-0.90136
	KPSS	0.44199	0.38813	0.37719	0.29286	0.29256	0.3417	0.19038	0.2716	0.22855	0.16807

Table 20: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the first differenced narrow band FDLS residuals. For all possible bivariate systems $X-Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the first differenced residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $l=0$ and $J=1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m=23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\Delta\hat{u})$	1.0147	0.98232	0.96133	1.0723	0.96185	0.962	1.0657	0.90746	0.99318	0.99013
	$\tau_{d=1}$	0.51848	-0.62488	-1.3665	2.5564*	-1.3483	-1.3429	2.3217*	-3.2699**	-0.24109	-0.34869
	KPSS	0.38731	0.39583	0.42804	0.22093	0.25929	0.29206	0.16819	0.26428	0.21831	0.16228
$m=51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\Delta\hat{u})$	1.015	0.97985	0.9618	1.0743	0.96162	0.9629	1.0662	0.90745	0.99437	0.98578
	$\tau_{d=1}$	0.53124	-0.71208	-1.3498	2.6247**	-1.3563	-1.3108	2.3384*	-3.2703**	-0.1988	-0.50235
	KPSS	0.38848	0.40183	0.42817	0.21943	0.25816	0.28866	0.16648	0.25599	0.21205	0.15926
$m=114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\Delta\hat{u})$	1.0152	0.9782	0.96191	1.0748	0.96148	0.96316	1.0663	0.90745	0.9946	0.98516
	$\tau_{d=1}$	0.53552	-0.77031	-1.346	2.6426**	-1.3612	-1.3018	2.3424*	-3.2705**	-0.19084	-0.52444
	KPSS	0.38884	0.4055	0.42828	0.21905	0.25755	0.28767	0.16608	0.25512	0.21096	0.15874
$m=252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\Delta\hat{u})$	1.0152	0.97738	0.96196	1.075	0.96141	0.96329	1.0663	0.90744	0.99467	0.98501
	$\tau_{d=1}$	0.53778	-0.79919	-1.3442	2.6516**	-1.3638	-1.2972	2.3442*	-3.2706**	-0.18848	-0.52988
	KPSS	0.38902	0.40733	0.42835	0.21885	0.25725	0.28718	0.1659	0.25471	0.21063	0.1586
$m=556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\Delta\hat{u})$	1.0152	0.97742	0.96198	1.0752	0.96138	0.96336	1.0664	0.90744	0.99472	0.98488
	$\tau_{d=1}$	0.53887	-0.79803	-1.3434	2.6566**	-1.3649	-1.2948	2.3453*	-3.2707**	-0.18644	-0.53426
	KPSS	0.38911	0.40735	0.42839	0.21874	0.25713	0.28692	0.16579	0.2543	0.21035	0.15849
$m=1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\Delta\hat{u})$	1.0153	0.97754	0.962	1.0753	0.96136	0.96338	1.0664	0.90744	0.99475	0.98484
	$\tau_{d=1}$	0.53936	-0.79364	-1.3429	2.6591**	-1.3654	-1.2939	2.3458*	-3.2707**	-0.18564	-0.53554
	KPSS	0.38915	0.40715	0.4284	0.21869	0.25707	0.28681	0.16573	0.25431	0.21023	0.15846

Table 21: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the first differenced narrow band FDLS residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the first differenced residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $J = l = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\Delta\hat{u})$	1.0089	0.97473	0.95241	1.0704	0.9693	0.96063	1.0694	0.90359	0.99315	0.99937
	$\tau_{d=1}$	0.30463	-0.86817	-1.635	2.4187*	-1.0545	-1.3524	2.3852*	-3.3121**	-0.23527	-0.02149
	KPSS	0.39919	0.41107	0.44648	0.22347	0.24959	0.2939	0.16459	0.26739	0.21834	0.15427
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\Delta\hat{u})$	1.0092	0.97197	0.95273	1.0722	0.96912	0.96161	1.0699	0.90362	0.99448	0.9949
	$\tau_{d=1}$	0.31493	-0.96309	-1.6238	2.481*	-1.0609	-1.3187	2.4029*	-3.311**	-0.1898	-0.1753
	KPSS	0.40048	0.41776	0.44683	0.22211	0.24847	0.29037	0.16288	0.25891	0.21195	0.15159
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\Delta\hat{u})$	1.0093	0.97016	0.95279	1.0727	0.969	0.96189	1.0701	0.90362	0.99472	0.99425
	$\tau_{d=1}$	0.31844	-1.025	-1.6218	2.4972*	-1.065	-1.3091	2.4071*	-3.3111**	-0.1813	-0.1974
	KPSS	0.40087	0.4218	0.44701	0.22177	0.24786	0.28935	0.16248	0.25803	0.21084	0.15112
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\Delta\hat{u})$	1.0093	0.96927	0.95282	1.0729	0.96894	0.96203	1.0701	0.90362	0.9948	0.9941
	$\tau_{d=1}$	0.32029	-1.0557	-1.6208	2.5054*	-1.0671	-1.3044	2.409*	-3.3111**	-0.17878	-0.20284
	KPSS	0.40107	0.42382	0.44711	0.22159	0.24757	0.28884	0.1623	0.25761	0.2105	0.151
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\Delta\hat{u})$	1.0093	0.96929	0.95283	1.0731	0.96891	0.9621	1.0702	0.90362	0.99486	0.99397
	$\tau_{d=1}$	0.3212	-1.0549	-1.6204	2.5099*	-1.068	-1.3018	2.4101*	-3.3111**	-0.1766	-0.20723
	KPSS	0.40116	0.42385	0.44717	0.22149	0.24745	0.28857	0.16219	0.25719	0.21021	0.1509
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\Delta\hat{u})$	1.0094	0.96942	0.95284	1.0731	0.9689	0.96213	1.0702	0.90362	0.99488	0.99393
	$\tau_{d=1}$	0.3216	-1.0505	-1.6202	2.5123*	-1.0684	-1.3008	2.4107*	-3.3111**	-0.17574	-0.20851
	KPSS	0.4012	0.42363	0.44719	0.22144	0.24739	0.28846	0.16214	0.2572	0.2101	0.15087

Table 22: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the narrow band FDLS tapered residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the tapered residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $l = 0$ and $J = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\hat{u})$	0.96096	1.0136	0.99213	1.0513	0.93205	0.98667	1.0286	0.83335	0.9934	0.98539
	$t_{d=1}$	-1.3797	0.47901	-0.2781	1.8138	-2.401*	-0.47102	1.0104	-5.8889**	-0.23329	-0.51639
	KPSS	0.50544*	0.33671	0.36775	0.24961	0.30034	0.25993	0.20713	0.32178	0.21808	0.1665
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\hat{u})$	0.96307	1.0152	0.99303	1.0553	0.93111	0.98661	1.0296	0.83248	0.99252	0.9847
	$t_{d=1}$	-1.305	0.53794	-0.24613	1.9527	-2.4343*	-0.47305	1.0445	-5.9197**	-0.26445	-0.54065
	KPSS	0.50223*	0.3349	0.36732	0.24507	0.29997	0.25818	0.20435	0.311	0.2139	0.16019
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\hat{u})$	0.96373	1.0158	0.99325	1.0564	0.93059	0.9866	1.0295	0.83239	0.99237	0.9846
	$t_{d=1}$	-1.2815	0.55745	-0.23868	1.9935*	-2.4527*	-0.47344	1.0412	-5.9228**	-0.26963	-0.54433
	KPSS	0.50123*	0.33431	0.36732	0.24373	0.29975	0.25765	0.20405	0.30987	0.21317	0.15922
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\hat{u})$	0.96408	1.016	0.99334	1.057	0.93034	0.9866	1.0294	0.83235	0.99233	0.98457
	$t_{d=1}$	-1.2694	0.56693	-0.23527	2.0137*	-2.4616*	-0.4736	1.04	-5.9243**	-0.2712	-0.54525
	KPSS	0.50072*	0.33404	0.36733	0.24307	0.29964	0.25738	0.20391	0.30934	0.21295	0.15897
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\hat{u})$	0.96424	1.0162	0.99339	1.0573	0.93023	0.9866	1.0296	0.83231	0.99229	0.98455
	$t_{d=1}$	-1.2637	0.57136	-0.23364	2.0247*	-2.4654*	-0.47368	1.0455	-5.9258**	-0.27258	-0.546
	KPSS	0.50047*	0.33392	0.36735	0.24271	0.29959	0.25724	0.20363	0.3088	0.21275	0.15878
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\hat{u})$	0.96431	1.0162	0.99341	1.0575	0.93019	0.98659	1.0297	0.83231	0.99227	0.98454
	$t_{d=1}$	-1.2611	0.57354	-0.23289	2.0303*	-2.4671*	-0.47371	1.0497	-5.9257**	-0.27312	-0.54622
	KPSS	0.50037*	0.33385	0.36736	0.24252	0.29957	0.25718	0.20344	0.30881	0.21268	0.15872

Table 23: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the narrow band FDLS tapered residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the tapered residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $J = l = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\hat{u})$	0.93611	0.98865	0.96550	1.02754	0.9067	0.96202	1.00633	0.81032	0.97528	0.97083
	$t_{d=1}$	-2.1948*	-0.38976	-1.18531	0.94626	-3.20515**	-1.30488	0.21750	-6.5163**	-0.84922	-1.00213
	KPSS	0.5659*	0.38336	0.41957	0.28505	0.33809	0.29203	0.23319	0.33849	0.23721	0.17992
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\hat{u})$	0.93841	0.99043	0.96644	1.03168	0.90578	0.96204	1.00749	0.80973	0.97474	0.97060
	$t_{d=1}$	-2.1157*	-0.32876	-1.15295	1.088201	-3.2369**	-1.3039	0.257192	-6.5365**	-0.86762	-1.00998
	KPSS	0.5617*	0.381031	0.418799	0.279637	0.337499	0.289797	0.229751	0.326608	0.232049	0.172625
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\hat{u})$	0.93914	0.99102	0.96666	1.03289	0.90527	0.96206	1.00741	0.80967	0.97466	0.97057
	$t_{d=1}$	-2.091*	-0.30854	-1.14545	1.129896	-3.2544**	-1.30339	0.254549	-6.5385**	-0.87064	-1.01113
	KPSS	0.5605*	0.38028	0.418702	0.27805	0.337153	0.289128	0.229367	0.325368	0.231145	0.171502
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\hat{u})$	0.93951	0.99130	0.96676	1.03349	0.90502	0.96207	1.00738	0.80964	0.97463	0.97056
	$t_{d=1}$	-2.078*	-0.29871	-1.14201	1.150492	-3.2629**	-1.30309	0.253617	-6.5395**	-0.87157	-1.01142
	KPSS	0.5598*	0.379923	0.418676	0.277266	0.336981	0.28879	0.229186	0.324783	0.230873	0.171221
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\hat{u})$	0.93969	0.99144	0.96681	1.03382	0.90492	0.96207	1.00756	0.80962	0.97461	0.97055
	$t_{d=1}$	-2.0719*	-0.29411	-1.14037	1.161758	-3.2665**	-1.30292	0.259702	-6.5404**	-0.87238	-1.01166
	KPSS	0.5595*	0.379759	0.41867	0.276837	0.336907	0.288608	0.228846	0.324188	0.230637	0.170993
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\hat{u})$	0.93977	0.99150	0.96683	1.03399	0.90487	0.96208	1.00769	0.80962	0.97460	0.97055
	$t_{d=1}$	-2.069*	-0.29186	-1.1396	1.167526	-3.2681**	-1.30285	0.264278	-6.5404**	-0.8727	-1.01173
	KPSS	0.5593*	0.379679	0.418668	0.276618	0.336874	0.288535	0.228623	0.324204	0.230543	0.170926

Table 24: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the first differenced narrow band FDLS tapered residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the first differenced tapered residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $l = 0$ and $J = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\Delta\hat{u})$	0.96222	1.0117	0.98838	1.0455	0.91288	0.97626	1.0175	0.83796	0.98258	0.95788
	$\tau_{d=1}$	-1.3349	0.41268	-0.41056	1.6089	-3.0786**	-0.83895	0.61834	-5.7259**	-0.61562	-1.4883
	KPSS	0.50244*	0.34009	0.3748	0.25796	0.32866	0.27324	0.21987	0.31837	0.2294	0.19246
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\Delta\hat{u})$	0.96305	1.0127	0.989	1.0472	0.91221	0.97614	1.0195	0.83736	0.98165	0.95679
	$\tau_{d=1}$	-1.3056	0.4473	-0.3886	1.6682	-3.1023**	-0.8433	0.68919	-5.7471**	-0.64832	-1.527
	KPSS	0.50227*	0.3395	0.37487	0.25652	0.32774	0.27142	0.21568	0.30759	0.2249	0.18542
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\Delta\hat{u})$	0.9633	1.013	0.98915	1.0478	0.91186	0.9761	1.02	0.8373	0.98148	0.95698
	$\tau_{d=1}$	-1.2967	0.45826	-0.38329	1.689	-3.1147**	-0.8447	0.70655	-5.7492**	-0.65436	-1.5203
	KPSS	0.50224*	0.33934	0.37497	0.25595	0.32719	0.27089	0.21468	0.30646	0.22413	0.184
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\Delta\hat{u})$	0.96343	1.0131	0.98922	1.0481	0.91169	0.97608	1.0202	0.83728	0.98143	0.95703
	$\tau_{d=1}$	-1.2922	0.46343	-0.3808	1.6995	-3.1206**	-0.84543	0.71392	-5.7502**	-0.65619	-1.5184
	KPSS	0.50223*	0.33928	0.37503	0.25565	0.32691	0.27062	0.21425	0.30592	0.22389	0.18364
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\Delta\hat{u})$	0.96349	1.0132	0.98926	1.0483	0.91162	0.97606	1.0203	0.83725	0.98138	0.95708
	$\tau_{d=1}$	-1.2901	0.46581	-0.37959	1.7053	-3.123**	-0.84582	0.71832	-5.7512**	-0.6578	-1.5167
	KPSS	0.50222*	0.33926	0.37507	0.25549	0.32679	0.27048	0.21399	0.30538	0.22369	0.18334
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\Delta\hat{u})$	0.96352	1.0132	0.98927	1.0483	0.91159	0.97606	1.0204	0.83725	0.98137	0.95709
	$\tau_{d=1}$	-1.2891	0.46696	-0.37902	1.7083	-3.1241**	-0.84598	0.72048	-5.7512**	-0.65844	-1.5162
	KPSS	0.50222*	0.33925	0.37508	0.2554	0.32673	0.27042	0.21387	0.3054	0.22361	0.18325

Table 25: Fractional cointegration analysis: Dittmann (2004) estimation procedure for bivariate systems using the first differenced narrow band FDLS tapered residuals. For all possible bivariate systems $X - Y$ first the narrow band FDLS estimator (32) of the regression of Y on X is computed for different m. The last one is the OLS estimator (with intercept). Next the GPH estimator and the test statistic are computed for the first differenced tapered residuals. The bandwidth parameter in the GPH estimation is set to $[n^{4/5}]$ and we choose $J = l = 1$. Finally the KPSS test without trend and with the Bartlett kernel and bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ for the estimation of the long run variance, is performed on the fractionally differenced residuals (42).

		T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
$m = 23$	$\hat{\beta}_m$	0.74243	0.63936	0.56868	0.54938	0.86181	0.82778	0.78703	0.96271	0.92503	0.9561
	$\widehat{\delta}(\Delta\hat{u})$	0.95592	1.00877	0.98279	1.04424	0.92332	0.97375	1.01981	0.82550	0.98235	0.96334
	$\tau_{d=1}$	-1.51423	0.301343	-0.59129	1.519698	-2.6341**	-0.90169	0.680466	-5.9945**	-0.60634	-1.25953
	KPSS	0.5174*	0.345378	0.385468	0.259843	0.313055	0.276501	0.217174	0.327542	0.229647	0.187109
$m = 51$	$\hat{\beta}_m$	0.74829	0.6464	0.57828	0.56105	0.86315	0.83101	0.7934	0.96457	0.9304	0.96007
	$\widehat{\delta}(\Delta\hat{u})$	0.95667	1.00963	0.98324	1.04580	0.92270	0.97362	1.02202	0.82489	0.98147	0.96231
	$\tau_{d=1}$	-1.48852	0.330942	-0.57585	1.573294	-2.6556**	-0.90637	0.756305	-6.0156**	-0.63649	-1.29483
	KPSS	0.5173*	0.344974	0.38583	0.258566	0.312159	0.274662	0.21281	0.316263	0.225089	0.180234
$m = 114$	$\hat{\beta}_m$	0.75012	0.64879	0.58112	0.56418	0.86389	0.83195	0.79485	0.96476	0.93129	0.96067
	$\widehat{\delta}(\Delta\hat{u})$	0.95690	1.00990	0.98333	1.04636	0.92237	0.97357	1.02256	0.82483	0.98131	0.96253
	$\tau_{d=1}$	-1.48077	0.340172	-0.57253	1.59264	-2.6668**	-0.90787	0.774878	-6.0177**	-0.64211	-1.28735
	KPSS	0.5174*	0.344886	0.386015	0.258022	0.31162	0.274122	0.211773	0.315084	0.224305	0.178825
$m = 252$	$\hat{\beta}_m$	0.75106	0.64999	0.58257	0.56568	0.86425	0.83242	0.7955	0.96485	0.93156	0.96081
	$\widehat{\delta}(\Delta\hat{u})$	0.95701	1.01003	0.98338	1.04665	0.92222	0.97355	1.02279	0.82480	0.98126	0.96259
	$\tau_{d=1}$	-1.47683	0.344475	-0.57107	1.602464	-2.672**	-0.90864	0.782764	-6.0187**	-0.64382	-1.28519
	KPSS	0.5174*	0.344862	0.386123	0.257739	0.311349	0.273851	0.211325	0.314528	0.224069	0.178465
$m = 556$	$\hat{\beta}_m$	0.75151	0.65056	0.58331	0.56649	0.8644	0.83267	0.79589	0.96494	0.93179	0.96093
	$\widehat{\delta}(\Delta\hat{u})$	0.95707	1.01008	0.98340	1.04680	0.92215	0.97354	1.02292	0.82477	0.98121	0.96264
	$\tau_{d=1}$	-1.47495	0.346428	-0.57038	1.607897	-2.6742**	-0.90906	0.787474	-6.0197**	-0.64533	-1.28336
	KPSS	0.5174*	0.344856	0.386182	0.257582	0.311233	0.273705	0.211055	0.313961	0.223864	0.178171
$m = 1351$	$\hat{\beta}_m$	0.75171	0.65084	0.58367	0.5669	0.86447	0.83277	0.79608	0.96494	0.93189	0.96097
	$\widehat{\delta}(\Delta\hat{u})$	0.95709	1.01011	0.98341	1.04689	0.92213	0.97353	1.02299	0.82477	0.98120	0.96266
	$\tau_{d=1}$	-1.47413	0.347369	-0.57006	1.610692	-2.6752**	-0.90923	0.789783	-6.0196**	-0.64593	-1.28281
	KPSS	0.5174*	0.344854	0.386212	0.257501	0.311181	0.273647	0.210922	0.313976	0.223783	0.178084

Table 26: Summary Statistics and Normality Tests for yields and spreads. Both the Jarque–Bera test and the Normality test proposed by Doornik and Hansen (1994) are computed. In both cases the null hypothesis is that the series is normally distributed and the test statistic is χ^2_2. The p-value is in square bracket. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	mean	std	skewness	ex. Kurtosis	JB test	DH test
T	3.7971	1.0329	0.49184	-0.85435	184.82	491.32
					[0.0000]**	[0.0000]**
Aaa	5.0023	1.3531	-0.1839	-1.1172	150.63	280.97
					[0.0000]**	[0.0000]**
Aa	5.2922	1.2345	-0.1256	-1.0434	125.41	209.97
					[0.0000]**	[0.0000]**
A	5.884	1.0786	-0.17544	-0.89106	99.848	164.72
					[0.0000]**	[0.0000]**
Baa	6.8015	1.0963	-0.14954	-0.84478	87.437	136.51
					[0.0000]**	[0.0000]**
sTAaa	1.2052	0.67704	0.35848	-0.79221	124.3	261.52
					[0.0000]**	[0.0000]**
sTAa	1.4951	0.6473	0.40889	-0.84553	150.65	350.45
					[0.0000]**	[0.0000]**
sTA	2.0869	0.68766	-0.07889	-1.1279	141.21	238.96
					[0.0000]**	[0.0000]**
sTBaa	3.0044	0.86577	-0.04827	-0.55195	34.183	41.948
					[0.0000]**	[0.0000]**
sAaaAa	0.28991	0.21031	1.3317	1.1194	908.8	1990
					[0.0000]**	[0.0000]**
sAaaA	0.88171	0.55901	1.1248	0.39386	567.9	1578.5
					[0.0000]**	[0.0000]**
sAaaBaa	1.7992	0.85093	1.1255	1.1131	686.55	954.28
					[0.0000]**	[0.0000]**
sAaA	0.5918	0.37765	1.0959	0.39871	540.33	1413.8
					[0.0000]**	[0.0000]**
sAaBaa	1.5093	0.67882	1.0193	1.2905	633.75	570.69
					[0.0000]**	[0.0000]**
sABaa	0.9175	0.39532	1.5203	5.4625	4255.4	633.83
					[0.0000]**	[0.0000]**

Table 27: Summary Statistics and Normality Tests for first differences of yields and spreads. Both the Jarque–Bera test and the Normality test proposed by Doornik and Hansen (1994) are computed. In both cases the null hypothesis is that the series is normally distributed and the test statistic is χ^2_2. The p-value is in square bracket. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	mean	std	skewness	ex. Kurtosis	JB test	DH test
ΔT	-0.00102	0.030141	-0.50965	13.697	20530	3560.2
					[0.0000]**	[0.0000]**
ΔAaa	-0.00044	0.056111	0.11038	4.5905	2298.7	941.74
					[0.0000]**	[0.0000]**
ΔAa	-0.00036	0.054982	0.12057	3.952	1706.1	757.68
					[0.0000]**	[0.0000]**
ΔA	-0.00035	0.056562	-0.22236	6.0169	3961.7	1335
					[0.0000]**	[0.0000]**
ΔBaa	-0.00054	0.095544	-5.9017	425.03	1.97E+07	85499
					[0.0000]**	[0.0000]**
ΔsTAaa	5.74E-04	0.039787	0.06011	4.0579	1793.7	795.57
					[0.0000]**	[0.0000]**
ΔsTAa	6.62E-04	0.039436	-0.05592	5.0338	2759.1	1080.1
					[0.0000]**	[0.0000]**
ΔsTA	6.70E-04	0.042533	-0.30854	8.9381	8736	2206.3
					[0.0000]**	[0.0000]**
ΔsTBaa	4.79E-04	0.09024	-6.9097	534.99	3.12E+07	1.01E+05
					[0.0000]**	[0.0000]**
ΔsAaaAa	8.81E-05	0.016003	-6.9689	211.46	4.89E+06	9446.6
					[0.0000]**	[0.0000]**
ΔsAaaA	9.57E-05	0.025676	-1.8983	56.592	3.50E+05	12937
					[0.0000]**	[0.0000]**
ΔsAaaBaa	-9.57E-05	0.087108	-9.3362	638.72	4.44E+07	79671
					[0.0000]**	[0.0000]**
ΔsAaA	7.66E-06	0.023299	-2.7988	67.146	4.94E+05	10512
					[0.0000]**	[0.0000]**
ΔsAaBaa	-1.84E-04	0.086607	-9.6012	665.31	4.82E+07	81236
					[0.0000]**	[0.0000]**
ΔsABaa	-1.91E-04	0.083233	-5.8308	578.27	3.64E+07	1.37E+05
					[0.0000]**	[0.0000]**

Table 28: Unit root and stationarity tests for yields and spreads. The Dickey-Fuller (DF) or augmented Dickey-Fuller with the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out. n is the lag length in the ADF and it is chosen by the AIC. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	DF-ADF	n	KPSS	KPSS HFO	PP
T	-1.653	100	21.939**	5.8588**	-2.4475
Aaa	-1.3355	5	15.53**	4.1328**	-1.3556
Aa	-1.4069	1	15.243**	4.0624**	-1.3791
A	-1.4417	4	14.781**	3.948**	-1.4918
Baa	-1.3117	100	14.614**	3.935**	-2.0929
sTAaa	-1.2298	20	3.0473**	0.81407**	-1.3345
sTAa	-1.5499	14	3.1227**	0.8378**	-1.6637
sTA	-2.0273	9	5.7085**	1.5349**	-2.0796
sTBaa	-2.395	100	5.7847**	1.5748**	-2.6558
sAaaAa	-1.5907	40	8.2945**	2.2227**	-1.8038
sAaaA	-1.5777	12	8.2115**	2.1942**	-1.4043
sAaaBaa	-1.637	100	6.0188**	1.6326**	-2.3115
sAaA	-1.9143	13	7.1743**	1.9317**	-1.6063
sAaBaa	-1.6902	100	5.1076**	1.4035**	-2.8231
sABaa	-2.5695	100	2.5392**	0.75018**	-5.0142

Table 29: Unit root and stationarity tests for the first differences of yields and spreads. The Dickey-Fuller (DF) or augmented Dickey-Fuller with the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out. n is the lag length in the ADF and it is chosen by the AIC. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

	DF-ADF	n	KPSS	KPSS HFO	PP
ΔT	-5.0327**	100	0.49783*	0.57193*	-46.559**
ΔAaa	-31.028**	2	0.28466	0.28348	-48.534**
ΔAa	-48.939**	0	0.23192	0.23514	-48.9**
ΔA	-30.982**	2	0.13926	0.13961	-49.026**
ΔBaa	-5.4058**	100	0.043328	0.067559	-51.144**
ΔsTAaa	-12.988**	15	0.2278	0.23711	-55.581**
ΔsTAa	-16.111**	9	0.20595	0.20595	-55.308**
ΔsTA	-17.149**	8	0.2161	0.20603	-53.492**
ΔsTBaa	-5.4791**	100	0.087047	0.17655	-52.481**
ΔsAaaAa	-9.8791**	27	0.13711	0.14612	-58.483**
ΔsAaaA	-47.969**	0	0.2518	0.23199	-48.007**
ΔsAaaBaa	-5.5451**	100	0.085492	0.15935	-51.546**
ΔsAaA	-12.846**	11	0.16635	0.15742	-49.314**
ΔsAaBaa	-5.9663**	100	0.0595	0.12429	-51.976**
ΔsABaa	-6.2875**	100	0.018934	0.049348	-52.222**

Table 30: d estimates for the yields with $l = 0$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.2581	1.2515	1.1604	1.083	1.0871
		$\tau_{d=1}$	1.5604	1.5201	0.9697	0.50184	0.52656
	$m = 51$	$\hat{d}(\Delta X)$	1.1294	1.1236	1.0096	0.96692	1.0217
		$\tau_{d=1}$	1.2699	1.213	0.093876	-0.32475	0.21305
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	1.0682	1.0422	0.99158	1.0198	0.85643
		$\tau_{d=1}$	1.0449	0.64641	-0.12902	0.30295	-2.1994*
	$m = 246$	$\hat{d}(\Delta X)$	1.0208	1.0409	1.0359	1.0575	0.74505
		$\tau_{d=1}$	0.4864	0.95613	0.83888	1.3435	-5.956**
	$m = 556$	$\hat{d}(\Delta X)$	1.0148	0.99107	1.0113	1	0.97023
		$\tau_{d=1}$	0.5147	-0.31107	0.39283	0.0015018	-1.0371
	$m = 22$	$\hat{d}(\Delta X)$	1.1826	1.2629	1.2284	1.078	0.96728
		$\tau_{d=1}$	0.67096	0.96584	0.83919	0.28667	-0.12023
	$m = 50$	$\hat{d}(\Delta X)$	1.1465	1.1281	1.0176	0.92094	0.98424
		$\tau_{d=1}$	0.93375	0.81619	0.11237	-0.50375	-0.10041
$J = 2$	$m = 112$	$\hat{d}(\Delta X)$	1.0602	1.0223	0.99915	0.99495	0.85764
		$\tau_{d=1}$	0.62374	0.23103	-0.0087573	-0.052284	-1.4748
	$m = 246$	$\hat{d}(\Delta X)$	1.033	1.022	1.0354	1.034	0.78248
		$\tau_{d=1}$	0.53074	0.35311	0.56936	0.54746	-3.4981**
	$m = 540$	$\hat{d}(\Delta X)$	1.0267	0.9821	1.0001	0.98612	0.97088
		$\tau_{d=1}$	0.64605	-0.43349	0.0034196	-0.33598	-0.70518
	$m = 21$	$\hat{d}(\Delta X)$	1.1145	1.2036	1.2009	1.1472	1.0003
		$\tau_{d=1}$	0.29945	0.53258	0.52556	0.38513	0.00079007
	$m = 51$	$\hat{d}(\Delta X)$	1.1662	1.0994	1.0029	0.90823	0.88635
		$\tau_{d=1}$	0.82486	0.49336	0.014482	-0.45542	-0.56399
$J = 3$	$m = 111$	$\hat{d}(\Delta X)$	1.0624	1.0168	0.98903	0.98403	0.84686
		$\tau_{d=1}$	0.50589	0.13617	-0.088924	-0.12953	-1.2418
	$m = 246$	$\hat{d}(\Delta X)$	1.0217	1.0077	1.0134	1.0205	0.78128
		$\tau_{d=1}$	0.27872	0.098729	0.17264	0.26338	-2.8083**
	$m = 540$	$\hat{d}(\Delta X)$	1.0076	0.97012	0.97491	0.96528	0.96627
		$\tau_{d=1}$	0.14766	-0.58275	-0.48925	-0.67707	-0.65778

Table 31: d estimates for the spreads with $l = 0$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	1.2842	1.2771	1.0939	0.90145	1.2824	1.001	1.0814	0.84378	0.97525	0.91161
		$\tau_{d=1}$	1.7178	1.6751	0.56784	-0.59567	1.707	0.0061963	0.49206	-0.94429	-0.14959	-0.53428
	$m = 51$	$\hat{d}(\Delta X)$	1.0892	1.0207	0.86605	1.003	1.084	0.99138	1.0394	0.90515	0.97152	0.9241
		$\tau_{d=1}$	0.87569	0.20275	-1.315	0.029142	0.82471	-0.084613	0.38672	-0.93115	-0.27958	-0.74513
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	0.98681	0.93182	1.014	0.7928	1.0093	1.0548	0.8507	1.0269	0.81457	0.73488
		$\tau_{d=1}$	-0.2021	-1.0445	0.21447	-3.1741**	0.14258	0.83966	-2.2871	0.41208	-2.8405**	-4.0613**
	$m = 246$	$\hat{d}(\Delta X)$	0.97249	0.97061	0.97969	0.67172	0.97788	1.0765	0.68616	1.0103	0.66432	0.60893
		$\tau_{d=1}$	-0.64277	-0.68661	-0.47458	-7.6691**	-0.51672	1.7879	-7.3318	0.24046	-7.8419**	-9.136**
	$m = 541$	$\hat{d}(\Delta X)$	0.91012	0.91467	0.91893	0.92445	0.92679	1.0143	0.94304	0.95264	0.94656	0.96151
		$\tau_{d=1}$	-3.1308**	-2.9722**	-2.8237**	-2.6316**	-2.55*	0.49733	-1.9842*	-1.6495	-1.8615	-1.3408
	$m = 22$	$\hat{d}(\Delta X)$	1.2269	1.2556	1.0385	0.928	1.2656	1.0239	1.0166	0.81106	0.90499	0.84669
		$\tau_{d=1}$	0.83364	0.93919	0.14144	-0.26454	0.9759	0.087669	0.060898	-0.6942	-0.34907	-0.56329
	$m = 50$	$\hat{d}(\Delta X)$	1.0579	0.95946	0.88081	0.95231	1.0661	0.94956	0.9755	0.83398	0.91914	0.88095
		$\tau_{d=1}$	0.36926	-0.25833	-0.7595	-0.30388	0.42115	-0.32143	-0.1561	-1.0579	-0.51526	-0.75863
$J = 2$	$m = 112$	$\hat{d}(\Delta X)$	0.94896	0.91678	0.98779	0.78335	1.0151	1.0305	0.8053	0.99712	0.76965	0.68177
		$\tau_{d=1}$	-0.5288	-0.86217	-0.12648	-2.2444*	0.15628	0.31604	-2.0171	-0.029806	-2.3863*	-3.2967**
	$m = 246$	$\hat{d}(\Delta X)$	0.938	0.95109	0.96599	0.70733	0.97893	1.0677	0.68778	1.0183	0.66653	0.60789
		$\tau_{d=1}$	-0.99713	-0.78661	-0.54696	-4.7066**	-0.33879	1.0887	-5.0211	0.29487	-5.3627**	-6.3058**
	$m = 540$	$\hat{d}(\Delta X)$	0.89623	0.90968	0.91502	0.95424	0.91996	1.0055	0.95067	0.95856	0.9486	0.95037
		$\tau_{d=1}$	-2.5127*	-2.187*	-2.0578*	-1.1081	-1.9382	0.13396	-1.1946	-1.0033	-1.2447	-1.2018
	$m = 21$	$\hat{d}(\Delta X)$	1.2875	1.3558	1.1515	0.97172	1.3603	1.0681	1.027	0.8034	0.89875	0.80958
		$\tau_{d=1}$	0.75206	0.93062	0.39632	-0.073973	0.94241	0.17825	0.070726	-0.51424	-0.26485	-0.49808
	$m = 51$	$\hat{d}(\Delta X)$	1.102	1.0298	0.86285	0.87513	1.0528	0.94813	0.9567	0.84687	0.88577	0.82421
		$\tau_{d=1}$	0.50608	0.14808	-0.68059	-0.61967	0.26222	-0.25739	-0.2149	-0.75993	-0.56686	-0.87238
$J = 3$	$m = 111$	$\hat{d}(\Delta X)$	0.97016	0.96446	0.96795	0.797	0.99019	1.0275	0.8078	1.0098	0.77858	0.69052
		$\tau_{d=1}$	-0.24195	-0.28824	-0.25993	-1.6463	-0.079541	0.22333	-1.5587	0.079266	-1.7956	-2.5097*
	$m = 246$	$\hat{d}(\Delta X)$	0.93912	0.97025	0.97433	0.71867	0.97007	1.075	0.68909	1.0317	0.67452	0.6193
		$\tau_{d=1}$	-0.78171	-0.38197	-0.32961	-3.6123**	-0.38434	0.96264	-3.9921**	0.40688	-4.1791**	-4.8882**
	$m = 540$	$\hat{d}(\Delta X)$	0.90421	0.91539	0.91342	0.9544	0.91137	1.002	0.95337	0.96212	0.95584	0.96029
		$\tau_{d=1}$	-1.8682	-1.6501	-1.6886	-0.88943	-1.7285	0.039412	-0.90953	-0.73876	-0.86122	-0.77449

Table 32: d estimates for the yields with $l = 1$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.3759	1.2371	1.0897	1.0369	1.1007
		$\tau_{d=1}$	1.8313	1.1554	0.43725	0.17976	0.49064
	$m = 51$	$\hat{d}(\Delta X)$	1.1562	1.1007	0.95973	0.93141	1.014
		$\tau_{d=1}$	1.346	0.86782	-0.34698	-0.59092	0.12019
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	1.0733	1.0219	0.96663	1.0101	0.83367
		$\tau_{d=1}$	1.0377	0.30963	-0.47219	0.1432	-2.3536*
	$m = 246$	$\hat{d}(\Delta X)$	1.0202	1.031	1.027	1.0554	0.72618
		$\tau_{d=1}$	0.45048	0.69009	0.60237	1.233	-6.0982**
	$m = 541$	$\hat{d}(\Delta X)$	1.0143	0.98406	1.0058	0.99653	0.97097
		$\tau_{d=1}$	0.48299	-0.53958	0.19782	-0.11746	-0.98258
	$m = 23$	$\hat{d}(\Delta X)$	1.3195	1.1994	1.0626	0.96704	0.95092
		$\tau_{d=1}$	1.0122	0.63175	0.19826	-0.10442	-0.15548
	$m = 51$	$\hat{d}(\Delta X)$	1.2089	1.0938	0.97749	0.88543	0.91273
		$\tau_{d=1}$	1.2087	0.54283	-0.13025	-0.66289	-0.50492
$J = 2$	$m = 111$	$\hat{d}(\Delta X)$	1.0668	1.007	0.96148	0.96577	0.84231
		$\tau_{d=1}$	0.64348	0.067874	-0.37124	-0.32985	-1.5196
	$m = 245$	$\hat{d}(\Delta X)$	1.0198	0.99279	1.0003	1.0351	0.76307
		$\tau_{d=1}$	0.3049	-0.11113	0.0039196	0.54125	-3.6502**
	$m = 541$	$\hat{d}(\Delta X)$	0.99835	0.96241	0.96757	0.96695	0.96452
		$\tau_{d=1}$	-0.038898	-0.88873	-0.76679	-0.78145	-0.83882
	$m = 22$	$\hat{d}(\Delta X)$	1.2125	1.1807	1.1392	1.0175	0.94072
		$\tau_{d=1}$	0.48897	0.41577	0.3202	0.040309	-0.13638
	$m = 49$	$\hat{d}(\Delta X)$	1.1523	1.0693	0.98725	0.89863	0.92421
		$\tau_{d=1}$	0.66643	0.30319	-0.055805	-0.44367	-0.3317
$J = 3$	$m = 112$	$\hat{d}(\Delta X)$	1.0865	1.0385	1.0228	1.0132	0.84473
		$\tau_{d=1}$	0.66476	0.29592	0.17539	0.10168	-1.1932
	$m = 244$	$\hat{d}(\Delta X)$	1.0335	1.0266	1.0473	1.0419	0.78075
		$\tau_{d=1}$	0.41309	0.32814	0.58284	0.51635	-2.7021**
	$m = 541$	$\hat{d}(\Delta X)$	1.0209	0.98464	0.99679	0.97531	0.9764
		$\tau_{d=1}$	0.39919	-0.29327	-0.061296	-0.47139	-0.4506

Table 33: d estimates for the spreads with $l = 1$ and non-tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	1.1701	1.1398	0.9199	0.7483	1.3607	0.90304	1.0415	0.68071	0.91293	0.95979
		$\tau_{d=1}$	0.82871	0.68096	-0.39028	-1.2263	1.7573	-0.47238	0.20203	-1.5557	-0.4242	-0.19593
	$m = 51$	$\hat{d}(\Delta X)$	1.0141	0.92667	0.7551	0.95332	1.0829	0.94366	1.0112	0.84406	0.94138	0.944
		$\tau_{d=1}$	0.12169	-0.63181	-2.1099*	-0.40215	0.71385	-0.4854	0.096296	-1.3435	-0.50508	-0.48249
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	0.94058	0.87891	0.98142	0.74608	1.0006	1.0396	0.81616	1.0124	0.78285	0.72336
		$\tau_{d=1}$	-0.84075	-1.7135	-0.26292	-3.5929**	0.0087585	0.56067	-2.6013**	0.1754	-3.0727**	-3.9145**
	$m = 246$	$\hat{d}(\Delta X)$	0.94903	0.94795	0.96128	0.64073	0.9713	1.0699	0.65802	1.0011	0.63845	0.5946
		$\tau_{d=1}$	-1.1351	-1.1591	-0.86233	-8.0013**	-0.63918	1.5568	-7.616**	0.024421	-8.052**	-9.0286**
	$m = 541$	$\hat{d}(\Delta X)$	0.89596	0.90114	0.90726	0.92044	0.92141	1.0082	0.94063	0.94549	0.94637	0.97006
		$\tau_{d=1}$	-3.5212**	-3.3459**	-3.1386**	-2.6926**	-2.6597**	0.27795	-2.0093*	-1.8447	-1.815	-1.0133
	$m = 23$	$\hat{d}(\Delta X)$	1.1189	1.1007	0.81821	0.79465	1.2277	0.91241	0.99823	0.69297	0.86705	0.83902
		$\tau_{d=1}$	0.3766	0.31905	-0.57593	-0.65057	0.72131	-0.27749	-0.0056051	-0.97269	-0.42119	-0.51
	$m = 51$	$\hat{d}(\Delta X)$	1.0578	1.0043	0.77206	0.88315	1.0976	0.89983	0.9462	0.79136	0.86697	0.88421
		$\tau_{d=1}$	0.33429	0.024886	-1.3188	-0.67608	0.5647	-0.5796	-0.31129	-1.2072	-0.76971	-0.66997
$J = 2$	$m = 111$	$\hat{d}(\Delta X)$	0.94313	0.9224	0.91163	0.77666	0.99698	1.0258	0.79501	0.99304	0.75674	0.70038
		$\tau_{d=1}$	-0.54809	-0.74788	-0.85159	-2.1524*	-0.029142	0.24891	-1.9755	-0.067079	-2.3443*	-2.8874**
	$m = 245$	$\hat{d}(\Delta X)$	0.9395	0.96127	0.96641	0.68919	0.96173	1.0567	0.6659	1.0237	0.64683	0.61228
		$\tau_{d=1}$	-0.93212	-0.59672	-0.51751	-4.7884**	-0.5896	0.87381	-5.147**	0.36503	-5.4409**	-5.9731**
	$m = 541$	$\hat{d}(\Delta X)$	0.90036	0.91179	0.91001	0.94825	0.91302	0.99138	0.94922	0.9513	0.95134	0.96816
		$\tau_{d=1}$	-2.356*	-2.0858*	-2.1277*	-1.2237	-2.0566*	-0.20391	-1.2007	-1.1515	-1.1506	-0.75294
	$m = 22$	$\hat{d}(\Delta X)$	1.0739	0.96073	0.79603	0.81196	1.327	0.98387	0.97927	0.66956	0.82034	0.822
		$\tau_{d=1}$	0.16997	-0.090348	-0.46929	-0.43262	0.75237	-0.037114	-0.047689	-0.76026	-0.41335	-0.40954
	$m = 49$	$\hat{d}(\Delta X)$	1.0315	0.94349	0.71925	0.84775	1.0649	0.91049	0.89909	0.78573	0.84542	0.87466
		$\tau_{d=1}$	0.13793	-0.24731	-1.2288	-0.66636	0.28392	-0.39175	-0.44164	-0.93782	-0.67657	-0.54858
$J = 3$	$m = 112$	$\hat{d}(\Delta X)$	0.95327	0.91758	0.93779	0.7614	1.0046	1.0455	0.78145	0.98456	0.75053	0.68538
		$\tau_{d=1}$	-0.35913	-0.63334	-0.47803	-1.8335	0.034985	0.34953	-1.6795	-0.11867	-1.917	-2.4177*
	$m = 244$	$\hat{d}(\Delta X)$	0.92885	0.9427	0.9518	0.70227	0.96135	1.0662	0.6788	1.0337	0.65873	0.61628
		$\tau_{d=1}$	-0.87684	-0.70623	-0.59401	-3.6694**	-0.47632	0.81533	-3.9586**	0.41525	-4.206**	-4.7291**
	$m = 541$	$\hat{d}(\Delta X)$	0.90126	0.90693	0.89682	0.9601	0.90361	0.99342	0.95766	0.96526	0.9581	0.96811
		$\tau_{d=1}$	-1.8852	-1.777	-1.97*	-0.7618	-1.8403	-0.12563	-0.80836	-0.66331	-0.79987	-0.60893

Table 34: d estimates for the yields with $l = 0$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m = 23$	$\hat{d}(\Delta X)$	1.2899	1.478	1.3751	1.1379	1.0869
		$\tau_{d=1}$	1.7523	2.8896	2.2672	0.83366	0.52532
	$m = 51$	$\hat{d}(\Delta X)$	1.1664	1.187	1.1156	0.94489	0.96329
		$\tau_{d=1}$	1.6333	1.8361	1.1345	-0.54101	-0.36041
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	1.1419	1.0675	1.0548	1.0088	0.99412
		$\tau_{d=1}$	2.1731*	1.0343	0.8397	0.13463	-0.090087
	$m = 246$	$\hat{d}(\Delta X)$	1.0404	1.029	1.0417	1.0339	1.0181
		$\tau_{d=1}$	0.94409	0.67689	0.97454	0.79283	0.4221
	$m = 541$	$\hat{d}(\Delta X)$	1.0165	0.98709	0.98741	0.94704	1.0251
		$\tau_{d=1}$	0.57379	-0.44964	-0.43852	-1.8447	0.87521
	$m = 22$	$\hat{d}(\Delta X)$	1.2432	1.3822	1.3452	1.2135	1.1297
		$\tau_{d=1}$	0.89372	1.4041	1.2683	0.7846	0.47637
	$m = 50$	$\hat{d}(\Delta X)$	1.1651	1.16	1.0681	0.92461	0.98681
		$\tau_{d=1}$	1.0521	1.0197	0.43396	-0.4804	-0.08407
$J = 2$	$m = 112$	$\hat{d}(\Delta X)$	1.1196	1.06	1.0417	1.0011	0.98983
		$\tau_{d=1}$	1.2388	0.6212	0.43199	0.011856	-0.10536
	$m = 246$	$\hat{d}(\Delta X)$	1.0399	1.0347	1.0441	1.0432	1.0183
		$\tau_{d=1}$	0.64198	0.55747	0.70873	0.69545	0.29416
	$m = 540$	$\hat{d}(\Delta X)$	1.0241	0.98924	0.99199	0.96007	1.0372
		$\tau_{d=1}$	0.58437	-0.26057	-0.19386	-0.96678	0.90108
	21	$\hat{d}(\Delta X)$	1.1945	1.3976	1.3983	1.2326	1.1202
		$\tau_{d=1}$	0.50878	1.0401	1.0417	0.60846	0.31449
	$m = 51$	$\hat{d}(\Delta X)$	1.1237	1.1377	1.0709	0.91235	0.92345
		$\tau_{d=1}$	0.61391	0.68335	0.35206	-0.43496	-0.37986
$J = 3$	$m = 111$	$\hat{d}(\Delta X)$	1.0867	1.0324	1.0131	0.9748	0.98964
		$\tau_{d=1}$	0.7033	0.26269	0.10607	-0.20434	-0.084021
	$m = 246$	$\hat{d}(\Delta X)$	1.0327	1.021	1.0283	1.0346	1.0203
		$\tau_{d=1}$	0.41999	0.27008	0.36347	0.44408	0.26123
	$m = 540$	$\hat{d}(\Delta X)$	1.023	0.97857	0.97851	0.95116	1.0339
		$\tau_{d=1}$	0.44773	-0.41788	-0.41917	-0.95264	0.66133

Table 35: d estimates for the spreads with $l = 0$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	1.3271	1.2381	0.99842	0.89355	1.3041	0.96375	1.0554	0.80029	0.94746	0.70537
		$\tau_{d=1}$	1.9769	1.4394	-0.0095709	-0.64347	1.8381	-0.21911	0.33513	-1.2072	-0.31758	-1.7809
	$m = 51$	$\hat{d}(\Delta X)$	1.1777	1.0788	0.88128	0.95244	1.0411	0.96008	1.002	0.85356	0.93172	0.88863
		$\tau_{d=1}$	1.7446	0.77363	-1.1655	-0.46688	0.40361	-0.39187	0.019709	-1.4376	-0.67031	-1.0934
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	1.0392	1.0195	0.99114	0.96512	0.97944	1.022	0.96871	0.95967	0.93177	0.87603
		$\tau_{d=1}$	0.60025	0.29886	-0.13579	-0.53432	-0.31491	0.33651	-0.47934	-0.61787	-1.0452	-1.8991
	$m = 246$	$\hat{d}(\Delta X)$	1.0136	1.0084	0.99483	0.98757	0.99336	1.0938	1.0029	1.0487	0.97516	0.93288
		$\tau_{d=1}$	0.31741	0.19508	-0.12067	-0.29032	-0.15505	2.1925*	0.067027	1.1367	-0.58031	-1.5681
	$m = 541$	$\hat{d}(\Delta X)$	0.91863	0.90536	0.90688	1.0478	0.94185	1.054	1.0762	1.0442	1.0736	1.0164
		$\tau_{d=1}$	-2.8344**	-3.2966**	-3.2434**	1.6645	-2.0254*	1.8825	2.6549**	1.5408	2.5629*	0.57166
	$m = 22$	$\hat{d}(\Delta X)$	1.2826	1.2477	1.0005	0.9065	1.3904	0.98571	1.0535	0.76995	0.91197	0.72771
		$\tau_{d=1}$	1.0385	0.91019	0.0018814	-0.34352	1.4345	-0.052492	0.1964	-0.84526	-0.32345	-1.0004
	$m = 50$	$\hat{d}(\Delta X)$	1.1168	1.017	0.86575	0.94063	1.0378	0.9601	1.0092	0.84993	0.95273	0.91688
		$\tau_{d=1}$	0.74401	0.10859	-0.85545	-0.37829	0.24091	-0.25422	0.058911	-0.9563	-0.30124	-0.52968
$J = 2$	$m = 112$	$\hat{d}(\Delta X)$	1.0106	0.99182	0.98783	0.95452	0.9718	1.0009	0.95521	0.95122	0.93203	0.852
		$\tau_{d=1}$	0.11014	-0.084713	-0.12603	-0.47115	-0.29217	0.0088505	-0.46396	-0.50532	-0.70419	-1.5332
	$m = 246$	$\hat{d}(\Delta X)$	1.0048	1.0096	0.99889	0.97406	0.99031	1.0843	0.9677	1.044	0.94718	0.90705
		$\tau_{d=1}$	0.07735	0.15401	-0.017772	-0.41715	-0.15576	1.3556	-0.51949	0.70838	-0.8495	-1.4947
	$m = 540$	$\hat{d}(\Delta X)$	0.9171	0.91342	0.91182	1.0423	0.93293	1.0575	1.0674	1.0441	1.0668	1.0077
		$\tau_{d=1}$	-2.0073*	-2.0964*	-2.1351*	1.0241	-1.6239	1.3935	1.6321	1.0682	1.6175	0.1855
	$m = 21$	$\hat{d}(\Delta X)$	1.351	1.3885	1.0363	0.9003	1.4829	1.0083	1.0846	0.73547	0.92579	0.7267
		$\tau_{d=1}$	0.91821	1.0161	0.094966	-0.26078	1.263	0.02161	0.22132	-0.69194	-0.1941	-0.71486
	$m = 51$	$\hat{d}(\Delta X)$	1.1372	1.0313	0.85755	0.91093	1.0163	0.94668	1.0043	0.8557	0.93772	0.89994
		$\tau_{d=1}$	0.68098	0.1551	-0.70693	-0.44203	0.08068	-0.26459	0.021183	-0.71608	-0.30905	-0.49655
$J = 3$	$m = 111$	$\hat{d}(\Delta X)$	0.99048	0.97958	0.98208	0.98401	0.94119	0.99573	0.96683	0.9722	0.94592	0.87009
		$\tau_{d=1}$	-0.077231	-0.16561	-0.1453	-0.12969	-0.47695	-0.034667	-0.26901	-0.22548	-0.43859	-1.0535
	$m = 246$	$\hat{d}(\Delta X)$	0.99393	1.007	0.9989	0.98782	0.9925	1.0923	0.967	1.0544	0.95059	0.9106
		$\tau_{d=1}$	-0.077906	0.090217	-0.014102	-0.15634	-0.096248	1.1846	-0.42368	0.69834	-0.63448	-1.1479
	$m = 540$	$\hat{d}(\Delta X)$	0.91317	0.91343	0.90697	1.0405	0.93741	1.0555	1.0629	1.0413	1.0647	1.0066
		$\tau_{d=1}$	-1.6935	-1.6884	-1.8144	0.79034	-1.2207	1.082	1.2266	0.80494	1.2625	0.12917

Table 36: d estimates for the yields with $l=1$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X)+1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			T	Aaa	Aa	A	Baa
	$m=23$	$\hat{d}(\Delta X)$	1.3479	1.5424	1.4008	1.0428	1.0549
		$\tau_{d=1}$	1.6949	2.6425	1.9528	0.2083	0.26751
	$m=51$	$\hat{d}(\Delta X)$	1.1696	1.1666	1.0842	0.87385	0.9261
		$\tau_{d=1}$	1.4612	1.4357	0.72581	-1.0869	-0.63671
$J=1$	$m=112$	$\hat{d}(\Delta X)$	1.1392	1.0438	1.033	0.98458	0.97932
		$\tau_{d=1}$	1.9701	0.61991	0.46705	-0.21816	-0.29257
	$m=246$	$\hat{d}(\Delta X)$	1.0321	1.0147	1.0301	1.0238	1.0129
		$\tau_{d=1}$	0.71463	0.32735	0.67021	0.53023	0.28768
	$m=541$	$\hat{d}(\Delta X)$	1.0114	0.9783	0.97938	0.9383	1.0228
		$\tau_{d=1}$	0.38479	-0.7344	-0.69781	-2.0881*	0.77246
	$m=23$	$\hat{d}(\Delta X)$	1.3497	1.5899	1.4199	1.0796	0.97171
		$\tau_{d=1}$	1.1079	1.8688	1.3302	0.25232	-0.089613
	$m=51$	$\hat{d}(\Delta X)$	1.157	1.1578	1.0713	0.89413	0.92349
		$\tau_{d=1}$	0.90837	0.91311	0.41277	-0.61257	-0.4427
$J=2$	$m=111$	$\hat{d}(\Delta X)$	1.1032	1.0203	1.0108	0.94338	1.0037
		$\tau_{d=1}$	0.99439	0.19572	0.10443	-0.5457	0.0359
	$m=245$	$\hat{d}(\Delta X)$	1.0205	0.98666	1.0077	1.0182	1.0123
		$\tau_{d=1}$	0.31548	-0.20551	0.11819	0.2798	0.19018
	$m=541$	$\hat{d}(\Delta X)$	1.0127	0.95111	0.95982	0.93412	1.0132
		$\tau_{d=1}$	0.30058	-1.156	-0.95	-1.5577	0.31315
	$m=22$	$\hat{d}(\Delta X)$	1.2746	1.4167	1.3819	1.1234	1.1013
		$\tau_{d=1}$	0.6318	0.9588	0.87866	0.28399	0.23303
	$m=49$	$\hat{d}(\Delta X)$	1.114	1.1109	1.0156	0.84867	0.93783
		$\tau_{d=1}$	0.49898	0.48517	0.068277	-0.66234	-0.2721
$J=3$	$m=112$	$\hat{d}(\Delta X)$	1.1237	1.0571	1.0342	0.97818	0.99098
		$\tau_{d=1}$	0.95034	0.43902	0.26256	-0.16771	-0.069297
	$m=244$	$\hat{d}(\Delta X)$	1.0343	1.0225	1.0374	1.0326	1.0211
		$\tau_{d=1}$	0.42223	0.27695	0.46133	0.40197	0.25979
	$m=541$	$\hat{d}(\Delta X)$	1.0251	0.98493	0.99033	0.95028	1.0297
		$\tau_{d=1}$	0.47982	-0.28764	-0.18455	-0.94936	0.56727

Table 37: d estimates for the spreads with $l = 1$ and tapered data. For every series X, this table report the estimates $\hat{d}(\Delta X) \equiv \widehat{d-1}(\Delta X) + 1$ along with the test statistic (15). One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

			sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
	$m = 23$	$\hat{d}(\Delta X)$	1.3007	1.1786	0.83758	0.822	1.356	0.87772	1.0732	0.66035	0.95341	0.82651
		$\tau_{d=1}$	1.4651	0.87002	-0.79135	-0.86724	1.7345	-0.59579	0.35673	-1.6548	-0.227	-0.84528
	$m = 51$	$\hat{d}(\Delta X)$	1.1464	1.0314	0.79479	0.92742	1.0143	0.91852	0.99064	0.80057	0.9234	0.96588
		t	1.2613	0.27034	-1.7681	-0.62533	0.12354	-0.70198	-0.080601	-1.7182	-0.65996	-0.294
$J = 1$	$m = 112$	$\hat{d}(\Delta X)$	1.0089	0.99073	0.96517	0.95466	0.96	1.0089	0.95893	0.94687	0.92718	0.90929
		$\tau_{d=1}$	0.12555	-0.13119	-0.49282	-0.64151	-0.56607	0.12565	-0.58109	-0.75179	-1.0304	-1.2835
	$m = 246$	$\hat{d}(\Delta X)$	0.99685	0.99345	0.98227	0.98463	0.98508	1.0915	1.001	1.0476	0.97641	0.95388
		$\tau_{d=1}$	-0.070165	-0.14595	-0.39497	-0.3423	-0.3322	2.0383*	0.022721	1.0609	-0.5253	-1.0271
	$m = 541$	$\hat{d}(\Delta X)$	0.90638	0.89358	0.89686	1.0488	0.93568	1.0511	1.0783	1.0433	1.0783	1.0303
		$\tau_{d=1}$	-3.1686**	-3.6018**	-3.4908**	1.6503	-2.1769*	1.7283	2.6515**	1.466	2.6506**	1.0258
	$m = 23$	$\hat{d}(\Delta X)$	1.3008	1.1765	0.77997	0.78599	1.302	0.89014	1.0532	0.66062	0.91643	0.81789
		$\tau_{d=1}$	0.95306	0.55917	-0.69708	-0.67801	0.95684	-0.34804	0.16865	-1.0752	-0.26475	-0.57695
	$m = 51$	$\hat{d}(\Delta X)$	1.1321	1.0273	0.81764	0.9077	1.001	0.92185	0.98761	0.82383	0.92797	0.96754
		$\tau_{d=1}$	0.76435	0.1579	-1.0551	-0.53407	0.0056384	-0.45218	-0.071701	-1.0193	-0.41677	-0.1878
$J = 2$	$m = 111$	$\hat{d}(\Delta X)$	1.0024	0.9762	0.94833	0.97485	0.94399	1.0073	0.96285	0.94568	0.9403	0.90483
		$\tau_{d=1}$	0.023354	-0.22932	-0.49795	-0.24234	-0.53978	0.069986	-0.35799	-0.52348	-0.57532	-0.91718
	$m = 245$	$\hat{d}(\Delta X)$	0.99183	0.99854	0.98705	0.97881	0.98026	1.0904	0.95546	1.0501	0.94464	0.91985
		$\tau_{d=1}$	-0.12589	-0.022423	-0.19945	-0.32653	-0.30409	1.3921	-0.68612	0.77173	-0.8528	-1.2348
	$m = 541$	$\hat{d}(\Delta X)$	0.89748	0.89466	0.89256	1.0374	0.93872	1.0463	1.062	1.0406	1.0719	1.0224
		$\tau_{d=1}$	-2.4241*	-2.4907*	-2.5404*	0.88403	-1.4489	1.0958	1.467	0.95942	1.7003	0.53052
	$m = 22$	$\hat{d}(\Delta X)$	1.2709	1.1372	0.81315	0.84481	1.4099	0.92604	1.0592	0.61262	0.89641	0.772
		$\tau_{d=1}$	0.62334	0.31562	-0.4299	-0.35706	0.94305	-0.17016	0.13614	-0.89125	-0.23834	-0.52456
	$m = 49$	$\hat{d}(\Delta X)$	1.1107	0.99025	0.75007	0.9255	0.99361	0.94636	1.01	0.80781	0.95436	0.99143
		$\tau_{d=1}$	0.48443	-0.042662	-1.0939	-0.32607	-0.027978	-0.23478	0.043688	-0.84117	-0.19977	-0.0375
$J = 3$	$m = 112$	$\hat{d}(\Delta X)$	1.0276	0.98392	0.96025	0.96525	0.94343	1.0237	0.95631	0.94652	0.93411	0.87899
		$\tau_{d=1}$	0.21243	-0.12353	-0.30546	-0.26703	-0.43473	0.18178	-0.33577	-0.41095	-0.50634	-0.92993
	$m = 244$	$\hat{d}(\Delta X)$	1.0033	1.0084	0.99464	0.98245	0.96651	1.0981	0.96678	1.0598	0.95237	0.92317
		$\tau_{d=1}$	0.040185	0.10374	-0.06602	-0.21631	-0.41279	1.2094	-0.40947	0.73702	-0.58703	-0.94692
	$m = 541$	$\hat{d}(\Delta X)$	0.90561	0.90769	0.89949	1.0443	0.91663	1.0469	1.0719	1.0481	1.0725	1.0215
		$\tau_{d=1}$	-1.8021	-1.7624	-1.919	0.84512	-1.5917	0.89498	1.3727	0.91814	1.3843	0.41125

Table 38: Local Whittle estimates for the yields. For every series X, this table report the estimates $\tilde{d}(X)$ and $\tilde{d}(\Delta X) \equiv \widetilde{d-1}(\Delta X) + 1$ along with the test statistics (26) and (27) respectively. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

		T	Aaa	Aa	A	Baa
$m = 23$	$\tilde{d}(X)$	1.0246	1.1363	1.1138	1.0436	0.94433
	$t_{d=1}$	0.23583	1.307	1.0918	0.41784	-0.53401
$m = 51$	$\tilde{d}(X)$	1.0562	1.0969	1.0531	1.0012	0.95645
	$t_{d=1}$	0.80258	1.3841	0.759	0.017322	-0.62198
$m = 112$	$\tilde{d}(X)$	1.035	1.0439	1.0405	1.0134	0.87084
	$t_{d=1}$	0.73995	0.92852	0.85753	0.28369	-2.7338**
$m = 246$	$\tilde{d}(X)$	0.9958	1.0184	1.0338	1.0263	0.81509
	$t_{d=1}$	-0.13178	0.57583	1.0604	0.82409	-5.8004**
$m = 541$	$\tilde{d}(X)$	0.98032	0.96551	0.97165	0.952	0.93213
	$t_{d=1}$	-0.9156	-1.6044	-1.319	-2.2331*	-3.1573**
$m = 23$	$\tilde{d}(\Delta X)$	1.1145	1.1786	1.1422	1.0511	0.9164
	$\tau_{d=1}$	1.0979	1.7132	1.3638	0.48989	-0.80189
$m = 51$	$\tilde{d}(\Delta X)$	1.1035	1.086	1.0387	0.9757	0.91543
	$\tau_{d=1}$	1.4776	1.2285	0.55322	-0.34705	-1.2079
$m = 112$	$\tilde{d}(\Delta X)$	1.0497	1.041	1.0353	1.0084	0.86556
	$\tau_{d=1}$	1.0519	0.86829	0.74792	0.17851	-2.8455**
$m = 246$	$\tilde{d}(\Delta X)$	1.0202	1.0224	1.0378	1.0319	0.81136
	$\tau_{d=1}$	0.63507	0.70141	1.1865	1	-5.9175**
$m = 541$	$\tilde{d}(\Delta X)$	1.0145	0.98156	0.98845	0.96748	0.95415
	$\tau_{d=1}$	0.67399	-0.85789	-0.5375	-1.5128	-2.1328*

Table 39: Local Whittle estimates for the spreads. For every series X, this table report the estimates $\tilde{d}(X)$ and $\tilde{d}(\Delta X) \equiv \widetilde{d-1}(\Delta X)+1$ along with the test statistics (26) and (27) respectively. One asterisk denotes significance at 5% level and two asterisks denote significance at 1% level.

		sTAaa	sTAa	sTA	sTBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
$m=23$	$\tilde{d}(X)$	1.1717	1.1117	0.99789	0.90988	1.0813	1.0066	0.96996	0.91087	0.89467	0.72878
	$t_{d=1}$	1.6467	1.0709	-0.02021	-0.86441	0.77936	0.062928	-0.28811	-0.85494	-1.0103	-2.6015**
$m=51$	$\tilde{d}(X)$	1.0131	0.97652	0.9152	0.9057	1.0125	0.96721	0.93819	0.90455	0.88818	0.78432
	$t_{d=1}$	0.18725	-0.33534	-1.2113	-1.3468	0.17916	-0.46828	-0.88287	-1.3634	-1.5971	-3.0805**
$m=112$	$\tilde{d}(X)$	1.0023	0.99495	0.98792	0.86258	1.0004	1.0323	0.843	1.0045	0.79136	0.69449
	$t_{d=1}$	0.049605	-0.10684	-0.25558	-2.9085**	0.008162	0.68448	-3.3231**	0.09521	-4.4161**	-6.4665**
$m=246$	$\tilde{d}(X)$	0.97093	0.98067	0.97815	0.79325	0.97008	1.0266	0.75687	1.0132	0.71827	0.64924
	$t_{d=1}$	-0.91182	-0.60632	-0.68533	-6.4856**	-0.9384	0.83476	-7.6267**	0.415	-8.8376**	-11.003**
$m=541$	$\tilde{d}(X)$	0.93805	0.94879	0.93947	0.9488	0.90683	0.98603	0.93777	0.96052	0.92321	0.93546
	$t_{d=1}$	-2.8821**	-2.3823*	-2.816**	-2.3817*	-4.3341**	-0.64987	-2.8949**	-1.8364	-3.572**	-3.0023**
$m=23$	$\tilde{d}(\Delta X)$	1.2312	1.1598	1.0153	0.90972	1.0769	1.0036	0.97134	0.91059	0.90072	0.74891
	$\tau_{d=1}$	2.2176*	1.5332	0.1463	-0.86596	0.7375	0.034742	-0.27488	-0.8576	-0.95224	-2.4084*
$m=51$	$\tilde{d}(\Delta X)$	1.0768	1.0244	0.94109	0.9101	1.0062	0.96765	0.93845	0.90476	0.89026	0.79554
	$\tau_{d=1}$	1.0966	0.34857	-0.84145	-1.2841	0.087909	-0.46202	-0.87916	-1.3602	-1.5674	-2.9203**
$m=112$	$\tilde{d}(\Delta X)$	1.0147	1.0112	0.98043	0.82468	1.0069	1.0307	0.84994	1.0049	0.80597	0.71468
	$\tau_{d=1}$	0.31031	0.2364	-0.41416	-3.7108**	0.14694	0.64987	-3.1762**	0.10289	-4.1068**	-6.0391
$m=246$	$\tilde{d}(\Delta X)$	0.98308	1.0077	0.99019	0.76281	0.97788	1.0307	0.76446	1.0173	0.73106	0.66569
	$\tau_{d=1}$	-0.53073	0.24222	-0.3077	-7.4404**	-0.69377	0.96332	-7.3886**	0.54124	-8.4363**	-10.487**
$m=541$	$\tilde{d}(\Delta X)$	0.92833	0.94244	0.9258	0.94646	0.91755	1.0022	0.9564	0.97652	0.94539	0.96025
	$\tau_{d=1}$	-3.3342**	-2.6775**	-3.4519**	-2.4908*	-3.8353**	0.10244	-2.0284*	-1.0923	-2.5402*	-1.8489

Table 40: Nielsen (2005) LM test for yields. For the univariate case, we set $d = 1$ in eq. (18). For the multivariate case, we set $\mathbf{d} = \iota$ in eq. (21). Panel A reports univariate tests whereas Panel B reports multivariate tests.

Panel A

		T	Aaa	Aa	A	Baa
$p = 0$	LM	15.039	1.5956	1.6096	0.58635	0.084594
	pval	0.000105	0.20653	0.20454	0.44383	0.77117
$p = 1$	LM	0.18465	0.57407	0.18801	0.95166	0.60887
	pval	0.66741	0.44864	0.66457	0.3293	0.43521
$p = 2$	LM	0.024477	0.1242	0.010532	0.18311	0.024272
	pval	0.87568	0.72453	0.91826	0.66871	0.8762
$p = 3$	LM	0.34213	0.20205	0.16713	0.019909	0.002732
	pval	0.5586	0.65308	0.68268	0.88779	0.95831
$p = 4$	LM	0.31699	0.23361	0.24655	0.087709	0.040775
	pval	0.57342	0.62886	0.61952	0.76711	0.83997

Panel B

$p = 0$	LM	10.155	LMK	48.357
	pval	0.001439	pval	3.00E-09
$p = 1$	LM	8.9621	LMK	12.561
	pval	0.002756	pval	0.027855
$p = 2$	LM	1.0671	LMK	3.6541
	pval	0.30159	pval	0.6002
$p = 3$	LM	0.029004	LMK	0.99808
	pval	0.86477	pval	0.96272
$p = 4$	LM	0.14536	LMK	1.5699
	pval	0.70301	pval	0.90487

Table 41: Nielsen (2005) LM test for spreads. For the univariate case, we set $d = 1$ in eq. (18). For the multivariate case, we set $\mathbf{d} = \iota$ in eq. (21). Panel A reports univariate tests whereas Panel B reports multivariate tests for spreads over Treasury only.

Panel A

	sAaa	sAa	sA	sBaa	sAaaAa	sAaaA	sAaaBaa	sAaA	sAaBaa	sABaa
$p = 0$	24.31339	20.0056	11.77286	2.852134	56.06828	10.53994	0.000524	1.78096	0.631179	8.051394
	8.19E-07	7.72E-06	0.000601	0.091253	6.99E-14	0.001168	0.981734	0.182031	0.426923	0.004547
$p = 1$	6.097408	3.493377	4.736794	0.958987	11.85417	0.744657	0.069806	0.003883	0.620024	3.682575
	0.013538	0.061615	0.029524	0.327442	0.000575	0.388173	0.791619	0.950312	0.431038	0.054984
$p = 2$	1.965735	0.430146	0.75116	0.022034	1.748351	0.000364	0.394406	0.00677	0.272301	0.055102
	0.160901	0.511917	0.386109	0.881996	0.186084	0.984788	0.529993	0.934424	0.601792	0.814412
$p = 3$	0.008452	0.013447	0.033195	0.000772	0.362768	0.241951	0.03548	0.229294	0.0001	0.213746
	0.926748	0.907685	0.85543	0.977834	0.546973	0.6228	0.850593	0.632048	0.99201	0.643847
$p = 4$	0.001935	0.08217	0.028961	0.207774	0.03761	0.331287	0.065499	0.033083	0.260934	0.788442
	0.964913	0.774378	0.864869	0.648517	0.846228	0.564902	0.798006	0.855671	0.609479	0.374572

Panel B

$p = 0$	LM	25.33604	LMK	28.98468
	pval	4.82E-07	pval	7.87E-06
$p = 1$	LM	12.97074	LMK	13.03932
	pval	0.000316	pval	0.011085
$p = 2$	LM	1.23618	LMK	2.509452
	pval	0.266209	pval	0.642944
$p = 3$	LM	0.227847	LMK	0.292957
	pval	0.633125	pval	0.990264
$p = 4$	LM	0.335473	LMK	1.22171
	pval	0.562454	pval	0.874511

Table 42: Cointegration analysis: unit root and stationarity tests for bivariate systems. Cointegration analysis is performed for all possible bivariate systems $X - Y$. For each pair of variables X and Y, the Dickey-Fuller with the constant (DF), augmented Dickey-Fuller with the constant (ADF), Phillips-Perron with the constant (PP), a two KPSS tests without trend are carried out on the estimated OLS residuals of the regression of Y on X and a constant. In the first KPSS test the Bartlett kernel with bandwidth parameter $\left[4\left(\frac{n}{100}\right)^{1/4}\right]$ is chosen for the estimation of the long run variance. In the second test the automatic bandwidth selection procedure of Hobijn et al. (1998) is considered.

	T - Aaa	T - Aa	T- A	T - Baa	Aaa - Aa	Aaa - A	Aaa - Baa	Aa - A	Aa - Baa	A - Baa
DF	-1.7982	-1.8528	-1.9715	-2.9225*	-2.479	-1.5925	-3.1137*	-1.8512	-3.546**	-5.3516**
ADF1	-1.6988	-1.7673	-1.9387	-2.8895*	-2.2334	-1.6715	-3.104*	-1.9196	-3.5216**	-5.2946**
ADF2	-1.642	-1.7077	-1.8504	-3.1018*	-2.0687	-1.6382	-3.4195*	-1.8036	-3.8791**	-5.9872**
ADF3	-1.5731	-1.666	-1.8045	-3.095*	-2.035	-1.6134	-3.4449**	-1.752	-3.9115**	-6.1183**
ADF4	-1.5627	-1.6539	-1.7772	-3.1314*	-1.987	-1.581	-3.4707**	-1.7518	-3.9604**	-6.2317**
ADF5	-1.5363	-1.6476	-1.7814	-2.3416	-1.9766	-1.6322	-2.4037	-1.769	-2.7033	-4.4134**
PP	-1.5876	-1.6955	-1.8444	-2.6246	-2.1118	-1.669	-2.7963	-1.841	-3.1658*	-5.0413**
KPSS	3.1**	3.2061**	4.2174**	5.0121**	3.3863**	4.9844**	5.4244**	4.8348**	4.9929**	2.8503**
KPSS HFO	0.82996**	0.86036**	1.1332**	1.3654**	0.91777**	1.3418**	1.4873**	1.3098**	1.3829**	0.84248**

Table 43: Sensitivity Analysis for the fractional Merton model.

Equity	Spread
$\frac{\partial}{\partial r} S_0 = DT\mathrm{e}^{-rT} N(d_2)$	$\frac{\partial}{\partial r} s = N(d_2)\mathrm{e}^{sT} - 1 = -\frac{V_0 N(-d_1)}{B}$
$\frac{\partial}{\partial \sigma} S_0 = V_0 T^H n(d_1)$	$\frac{\partial}{\partial \sigma} s = T^{H-1} n(d_2)\mathrm{e}^{sT}$
$\frac{\partial}{\partial \ell} S_0 = -V_0 N(d_2)$	$\frac{\partial}{\partial \ell} s = \frac{1-N(d_2)\mathrm{e}^{sT}}{T\ell} = \frac{N(-d_1)\mathrm{e}^{sT}}{T\ell^2}$
$\frac{\partial}{\partial T} S_0 = V_0 \sigma H T^{H-1} n(d_1) + rD\mathrm{e}^{-rT} N(d_2)$	$\frac{\partial}{\partial T} s = \sigma H T^{H-2} n(d_2)\mathrm{e}^{sT} + \frac{-s+r\left[\mathrm{e}^{sT} N(d_2)-1\right]}{T}$
$\frac{\partial}{\partial H} S_0 = V_0 \log T \sigma T^H n(d_1)$	$\frac{\partial}{\partial H} s = \log T \sigma T^{H-1} n(d_2)\mathrm{e}^{sT}$

Table 44: Sensitivity Analysis for the fractional Black and Cox model.

Equity	Spread
$\frac{\partial}{\partial\sigma} S_0 = T^H \left[V_0 n(d_1) - L n(y_1)\right]$	$\frac{\partial}{\partial\sigma} s = T^{H-1} \mathrm{e}^{sT} \left[n(d_2) - \frac{V_0}{L} n(y_2)\right]$
$\frac{\partial}{\partial\ell} S_0 = V_0 \left[N(y_2) - N(d_2)\right] + \frac{Ln(y_1)}{\sigma\ell T^H} \left[1 - \frac{L}{V_0}\right]$	$\frac{\partial}{\partial\ell} s = \frac{1}{T} \left[\frac{1}{\overline{B}(0,T)} \left(\frac{\partial S_0}{\partial\ell}\right) + \frac{1}{\ell}\right]$
$\frac{\partial}{\partial T} S_0 = \sigma H T^{H-1} \left[V_0 n(d_1) - L n(y_1)\right]$	$\frac{\partial}{\partial T} s = \sigma H T^{H-2} \mathrm{e}^{sT} \left[n(d_2) - \frac{V_0}{L} n(y_2)\right] - \frac{s}{T}$
$\frac{\partial}{\partial H} S_0 = \log T \sigma T^H \left[V_0 n(d_1) - L n(y_1)\right]$	$\frac{\partial}{\partial H} s = \log T \sigma T^{H-1} \mathrm{e}^{sT} \left[n(d_2) - \frac{V_0}{L} n(y_2)\right]$