Applications of the Dirac Delta Family Method in Implied Volatility, Risk-neutral Density, and High-dimensional Stochastic Control

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Overview

- Dirac Delta family method
- Explicit implied volatility formula
- Risk-neutral density estimation
- High-dimensional stochastic control
- Conclusion

The presentation is mainly based on the following recent research:

- Z. Cui, J. Kirkby, D. Nguyen, S. Taylor. A closed-form model-free implied volatility formula through Delta families. *Journal of Derivatives*. 2021 Summer Issue, 28(4), 111-127.
- Z. Cui, Y. Xu. A new approach to recover risk-neutral distributions from options. *Quantitative Finance*, 2022, forthcoming.
- J. Ma, Z. Lu, Z.Cui. Delta family approach for the stochastic control problems of utility maximization. *working paper*, 2022.

Main References

The Delta sequence is originated from statistics with focus on nonparametric density estimation or nonparametric regression:

- One-dimensional theory: Walter, G. and Blum, J. Probability density estimation using delta sequences. *Annals of Statistics*, 1979, 7(2), pp.328-340.
- Multi-dimensional theory: Susarla V, Walter G. Estimation of a multivariate density function using delta sequences. *Annals of Statistics*, 1981, 9(2), pp.347-355.
- Multivariate diffusion density estimation: Yang, N., Chen, N. and Wan, X. A new delta expansion for multivariate diffusions via the Ito-Taylor expansion. *Journal of Econometrics*, 2019, 209(2), pp.256-288.
- Regression applications: Strobl, E. and Visweswaran, S. Dirac Delta Regression: Conditional Density Estimation with Clinical Trials. *Proceedings of Machine Learning Research*, 2021, pp.1-48.

Dirac Delta Family Method

- For an open subset $\Omega \subset \mathbb{R}$, let $C_0^\infty(\Omega)$ denote the space of infinitely differentiable functions on Ω having compact support.
- A family of functions $\delta_\varepsilon \in L^\infty(\Omega \times \Omega)$ is said to be a delta family on Ω if for each $\varphi \in C_0^\infty(\Omega)$ and $x \in \Omega$, the following holds
$$\lim_{\varepsilon \to 0} \int_\Omega \delta_\varepsilon(x-y)\varphi(y)dy = \varphi(x).$$
- Recalling the sifting property of the Dirac Delta function, i.e., $\int_\Omega \delta(x-y)\varphi(y)dy = \varphi(x)$. The delta family essentially converges to the Dirac Delta function as $\varepsilon \to 0$.
- The next table lists several common examples of delta families.

For convenience, we will utilize the following Gaussian form

$$\delta_\varepsilon(x-y) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}, \tag{1}$$

which resembles the probability density function (pdf) of a normal random variable.

Types	delta families $\delta_\varepsilon(x-y)$
Normal density	$\frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}$
Lorentzian type	$\frac{1}{\pi}\frac{\varepsilon}{(x-y)^2+\varepsilon^2}$
Fourier integral	$\frac{1}{2\pi}\int_{-1/\varepsilon}^{1/\varepsilon} e^{i(x-y)t}dt$
Trigonometric function	$\frac{1}{2\pi}\frac{\sin\left[\left(\frac{1}{\varepsilon}+\frac{1}{2}\right)(x-y)\right]}{\sin\left(\frac{1}{2}(x-y)\right)}$

Table: Different types of delta families.

Orthogonal series representation

Besides the above converging (smooth) representation of Delta family, there exist exact distributional representations[2] of the Dirac Delta function through orthogonal series.

- Let $\{g_k\,(y)\}_{k=0}^{\infty}$ be a complete orthonomal basis, then

$$\delta(x-a) = \sum_{k=0}^{\infty} g_k(x) g_k\,(a)\,. \tag{2}$$

- Above holds in "distributional" sense and essentially equivalent to the *completeness* of the basis.

[2] Li, Y.T. and Wong, R. Integral and series representations of the Dirac delta function. *Communications on Pure and Applied Analysis*, 2008, 7(2), pp. 229-247.

- Three cases[3]:

$$\delta(x-a) = \sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)P_k(x)P_k(a),$$
$$\delta(x-a) = \frac{e^{-(x^2+a^2)/2}}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{1}{2^k k!}H_k(x)H_k(a),$$
$$\delta(x-a) = e^{-(x+a)/2}\sum_{k=0}^{\infty}L_k(x)L_k(a), \tag{3}$$

  where $P_k(x)$, $H_k(x)$ and $L_k(x)$ are respectively the Legendre, Hermite and Laguerre polynomials.
- We pick different orthogonal polynomials depending on the particular application, e.g. Hermite is suitable for transition density approximation, Laguerre is suitable for first passage time density approximation, and Legendre is suitable for compact domain.

[3] The following formulas appear respectively as formula (1.17.22), (1.17.24) and (1.17.23) in the NIST handbook of Mathematical Functions, a definite reference on special functions. `https://dlmf.nist.gov/1.17`

Definition

(Sursarla and Walter 1981): A delta family $\{\delta_\varepsilon\}$ *on* $\mathbb{R}$ *is said to be of positive type if* $\delta_\varepsilon \geq 0$ *holds and for each* $x \in \mathbb{R}$,

1. $\displaystyle\int_{\mathbb{R}} \delta_\varepsilon(x-y)dy = 1;$
2. $\displaystyle\sup_{r>0}\left\{\int_{|x-y|>r} \delta_\varepsilon(x-y)dy\right\} = \mathcal{O}(\varepsilon)$, *as* $\varepsilon \to 0;$
3. $||\delta_\varepsilon(x-y)||_\infty = \mathcal{O}(\varepsilon)$, *as* $\varepsilon \to 0;$
4. *For each* $\eta > 0$, $\sup\{\delta_\varepsilon(x-y)||x-y| > \eta\} \to 0$, *as* $\varepsilon \to 0$.

Implied Volatility Definition

- Consider Black Scholes model: $dS_t = (r-q)S_t dt + \sigma S_t dW_t$, where $r \in \mathbb{R}^+$ is the risk-free interest rate, $q \in \mathbb{R}^+$ is the dividend (convenience) yield.
- Recall the Black-Scholes formula $f_{BS}(\sigma)$:

$$f_{BS}(\sigma) \equiv f_{BS}(\sigma, r, q, K, S_0, T) = \begin{cases} S_0 e^{-qT}\mathcal{N}(d_1(\sigma)) - Ke^{-rT}\mathcal{N}(d_2(\sigma)), & \text{call,} \\ Ke^{-rT}\mathcal{N}(-d_2(\sigma)) - S_0 e^{-qT}\mathcal{N}(-d_1(\sigma)), & \text{put,} \end{cases}$$

and

$$d_1(\sigma) := \frac{\ln(S_0/K) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}, \quad d_2(\sigma) := d_1(\sigma) - \sigma\sqrt{T},$$

with $\mathcal{N}(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$.

- We can derive the Vega formula:

$$f'_{BS}(\sigma) = S_0\sqrt{T}\mathcal{N}'(d_1(\sigma)) > 0, \tag{4}$$

hence $f_{BS}(\sigma)$ is monotonically increasing in σ.

- Given an observed market price C of an European option, we aim to solve for the root of the following equation

$$f_{BS}(\sigma) = C. \tag{5}$$

Model-free Implied Volatility Bounds[4]

Lemma

Let

$$c := \frac{C}{e^{-qT}S_0}, \quad k := \ln \frac{K}{S_0 e^{(r-q)T}}.$$

Assume that $(1-e^k)^+ \leq c < 1$. *For* $k \geq 0$ *the following inequalities hold:*

$$\frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2}\right) \leq \sigma(C) \leq \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{1+e^k}\right),$$

and for $k < 0$ *we have*

$$\frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{2e^k}\right) \leq \sigma(C) \leq \frac{-2}{\sqrt{T}}\mathcal{N}^{-1}\left(\frac{1-c}{1+e^k}\right).$$

[4] Tehranchi, M.R. Uniform bounds for Black–Scholes implied volatility. *SIAM Journal on Financial Mathematics*, 2016, 7(1), pp.893-916.

Implied Volatility Representation

Let $C_l = f(\sigma_l)$ and $C_u = f(\sigma_u)$, which are defined in previous lemma.

1. $f_{BS}^{-1} \in L^p(C_l, C_u)$ for any $p \geq 0$.
2. $|f_{BS}^{-1}(C_1) - f_{BS}^{-1}(C_0)| \leq M_1 |C_1 - C_0|^\gamma$ for some $M_1 > 0$, and for any $\gamma \in (0, 1]$.
3. For a positive type $\{\delta_\varepsilon\}$ family,

$$\left| \int_{C_l}^{C_u} \delta_\varepsilon(x-y) f_{BS}^{-1}(y) dy - f_{BS}^{-1}(x) \right| = \mathcal{O}(\varepsilon), \quad \text{as} \quad \epsilon \to 0.$$

   This, in particular, implies that

$$f_{BS}^{-1}(x) = \lim_{\varepsilon \to 0} \int_{C_l}^{C_u} \delta_\varepsilon(x-y) f_{BS}^{-1}(y) dy.$$

   which establishes the convergence of $\int_{C_l}^{C_u} \delta_\varepsilon(x-y) f_{BS}^{-1}(y) dy$ to the true implied volatility value when we substitute $x = C$, with C being the market-observable price.

Assume that $\{\delta_\varepsilon\}$ is a delta family of positive type, then $f_{BS}^{-1}(x) = \lim_{\varepsilon\to 0+} \int_{\sigma_l}^{\sigma_u} \delta_\varepsilon(f_{BS}(s) - x) \cdot s f_{BS}'(s) ds =: \lim_{\varepsilon\to 0+} \sigma_{BS,\varepsilon}(x)$.

$$
\begin{aligned}
f_{BS}^{-1}(x) &= \int_{C_l}^{C_u} f_{BS}^{-1}(u)\delta(u-x)du \\
&= \int_{C_l}^{C_u} f_{BS}^{-1}(u) \left(\lim_{\varepsilon\to 0+} \delta_\varepsilon(u-x) \right) du \\
&= \lim_{\varepsilon\to 0+} \int_{C_l}^{C_u} f_{BS}^{-1}(u)\delta_\varepsilon(u-x)du \\
&= \lim_{\varepsilon\to 0+} \int_{\sigma_l}^{\sigma_u} f_{BS}^{-1}(f_{BS}(s))\delta_\varepsilon(f_{BS}(s)-x)df_{BS}(s) \\
&= \lim_{\varepsilon\to 0+} \int_{\sigma_l}^{\sigma_u} \delta_\varepsilon(f_{BS}(s)-x) \cdot s f_{BS}'(s)ds, \qquad (6)
\end{aligned}
$$

where in the fourth equality sign we have utilized the change of variable $u = f_{BS}(s)$, which maps the domain $u \in [C_l, C_u]$ to the domain $s \in [\sigma_l, \sigma_u]$ due to monotonicity.

- Consider the delta family $\delta_\varepsilon(x-y) = \frac{1}{2\sqrt{\pi\varepsilon}} e^{-\frac{(x-y)^2}{4\varepsilon}}$.
- Using the fact that

$$f'_{BS}(\sigma) = S_0\sqrt{T}\mathcal{N}'(d_1(\sigma)),$$

then we have

$$\begin{aligned}\sigma_{BS}(C) &= \lim_{\varepsilon\to 0} S_0\sqrt{T}\int_{\sigma_l}^{\sigma_u} \frac{s}{2\sqrt{\pi\varepsilon}} e^{-\frac{(f_{BS}(s)-C)^2}{4\varepsilon}} \mathcal{N}'(d_1(s))ds \\ &= \lim_{\varepsilon\to 0} \frac{S_0\sqrt{T}}{2\pi\sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s\cdot\exp\left(-\frac{(f_{BS}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right) ds.\end{aligned} \tag{7}$$

- The formula (7) is explicit, and does not involve iterative numerical recursions as in the Newton's method.
- We take a small ε and carry out numerical integration. The current implementation is still not comparable with the speed of Newton method with a well-chosen starting point.
- However, the explicit form facilitates computation of sensitivities of implied volatility, e.g.

$$\begin{aligned}\frac{\partial \sigma_{BS}(C)}{\partial K} &= \lim_{\varepsilon \to 0} \frac{S_0\sqrt{T}}{2\pi\sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s \cdot \frac{\partial}{\partial K} \exp\left(-\frac{(f_{BS}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right) ds \\ &= \lim_{\varepsilon \to 0} \frac{S_0\sqrt{T}}{2\pi\sqrt{2\varepsilon}} \int_{\sigma_l}^{\sigma_u} s \cdot \exp\left(-\frac{(f_{BS}(s)-C)^2}{4\varepsilon} - \frac{d_1^2(s)}{2}\right) \left(-\frac{f_{BS}(s)-C}{2\varepsilon} \frac{\partial f_{BS}(s)}{\partial K} - d_1(s) \frac{\partial d_1(s)}{\partial K}\right) ds.\end{aligned}$$

- Future research: develop efficient high-precision numerical integration methods for the integrals involved.

Numerical Examples

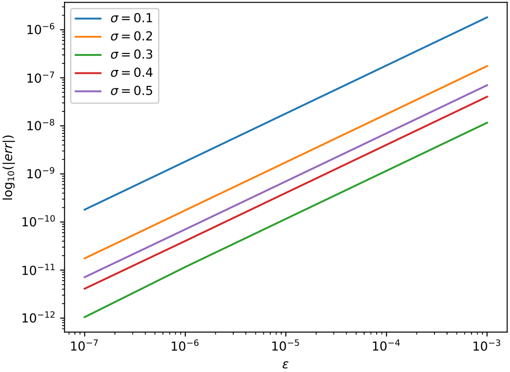


Figure: Implied volatility convergence of our formula as a function of ε for option prices generated from Black-Scholes model, with $\sigma \in \{0.1, \ldots, 0.5\}$, $T = 0.5$, $r = 0.05$, $K = S_0 = 100$

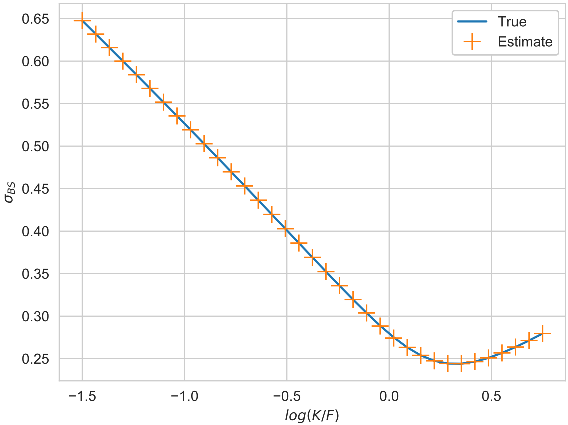


Figure: Implied volatility fit using our formula when data generating process is SABR model with parameters: $\alpha = 0.7, \beta = 0.8, \rho = -0.5, \nu = 0.7,\ T = 0.5,\ r = 0.00,\ F_0 = 100.$

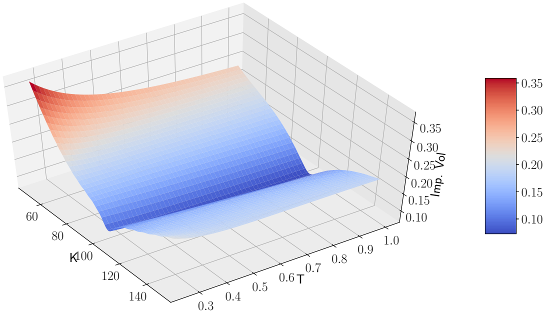


Figure: Heston model implied volatility surface computed from our formula with parameters $S_0 = 100$, $r = 3.19\%$, $q = 0\%$, $v_0 = 0.02$, $\rho = -0.1$, $\kappa = 0.15$, $\theta = 0.05$, $\sigma = 1.3$, and $\epsilon = 0.01$.

Risk-neutral density

- Risk-neutral density is central to derivatives valuation and asset pricing.

  1. Derivatives valuation: any European derivative price is the risk-neutral expectation of the payoff function[5].
  2. Asset pricing: assume a representative agent with time-separable utility $U(\cdot)$, then

$$f^{Q}\left(S_{T}\right) \propto \exp(rT)\beta \frac{U'\left(S_{T}\right)}{U'\left(S_{0}\right)} f^{P}\left(S_{T}\right) \tag{8}$$

[5] Andricopoulos, A. D., Widdicks, M., Duck, P. W., and Newton, D. P. Universal option valuation using quadrature methods. *Journal of Financial Economics*, 2003, 67(3), pp. 447–471.

- Option prices provide natural *forward-looking* information:
    1. Obtain risk-aversion from option prices (Bliss and Panigirtzolou 2004)
    2. Skewness and expected stock returns from options (Conrad, Dittmar and Ghysels 2013)
    3. Improve portfolio selection using options information (DeMiguel, Plyakha and Vilkov 2013)
    4. Market disasters implied from index options (Backus, Chernov and Martin 2011)
    5. Option implied correlations (Buss and Vilkov 2012)
- Option prices can be used to infer physical distributions under some assumptions, a stream of literature named *Ross recovery*, Ross (2015); Carr and Yu (2012); Bakshi, Chabi-Yo and Gao (2015); Jensen, Lando and Pedersen (2019).

Parametric Estimation Methods

Structural assumptions made either on the functional forms of the risk-neutral distributions or on the asset dynamics:

- Normal mixtures (Soderlind and Svensson 1997)
- Log-normal mixtures (Jarrow and Rudd 1982, Longstaff 1995)
- Edgeworth expansions (Rubinstein 1998)
- Hermite functions (Madan and Milne 1994, Xiu 2014)

Non-parametric or Semi-parametric Estimation Methods

- Implied tree/lattice (Jackwerth and Rubinstein 1996)
- Kernel-based method (Aït-Sahalia and Lo 1998)
- Maximum entropy (Buchen and Kelley 1996)
- Curve fitting method: based on Breeden and Litzenberger (1978)
  1. Fit options prices (Breeden and Litzenberger 1978)
  2. Fit implied volatility (Shimko 1993, Bliss and Panigzoglou 2004)

Key Motivation

- Most of the existing literature crucially rely on the result of Breeden and Litzenberger (1978):

$$\frac{\partial^2 C(K)}{\partial K^2} = e^{-rT} f\left(S_T\right). \tag{9}$$

- It is an elegant model-free result, however, it has clear drawbacks:

  1. The partial derivative has to be approximated by finite difference schemes, with errors.
  2. The recovered density may not be positive or smooth, hence interpolation is needed.

Inspired by VIX

- The CBOE (together with Goldman Sachs) revised its calculation methodology of the volatility index (VIX) in 2004:

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2 \qquad (10)$$

- Behind it is the celebrated Carr-Madan spanning formula: $f \in C^2$,

$$\begin{aligned} f(S) = f(\kappa) + f'(\kappa)(S-\kappa) + \int_0^{\kappa} f''(K)(K-S)^+ dK \\ + \int_{\kappa}^{\infty} f''(K)(S-K)^+ dK, \end{aligned} \qquad (11)$$

for any reference point κ.

- Our main research question is:

Can we recover the risk-neutral density similarly as the VIX?

- Answer turns out to be:

Yes, but we need to overcome a challenge.

Challenge (and solution!)

- Risk-neutral density function satisfies:

$$f(y \mid x) = \mathbb{E}_x[\delta(X_{t+\Delta} - y)], \tag{12}$$

  where $\delta(\cdot)$ is the Dirac Delta function.

- *Dirac Delta function* is not twice-differentiable, thus can not directly apply Carr-Madan spanning.

- We resort to an exact distributional representation of the Dirac Delta function using complete orthonormal basis of Hilbert space $\implies$ recover smoothness!

Key representation

- Above all, there is:

$$f(y \mid x) = \mathbb{E}_x[\delta(X_{t+\Delta} - y)] = \sum_{k=0}^{\infty} g_k(y)\, \mathbb{E}_x[g_k(X_{t+\Delta})].$$

- Interpretation: risk-neutral density expressed in terms of a series of weighted European style derivative contracts with **smooth** payoffs given by $g_k(x)$.
- Key observation: we can apply Carr-Madan spanning to $\mathbb{E}_x[g_k(X_{t+\Delta})]$!
- Essence: treat risk-neutral density as a hypothetical derivative (with Dirac Delta payoff) to be spanned by European call/put options.

Risk-neutral Density Estimator

- Under the risk-neutral measure, the put and call options are priced at $P(K) := e^{-r\Delta}\mathbb{E}_x[(K - X_{t+\Delta})^+]$, and $C(K) := e^{-r\Delta}\mathbb{E}_x[(X_{t+\Delta} - K)^+]$.
- The risk-neutral density is

$$
\begin{aligned}
f(y \mid x) \quad = \quad & \sum_{k=0}^{\infty} g_k(y)\, g_k(\kappa) + \sum_{k=0}^{\infty} g_k(y)\, g_k'(\kappa)\,(F - \kappa) \\
& + \sum_{k=0}^{\infty} g_k(y) \left(e^{r\Delta} \int_0^{\kappa} g_k''(K) P(K) dK \right) \\
& + \sum_{k=0}^{\infty} g_k(y) \left(e^{r\Delta} \int_{\kappa}^{\infty} g_k''(K) C(K) dK \right). \quad (13)
\end{aligned}
$$

Discretization in Strike Space

- In practice, there is no supply of options with a continuum of strikes. Thus we discretize the integrals in the strike space.
- This is similar in spirit with the discretization that leads to the final CBOE VIX formula.
- Choose $\kappa := K^*$, which is the first strike below the forward index level F. Assume available strike prices: $K_0 = 0 < K_1 < K_2 < \cdots < K_m = K^* < K_{m+1} < \cdots < K_M$ (unequal spacing allowed).
- Let $P(K_i)$ or $C(K_i)$ denote respectively the midpoint of the bid-ask spread for each put or call option with strike K_i.

- Define for a generic function g and a generic price function P:

$$\beta\left(a, b; g'', P\right) := P(a)\left(\frac{g\left(b\right) - g\left(a\right)}{b - a} - g'\left(a\right)\right) + P(b)\left[g'\left(b\right) - \frac{g\left(b\right) - g\left(a\right)}{b - a}\right]. \quad (14)$$

- There is

$$\int_0^{\kappa} g_k''(K)P(K)dK \simeq \sum_{i=1}^{m} \beta\left(K_{i-1}, K_i; g_k'', P\right). \quad (15)$$

- Similarly

$$\int_{\kappa}^{\infty} g_k''(K)C(K)dK \simeq \sum_{i=m+1}^{M} \beta\left(K_{i-1}, K_i; g_k'', C\right). \quad (16)$$

Risk-neutral Density Estimator

$$\hat{f}(y \mid x) = \underbrace{\sum_{k=0}^{\infty} g_k(y)\, g_k(\kappa)}_{\text{cash position}} + \underbrace{\left(\sum_{k=0}^{\infty} g_k(y)\, g_k'(\kappa)\right)(F - \kappa)}_{\text{forward position}} + A + B.$$

$$A := e^{r\Delta} \sum_{k=0}^{\infty} g_k(y) \sum_{i=1}^{m} \left[\underbrace{P(K_{i-1}) \left(\frac{g_k(K_i) - g_k(K_{i-1})}{K_i - K_{i-1}} - g_k'(K_{i-1}) \right)}_{\text{put option positions}} \right.$$

$$\left. + \underbrace{P(K_i) \left[g_k'(K_i) - \frac{g_k(K_i) - g_k(K_{i-1})}{K_i - K_{i-1}} \right]}_{\text{put option positions}} \right].$$

$$B :=$$

$$e^{r\Delta}\sum_{k=0}^{\infty} g_k(y) \sum_{i=m+1}^{M}\left[\underbrace{C(K_{i-1})\left(\frac{g_k(K_i)-g_k(K_{i-1})}{K_i-K_{i-1}} - g_k'(K_{i-1})\right)}_{\text{call option positions}}\right.$$

$$\left. + \underbrace{C(K_i)\left[g_k'(K_i) - \frac{g_k(K_i)-g_k(K_{i-1})}{K_i-K_{i-1}}\right]}_{\text{call option positions}}\right].$$

- In practice, we truncate the series at N terms:

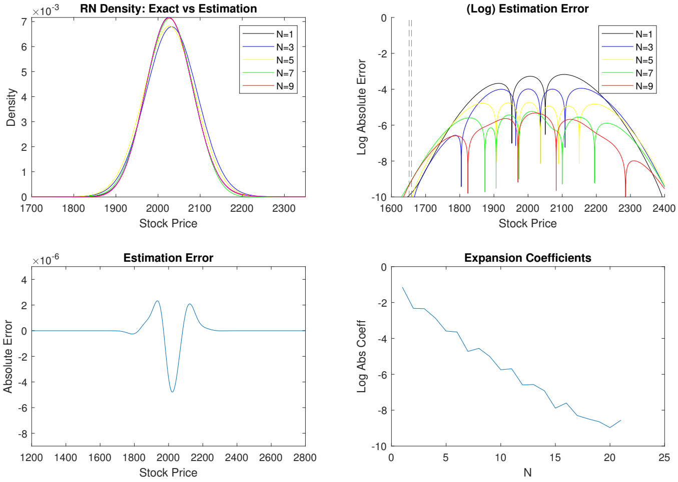
$$
\begin{aligned}
\hat{f}_N(y \mid x) \quad = \quad & \sum_{k=0}^{N} g_k\left(y\right) g_k(\kappa) + \left(\sum_{k=0}^{N} g_k\left(y\right) g_k'(\kappa)\right)(F-\kappa) \\
& + e^{r\Delta} \sum_{k=0}^{N} g_k\left(y\right) \sum_{i=1}^{m} \beta\left(K_{i-1}, K_i; g_k'', P\right) \\
& + e^{r\Delta} \sum_{k=0}^{N} g_k\left(y\right) \sum_{i=m+1}^{M} \beta\left(K_{i-1}, K_i; g_k'', C\right).
\end{aligned}
$$

- Remark: market-implied information **only** goes into the coefficients of the series.

Simulation Studies

- Choose a known model with exact risk-neutral density, simulate from it, and check how accurate our method recovers the risk-neutral density.
- Choose Black-Scholes option pricing model, although other candidates are entirely possible.
- We use the Black-Scholes model to generate simulated option prices for put and call options with strike prices $5 apart, which is the standard option strike price interval for S&P 500 index options.
- The simulations are carried out at maturities of (about) one month.
- Key parameters in the Black-Scholes model are obtained by roughly fitting the Black-Scholes model to the data of S&P 500 index options from OptionMetrics.

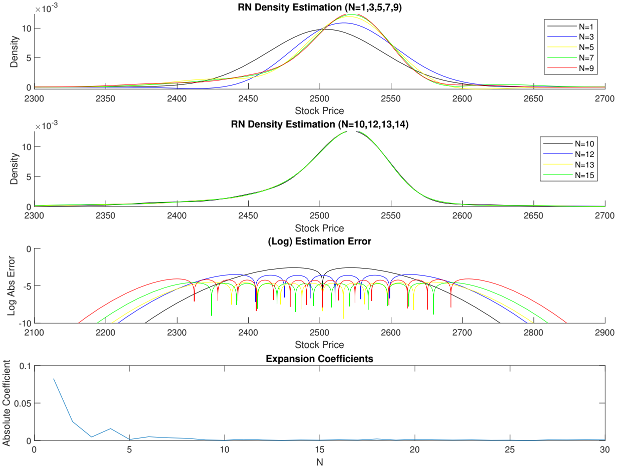
Simulation Results: \$5 strike, 1 month maturity



Empirical Studies

- We recover risk-neutral density from options prices obtained from the OptionMetrics database.
- We apply our method not only to the most commonly used maturities such as one month and one quarter, but also to the maturity of one week as a test of the power of our method.
- It is well-known that recovering risk-neutral density for shorter maturities is a harder task due to its narrow shape.
- The results reported here is for September 18, 2017, although we studied other dates as well with similar results.

Empirical Results: maturity of a month



Applications to Stochastic Control

- We have the following representation of the transition density function:

$$f(y \mid x) = \mathbb{E}_x[\delta(X_{t+h} - y)], \tag{17}$$

where $\delta(\cdot)$ is the Dirac Delta function.

- Recall the orthogonal series representation of Dirac Delta function:

$$\delta(x - a) = \sum_{k=0}^{\infty} g_k(x) g_k(a) \,. \tag{18}$$

- Combining the two expressions above, we have

$$f(y \mid x) = \sum_{k=0}^{\infty} g_k(y) \, \mathbb{E}_x[g_k(X_{t+h})]. \tag{19}$$

Intuitive development

- Consider a general stochastic control problem

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[U\left(X_T\right) \mid X_t = x\right].$$

where X_t depends on the unknown control π. The utility function does not depend on the unknown control.

- From Bellman's dynamical programming principle (DPP):

$$V(t,x) = \sup_{\pi \in \mathcal{A}} \mathbb{E}\left[V\left(t+h, X_{t+h}\right) \mid X_t = x\right]. \tag{20}$$

- The traditional approach proceeds as follows:
  1. Assume $V(t,x) \in C^{1,2}$, and then apply Ito's lemma to rewrite the right hand side of (20) into a differential form.
  2. Then take the limit $h \to 0$ to characterize the local behavior of the value function. This is achieved through obtaining the HJB PDE, which is a highly nonlinear PDE and challenging to solve.

- We shall take an alternative approach, and the idea is to rewrite the conditional expectation in (20) using the conditional transition density representation.
- The unknown control is only associated with the conditional density function.

$$
\begin{aligned}
\mathbb{E}[V(t+h, X_{t+h}) \mid X_t = x] &= \int_{\mathbb{R}_+} V(t+h, y) f(y \mid x) dy \\
&= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}_+} V(t+h, y) g_k(y)\, dy \right) \mathbb{E}_x[g_k(X_{t+h})] \\
&= \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}_+} V(t+h, y) g_k(y)\, dy \right) \left(\int_t^{t+h} \mathbb{E}_x[\mathcal{L}^{\pi} g_k(X_u)] du + g_k(x) \right), \quad (21)
\end{aligned}
$$

where $\mathcal{L}^{\pi}$ is the infinitesimal generator of the Markov process X, which contains the unknown control.
- Plugging (21) into (20), we have

$$
\begin{aligned}
V(t,x) &= \sup_{\pi \in \mathcal{A}} \mathbb{E}[V(t+h, X_{t+h}) \mid X_t = x] \\
&= \sup_{\pi \in \mathcal{A}} \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}_+} V(t+h, y) g_k(y)\, dy \right) \left(\int_t^{t+h} \mathbb{E}_x[\mathcal{L}^{\pi} g_k(X_u)] du + g_k(x) \right).
\end{aligned}
$$

- Assume that $h \to 0$ is very small, from the mean value theorem, we have the following approximation: $\pi_u \to \pi_t = \pi$, which is an unknown constant, and $X_u \to X_t = x$, i.e.

$$\int_t^{t+h} \mathbb{E}_x[\mathcal{L}^\pi g_k(X_u)]du \approx h \cdot \mathcal{L}^\pi g_k(x).$$

We have

$$V(t,x) \approx \sup_{\pi \in \mathcal{A}} \sum_{k=0}^{\infty} \left(\int_{\mathbb{R}_+} V(t+h,y) g_k(y)\, dy \right) \left(h \cdot \mathcal{L}^\pi g_k(x) + g_k(x) \right). \quad (22)$$

- The terminal condition is $V(T,x) = U(x)$, where $U(\cdot)$ is the known utility function.

- Let π^* be the solution of

$$\sum_{k=0}^{\infty}\left(\int_{\mathbb{R}_+} V(t+h,y)g_k(y)\,dy\right)\frac{\partial \mathcal{L}^{\pi} g_k(x)}{\partial \pi} = 0. \qquad (23)$$

- Then inserting $\pi = \pi^*$ into (22) gives that

$$V(t,x) \approx \sum_{k=0}^{\infty}\left(\int_{\mathbb{R}_+} V(t+h,y)g_k(y)\,dy\right)\left(h\cdot \mathcal{L}^{\pi^*} g_k(x) + g_k(x)\right). \qquad (24)$$

- Taking inner product of (24) with $g_j(x)$, $j = 0, 1, \ldots$ and denoting

$$V_j(t) := \int_{\mathbb{R}_+} V(t,x)g_j(x)dx, \quad j = 0, 1, \ldots,$$

  we obtain

$$V_j(t) \approx V_j(t+h) + h\sum_{k=0}^{\infty} V_k(t+h)\int_{\mathbb{R}_+} g_j(x)\cdot \mathcal{L}^{\pi^*} g_k(x)dx, \quad j = 0, 1, \ldots, \qquad (25)$$

  with π^* being the solution of (23) which can be rewritten as

$$\sum_{k=0}^{\infty} V_k(t+h)\frac{\partial \mathcal{L}^{\pi} g_k(x)}{\partial \pi} = 0. \qquad (26)$$

  Using the orthogonality of the basis functions $g_j(x)$, we can obtain that

$$V(t,x) = \sum_{j=0}^{\infty} V_j(t)g_j(x). \qquad (27)$$

Algorithm 1 (Delta algorithm for 1-D stochastic control)

1: Divide $[0, T]$ into N equal sub-intervals at a time step size of $h := T/N$. Denote $t_n := nh$ for $n = 0, 1, \ldots, N$ and $V_j^n \approx V_j(t_n)$ for $j = 0, 1, \ldots, M$.

2: There is the following recursion for $n = 0, 1, \ldots, N-1$,

$$V_j^n = V_j^{n+1} + h \sum_{k=0}^{M} V_k^{n+1} \int_{\mathbb{R}_+} g_j(x) \mathcal{L}^{\pi_{n+1}^*} g_k(x) dx, \quad j = 0, 1, \ldots, M, \tag{28}$$

with π_{n+1}^* being the solution of $\sum_{k=0}^{\infty} V_k^{n+1} \frac{\partial \mathcal{L}^{\pi} g_k(x)}{\partial \pi} = 0$, and terminal condition $V_j^N = \int_{\mathbb{R}_+} U(x) g_j(x) dx, \quad j = 0, 1, \ldots, M.$

3: Finally from (27), we obtain the approximation of the value functions

$$V(t_n, x) \approx \sum_{j=0}^{\infty} V_j^n g_j(x), \quad n = 0, 1, \ldots, N, \tag{29}$$

and the optimal strategies

$$\pi^*(t_n, x) \approx \pi_n^*(x), \quad n = 0, 1, \ldots, N. \tag{30}$$

Multidimensional stochastic control

Consider a multi-dimensional stochastic process $\mathbf{X}_t := (X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(n)})$. Denote its transition density by:

$$f(\mathbf{y} \mid \mathbf{x}) := P\left(\mathbf{X}_{t+h} = \mathbf{y} \mid \mathbf{X}_t = \mathbf{x}\right),$$

where $\mathbf{y} := (y_1, y_2, \cdots, y_n)$ and $\mathbf{x} := (x_1, x_2, \cdots, x_n)$.

Lemma

We have the following representation of the multi-dimensional transition density function:

$$f(\mathbf{y} \mid \mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[\prod_{i=1}^{n} \delta(X_{t+h}^{(i)} - y_i)\right], \tag{31}$$

where $\delta(\cdot)$ is the Dirac Delta function.

- We obtain

$$f(\mathbf{y} \mid \mathbf{x}) = \mathbb{E}_{\mathbf{x}}\left[\prod_{i=1}^{n} \delta(X_{t+h}^{(i)} - y_i)\right] = \mathbb{E}_{\mathbf{x}}\left[\prod_{i=1}^{n}\left(\sum_{k_i=0}^{\infty} g_{k_i}^{(i)}(X_{t+h}^{(i)}) g_{k_i}^{(i)}(y_i)\right)\right]$$
$$= \sum_{k_1,\ldots,k_n=0}^{\infty}\left(\prod_{i=1}^{n} g_{k_i}^{(i)}(y_i)\right)\mathbb{E}_{\mathbf{x}}\left[\prod_{i=1}^{n} g_{k_i}^{(i)}(X_{t+h}^{(i)})\right]. \tag{32}$$

- Consider a general stochastic control problem with its value function characterized by

$$V(t, \mathbf{x}) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[U(\mathbf{X}_T) \mid \mathbf{X_t} = \mathbf{x}]. \tag{33}$$

From Bellman's dynamical programming principle, we have the following characterization of the value function:

$$V(t, \mathbf{x}) = \sup_{\pi \in \mathcal{A}} \mathbb{E}[V(t+h, \mathbf{X}_{t+h}) \mid \mathbf{X_t} = \mathbf{x}]. \tag{34}$$

We utilize the density representation in (32) to rewrite the conditional expectations on the right hand side of (34) as follows.

$$\begin{aligned}
\mathbb{E}[V(t+h,\mathbf{X}_{t+h}) \mid \mathbf{X_t}=\mathbf{x}] &= \int_{\mathbb{R}^n_+} V(t+h,\mathbf{y}) f(\mathbf{y}\mid\mathbf{x}) d\mathbf{y} \\
&= \sum_{k_1,\dots,k_n=0}^{\infty} \left(\int_{\mathbb{R}^n_+} V(t+h,\mathbf{y}) \prod_{i=1}^{n} g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \mathbb{E}_{\mathbf{x}} \left[\prod_{i=1}^{n} g_{k_i}^{(i)}(X_{t+h}^{(i)}) \right] \qquad (35) \\
&= \sum_{k_1,\dots,k_n=0}^{\infty} \left(\int_{\mathbb{R}^n_+} V(t+h,\mathbf{y}) \prod_{i=1}^{n} g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \\
&\qquad \times \left(\prod_{i=1}^{n} g_{k_i}^{i}(x^i) + \int_t^{t+h} \mathbb{E}_{\mathbf{x}} \left[\mathcal{L}^{\pi} \left(\prod_{i=1}^{n} g_{k_i}^{(i)}(X_u^{(i)}) \right) \right] du \right),
\end{aligned}$$

where $\mathcal{L}^{\pi}$ is the infinitesimal generator of the multi-dimensional Markov process $\mathbf{X}$ and depends on the unknown control.

- Note that we can still use the one-dimensional mean value theorem in the time space, and arrive at the following first order approximation:

$$\int_t^{t+h} \mathbb{E}_{\mathbf{x}} \left[\mathcal{L}^\pi \left(\prod_{i=1}^n g_{k_i}^{(i)}(X_u^{(i)}) \right) \right] du \approx h \cdot \mathcal{L}^\pi \left(\prod_{i=1}^n g_{k_i}^{(i)}(x_i) \right). \tag{36}$$

- Then we finally reduce the problem into the optimization problem below, similar as the one-dimensional case:

$$\begin{aligned}
&V(t, \mathbf{x}) \\
&= \sup_{\pi \in \mathcal{A}} \sum_{k_1, \dots, k_n = 0}^{\infty} \left(\int_{\mathbb{R}_+^n} V(t+h, \mathbf{y}) \prod_{i=1}^n g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \int_t^{t+h} \mathbb{E}_{\mathbf{x}} \left[\mathcal{L}^\pi \left(\prod_{i=1}^n g_{k_i}^{(i)}(X_u^{(i)}) \right) \right] du \\
&\approx \sup_{\pi \in \mathcal{A}} \sum_{k_1, \dots, k_n = 0}^{\infty} \left(\int_{\mathbb{R}_+^n} V(t+h, \mathbf{y}) \prod_{i=1}^n g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \\
&\qquad \times \left(h \cdot \mathcal{L}^\pi \left(\prod_{i=1}^n g_{k_i}^{(i)}(x_i) \right) + \prod_{i=1}^n g_{k_i}^{i}(x^i) \right).
\end{aligned} \tag{37}$$

Let π^* be the solution of

$$\sum_{k_1,\ldots,k_n=0}^{\infty} \left(\int_{\mathbb{R}_+^n} V(t+h,\mathbf{y}) \prod_{i=1}^{n} g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \frac{\partial}{\partial \pi} \mathcal{L}^{\pi} \left(\prod_{i=1}^{n} g_{k_i}^{(i)}(x_i) \right) = 0. \tag{38}$$

Then inserting $\pi = \pi^*$ into (37) gives that

$$\begin{aligned} V(t,\mathbf{x}) \approx \sum_{k_1,\ldots,k_n=0}^{\infty} & \left(\int_{\mathbb{R}_+^n} V(t+h,\mathbf{y}) \prod_{i=1}^{n} g_{k_i}^{(i)}(y_i) d\mathbf{y} \right) \\ & \times \left(h \cdot \mathcal{L}^{\pi^*} \left(\prod_{i=1}^{n} g_{k_i}^{(i)}(x_i) \right) + \prod_{i=1}^{n} g_{k_i}^{i}(x^i) \right). \end{aligned} \tag{39}$$

Take inner product with $\prod_{j=1}^{n} g_{m_j}^{(j)}(x_j)$, $m_j = 0, 1, \ldots; j = 1, \ldots, n$ and denote

$$V_{\mathbf{m}}(t) := \int_{\mathbb{R}_+^n} V(t,\mathbf{x}) \prod_{j=1}^{n} g_{m_j}^{(j)}(x_j) d\mathbf{x}, \quad m_j = 0, 1, \ldots; j = 1, \ldots, n,$$

and $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $\mathbf{k} = (k_1, k_2, \ldots, k_n)$.

We obtain that for $m_j = 0, 1, \ldots; j = 1, \ldots, n$,

$$V_{\mathbf{m}}(t) \approx V_{\mathbf{m}}(t+h) + h \sum_{k_1,\ldots,k_n=0}^{\infty} V_{\mathbf{k}}(t+h) \int_{\mathbb{R}_+^n} \prod_{j=1}^{n} g_{m_j}^{(j)}(x_j) \mathcal{L}^{\pi^*} \left(\prod_{i=1}^{n} g_{k_i}^{(i)}(x_i) \right) d\mathbf{x}, \tag{40}$$

with π^* being the solution of (38), which can be further rewritten as

$$\sum_{k_1,\ldots,k_n=0}^{\infty} V_{\mathbf{k}}(t+h) \frac{\partial}{\partial \pi} \mathcal{L}^{\pi} \left(\prod_{i=1}^{n} g_{k_i}^{(i)}(x_i) \right) = 0. \tag{41}$$

Using the orthogonality of the basis functions $g_j(x)$, we can obtain that

$$V(t, \mathbf{x}) = \sum_{k_1,\ldots,k_n=0}^{\infty} V_{\mathbf{k}}(t) \prod_{i=1}^{n} g_{k_i}^{(i)}(x_i). \tag{42}$$

Truncating the sums in (40), (41) and (42) by finite number leads to the following time-stepping algorithm.

Algorithm 2 (Delta algorithm for multi-D stochastic control)

1: Denote $V^n_{\mathbf{m}} \approx V_{\mathbf{m}}(t_n)$ for $\mathbf{m} = (m_1, \ldots, m_n)$; $m_j = 0, 1, \ldots, M_j$; $j = 1, \ldots, n$; $n = 0, 1, \ldots, N$.

2: The algorithm is given by the following recursion for $n = 0, 1, \ldots, N$,

$$V^n_{\mathbf{m}} \approx V^{n+1}_{\mathbf{m}} + h \sum_{k_1=0}^{M_1} \cdots \sum_{k_n=0}^{M_n} V^{n+1}_{\mathbf{k}} \int_{\mathbb{R}^n_+} \prod_{j=1}^{n} g^{(j)}_{m_j}(x_j) \mathcal{L}^{\pi^*_{n+1}} \left(\prod_{i=1}^{n} g^{(i)}_{k_i}(x_i) \right) d\mathbf{x}, \quad (43)$$

with π^*_{n+1} being the solution of (38) which can be rewritten as $\sum_{k_1=0}^{M_1} \cdots \sum_{k_n=0}^{M_n} V^{n+1}_{\mathbf{k}} \frac{\partial}{\partial \pi} \mathcal{L}^{\pi} \left(\prod_{i=1}^{n} g^{(i)}_{k_i}(x_i) \right) = 0$. The terminal condition is $V^N_{\mathbf{m}} = \int_{\mathbb{R}^n_+} U(\mathbf{x}) \prod_{j=1}^{n} g^{(j)}_{m_j}(x_j) d\mathbf{x}$.

3: Finally from (42), we obtain the approximation of the value functions

$$V(t_n, \mathbf{x}) \approx \sum_{m_1=0}^{M_1} \cdots \sum_{m_n=0}^{M_n} V^n_{\mathbf{m}} \prod_{j=1}^{n} g^{(j)}_{m_j}(x_j), \quad (44)$$

and the optimal strategies

$$\pi^*(t_n, x) \approx \pi^*_n(x), \quad n = 0, 1, \ldots, N. \quad (45)$$

- Heston model:

$$\frac{dS_t}{S_t} = (r + \lambda \mathcal{V}_t)dt + \sqrt{\mathcal{V}_t}dW_t^{(1)}, \tag{46}$$

$$d\mathcal{V}_t = \kappa(\theta - \mathcal{V}_t)dt + \sigma\sqrt{\mathcal{V}_t}dW_t^{(2)}, \tag{47}$$

where $E[dW_t^{(1)}dW_t^{(2)}] = \rho dt$.

- The wealth process is

$$\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t} + (1 - \pi_t)rdt = \lambda \pi_t \mathcal{V}_t dt + \pi_t \sqrt{\mathcal{V}_t}dW_t^{(1)}, \tag{48}$$

where $\mathcal{V}$ is given by (47). The infinitesimal generator of (48) is given by

$$\mathcal{L}^\pi f(x, v) = \lambda \pi v x \frac{\partial f}{\partial x} + \kappa(\theta - v)\frac{\partial f}{\partial v} + \frac{\pi^2 v x^2}{2}\frac{\partial^2 f}{\partial x^2} + \frac{\sigma^2 v}{2}\frac{\partial^2 f}{\partial v^2} + \rho \pi \sigma x v \frac{\partial^2 f}{\partial x \partial v}.$$

Algorithm 3 (Optimal investment under Heston models)

1: Denote $V^n_{m_1 m_2} \approx V_{m_1 m_2}(t_n)$ for $m_1 = 0,1,\ldots,M_1$; $m_2 = 0,1,\ldots,M_2$; $n = 0,1,\ldots,N$.

2: The algorithm is given by the following recursion for $n = 0,1,\ldots,N-1$,

$$V^n_{m_1 m_2} \approx V^{n+1}_{m_1 m_2} + h \sum_{k_1=0}^{M_1} \sum_{k_2=0}^{M_2} V^{n+1}_{k_1 k_2} \int_{\Omega^2} g^{(1)}_{m_1}(y_1) g^{(2)}_{m_2}(y_2) \mathcal{L}^{\pi^*_{\mathbf{n}+\mathbf{1}}} \left(g^{(1)}_{k_1}(y_1) g^{(2)}_{k_2}(y_2) \right) d\mathbf{y},$$

where $\pi^*_{n+1} = \dfrac{-\sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} V^{n+1}_{k_1 k_2}\left(\lambda g^{(2)}_{k_2}(\hat{v}(v)) + \rho\sigma \frac{dg^{(2)}_{k_2}(\hat{v}(v))}{dv}\right)\frac{dg^{(1)}_{k_1}(\hat{x}(x))}{dx}}{x\sum_{k_1=0}^{M_1}\sum_{k_2=0}^{M_2} V^{n+1}_{k_1 k_2}\frac{d^2 g^{(1)}_{k_1}(\hat{x}(x))}{dx^2} g^{(2)}_{k_2}(\hat{v}(v))}$, and terminal condition $V^N_{m_1 m_2} = \int_{\Omega^2} U(\hat{x}(y_1)) g^{(1)}_{m_1}(\hat{x}(y_1)) g^{(2)}_{m_2}(\hat{v}(y_2)) d\mathbf{y}$, $m_1 = 0,1,\ldots,M_1$; $m_2 = 0,1,\ldots,M_2$.

3: Finally we obtain the approximation of the value functions $V(t_n,x,v) \approx \sum_{m_1=0}^{M_1}\sum_{m_2=0}^{M_2} V^n_{m_1 m_2} g^{(1)}_{m_1}(\hat{x}(x)) g^{(2)}_{m_2}(\hat{v}(v))$, for $m_1 = 0,1,\ldots,M_1$; $m_2 = 0,1,\ldots,M_2$;, $n = 0,1,\ldots,N$, and the optimal strategies $\pi^*(t_n,x,v) \approx \pi^*_n(x,v)$.

- We take parameters from Kraft (2005):
  $r = 5\%, \rho = -0.5, \kappa = 10, \theta = 0.05, \sigma = 0.5, A = 0.5,\ T = 1.$
- Then we calculate the maximum norm of numerical error.

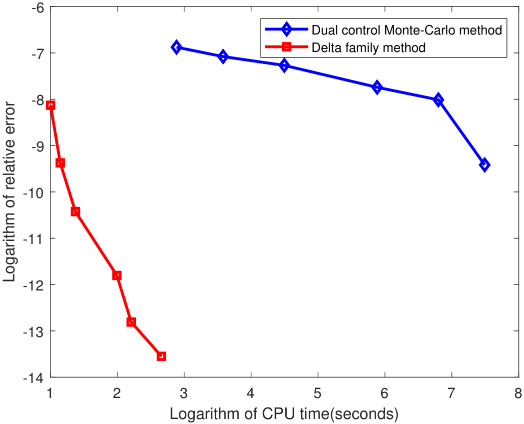
$$\|V_0(\cdot,\cdot) - V^*(0,\cdot,\cdot)\|_\infty = \max_{x\in[x_1,x_2],v\in[v_1,v_2]} |V_0(x,v) - V^*(0,x,v)|\,,$$

$$\|\pi_0^*(\cdot,\cdot) - \pi^*(0,\cdot,\cdot)\|_\infty = \max_{x\in[x_1,x_2],v\in[v_1,v_2]} |\pi_0^*(x,v) - \pi^*(0,x,v)|\,.$$

Table: Errors of delta family approach with $N = 2000$.

M	Errors		CPU time (Seconds)
	Prime value	Optimal strategy	
6	2.95e-4	6.08e-2	2.729
8	8.50e-5	9.46e-3	3.152
10	2.97e-5	2.93e-3	3.956
12	7.48e-6	1.94e-3	7.356
14	2.73e-6	1.12e-3	9.107
16	1.31e-6	4.84e-4	14.294

Comparison between delta family and dual Monte-Carlo[6]



[6] Ma, J., Li, W. and Zheng, H. (2020). Dual control Monte-Carlo method for tight bounds of value function under Heston stochastic volatility model. *European Journal of Operational Research*, 280, 428-440.

Optimal reinsurance-investment under (rough) Heston models

- Assume that the surplus process of the insurer satisfies

$$dR_t = c\left[\eta - \vartheta(1-\widehat{q}_t)\right]dt + b\widehat{q}_t dW_t, \tag{49}$$

  where c, b, η are positive constants representing the claim rate, volatility and safely loading of insurer, respectively, and W_t is the standard Brownian motion representing the uncertainty of the insurance market.

- Assume that the proportion of reinsurance is $q_t X_t$ at time t, which means that the insurer undertakes $100 q_t X_t\%$ of the claims while the reinsurer undertakes the rest.

- Meanwhile, the insurer has to pay a premium at the rate of $c\vartheta(1-q_t X_t)$ to the reinsurer due to the reinsurance business, where ϑ denotes the safety loading of the reinsurer.

- The insurer has an option to invest in the continuous-time financial market that consists of one risky asset and one risk-free asset with interest $r > 0$ to get more benefits.
- The wealth process $\widehat{X}_t$ evolves according to

$$d\widehat{X}_t = r\widehat{X}_t dt + \lambda\widehat{\pi}_t \mathcal{V}_t dt + \widehat{\pi}_t \sqrt{\mathcal{V}_t} dW_t^1 + c(\eta - \vartheta)dt + c\vartheta\widehat{q}_t dt + b\widehat{q}_t dW_t, \quad (50)$$

  with $\widehat{X}_t = \widehat{x}$. Here $(\widehat{q}_t, \widehat{\pi}_t)$ denotes admissible reinsurance-investment strategy. The set of all admissible strategies is denoted by $\widehat{\Pi}$.
- Aim of the insurer is to find the best strategy to maximize the expected utility of the terminal wealth with minimum guaranteed threshold $L \geq 0$, that is

$$\sup_{\widehat{u} \in \widehat{\Pi}} E\left[U(\widehat{X}_T - L)\right], \quad (51)$$

  where $\widehat{X}_t$ is subject to (52) and $\widehat{X}_T \geq L$.

Rough Heston model

- The wealth process is

$$dX_t = X_t\Big[(r+\lambda\pi_t\mathcal{V}_t)dt + \pi_t\sqrt{\mathcal{V}_t}dW_t^1 + c\vartheta q_t dt + bq_t dW_t\Big],$$

$$\mathcal{V}_t = \mathcal{V}_0 + \int_0^t K(t-s)\Big(\kappa(\theta-\mathcal{V}_s)ds + \sigma\sqrt{\mathcal{V}_s}dW_s^2\Big), \quad (52)$$

- Note that the fractional kernel can be expressed as a Laplace transform of a positive measure ζ

$$K(t) = \int_0^\infty e^{-\gamma t}\zeta(d\gamma), \quad \zeta(d\gamma) = \frac{\gamma^{-\alpha}}{\Gamma(\alpha)\Gamma(1-\alpha)}d\gamma.$$

- We then approximate ζ by a finite sum of Dirac measures $\zeta^\ell = \sum_i^\ell c_i^\ell \delta_{\gamma_i^\ell}$ with positive weights $(c_i^\ell)_{1\leq i\leq \ell}$ and mean reversions $(\gamma_i^\ell)_{1\leq i\leq\ell}$ for $\ell \geq 1$. This in turn yields an approximation of the fractional kernel by a sequence of smoothed kernels $(K^\ell)_{\ell\geq 1}$ given by

$$K^\ell(t) = \sum_{i=1}^{\ell} c_i^\ell e^{-\gamma_i^\ell t}, \quad \ell \geq 1.$$

- Therefore, the stock price process can be well approximated by the lifted Heston model[7] as follows

$$\frac{dS_t^\ell}{S_t^\ell} = (r + \lambda \mathcal{V}_t^\ell)dt + \sqrt{\mathcal{V}_t^\ell} dW_t^1, \tag{53}$$

- The variance process $\mathcal{V}_t^\ell$ is given by a fractional square-root process

$$\mathcal{V}_t^\ell = \xi(t) + \sum_{i=1}^{\ell} c_i^\ell \mathcal{V}_t^{\ell,i}, \tag{54}$$

and

$$d\mathcal{V}_t^{\ell,i} = (-\gamma_i^\ell \mathcal{V}_t^{\ell,i} - \kappa \mathcal{V}_t^\ell)dt + \sigma \sqrt{\mathcal{V}_t^\ell} dW_t^2, \tag{55}$$

where $\xi(t) = \mathcal{V}_0 + \kappa\theta \int_0^t K(t-s)ds$.

- We can approximate the wealth process as follows

$$dX_t^\ell = X_t^\ell \Big[(r + \lambda \pi_t \mathcal{V}_t^\ell)dt + \pi_t \sqrt{\mathcal{V}_t^\ell} dW_t^1 + c\vartheta q_t dt + b q_t dW_t \Big]. \tag{56}$$

[7] Abi Jaber, E. (2019). Lifting the Heston model. Quantitative Finance, 19(12), 1995-2013.

- The infinitesimal generator of (56) is given by

$$
\begin{aligned}
&\mathcal{L}^{q,\pi} f(x, v_1, \ldots, v_\ell) \\
&= \left[r + \lambda\pi \left(\xi(t) + \sum_{l=1}^{\ell} c_l^\ell v_l \right) + c\vartheta q \right] x f_x + \frac{1}{2} x^2 \left[\pi^2 \left(\xi(t) + \sum_{l=1}^{\ell} c_l^\ell v_l \right) + b^2 q^2 \right] f_{xx} \\
&- \sum_{i=1}^{\ell} \left(\gamma_i^\ell v_i + \kappa\xi(t) + \kappa \sum_{l=1}^{\ell} c_l^\ell v_l \right) f_{v_i} + \frac{1}{2} \sum_{i,j=1}^{\ell} \sigma^2 \left(\xi(t) + \sum_{l=1}^{\ell} c_l^n v_l \right) f_{v_i v_j} \\
&+ \sum_{i=1}^{\ell} \rho\sigma x\pi \left(\xi(t) + \sum_{l=1}^{\ell} c_l^\ell v_l \right) f_{x v_i}. \qquad (57)
\end{aligned}
$$

- The utility maximization problem is approximated by

$$
\sup_{\pi \in \Pi^\ell} \mathbb{E} \left[U\left(X_T^\ell - L \right) \right], \qquad (58)
$$

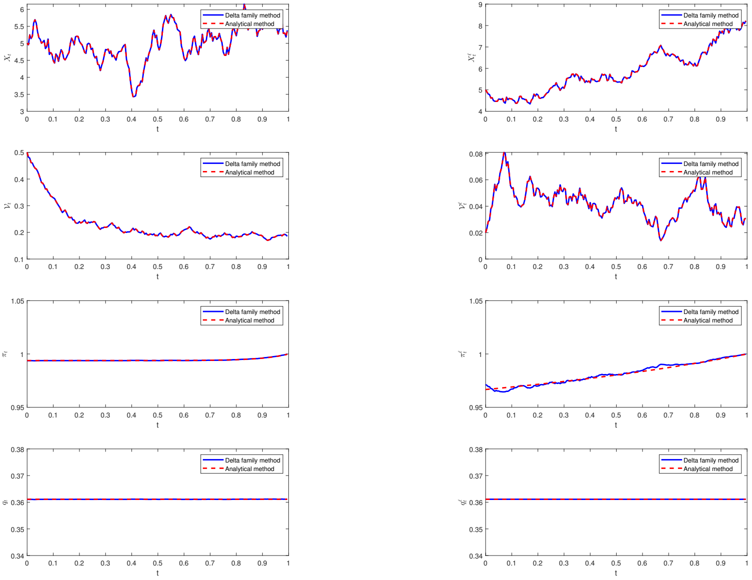
where X_t^ℓ is subject to (56).

Numerical Example

- For the reinsurance process, the model parameters are taken from Bi and Cai (2019): $c = 0.13$, $b = 0.6$, $\eta = 0.3$, $\vartheta = 0.5$. The minimum guaranteed threshold $L = 0$.
- For Heston model, The parameters are taken the same as previous example. For rough Heston model, we use multi-factor lifted model to approximate it. We set the number of factors $\ell = 3$ and the value of model parameters are taken from Abi Jaber (2019):

$$\mathcal{V}_0 = 0.02,\ \ell = 3,\ \lambda = 0.5,\ \rho = -0.7,\ \kappa = 0.3,\ \theta = 0.02,\ \sigma = 0.3.$$

- The interest rate is $r = 0.05$, the initial value of wealth $\widehat{x}_0 = 5$ and the investment horizon $T = 1$.
- The paths of wealth process and the strategies are drawn using the delta family methods and the analytical methods in Ma, Li and Zheng (2020) for the Heston model and Ma, Chen and Lu (2022) for the multi-factor stochastic volatility model.



The paths of wealth and optimal reinsurance-investment strategies under Heston model (left) and rough Heston model (right)

Optimal stopping investment problem under SLV models

Consider stochastic local volatility (SLV) model:

$$\frac{dS_t}{S_t} = \omega(S_t, \mathcal{V}_t)dt + \eta(\mathcal{V}_t)\Gamma(S_t)dW_t^1,$$
$$d\mathcal{V}_t = \beta(\mathcal{V}_t)dt + \zeta(\mathcal{V}_t)dW_s^2.$$

Some typical SLV models are listed as follows:

(i) Heston model

$$\omega(S_t, \mathcal{V}_t) = (r + \lambda\mathcal{V}_t)S_t, \quad \eta(\mathcal{V}_t) = \sqrt{\mathcal{V}_t}, \quad \Gamma(S_t) = S_t,$$
$$\beta(\mathcal{V}_t) = \kappa(\theta - \mathcal{V}_t), \quad \zeta(\mathcal{V}_t) = \sigma\sqrt{\mathcal{V}_t}. \tag{59}$$

(ii) 4/2 model

$$\omega(S_t, \mathcal{V}_t) = (r + \lambda\mathcal{V}_t)S_t, \quad \eta(\mathcal{V}_t) = a\sqrt{\mathcal{V}_t} + b/\sqrt{\mathcal{V}_t}, \quad \Gamma(S_t) = S_t,$$
$$\beta(\mathcal{V}_t) = \kappa(\theta - \mathcal{V}_t), \quad \zeta(\mathcal{V}_t) = \sigma\sqrt{\mathcal{V}_t}. \tag{60}$$

(iii) α-Hyper model

$$\omega(S_t, \mathcal{V}_t) = (r + \lambda\mathcal{V}_t)S_t, \quad \eta(\mathcal{V}_t) = \exp(\mathcal{V}_t), \quad \Gamma(S_t) = S_t,$$
$$\beta(\mathcal{V}_t) = \kappa(\theta - \exp(a\mathcal{V}_t)), \quad \zeta(\mathcal{V}_t) = \sigma\sqrt{\mathcal{V}_t}. \tag{61}$$

- Consider the following optimal stopping control problem:

$$V(t,x,s,v) = \sup_{\tau\in[t,T],\,\pi\in\mathcal{A}} \mathbb{E}[e^{-\gamma(\tau-t)}U(X_\tau - L) \mid X_t = x, \mathcal{V}_t = v, S_t = s], \tag{62}$$

  where $L > 0$ is the minimum wealth threshold value and γ is utility discount factor.

- The value function V satisfies the following HJB variational inequality

$$\min\Big\{-\frac{\partial V}{\partial t} - \sup_{\pi\in\mathcal{A}} \mathcal{L}^\pi[V],\ V - G(x)\Big\}, \tag{63}$$

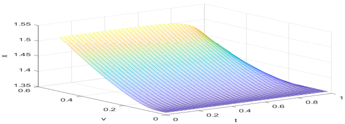
  where $G(x) = U(x - L)$ and

$$\begin{aligned}\mathcal{L}^\pi[f(t,x,s,v)] = &-\gamma f + \Big[r + \Big(\frac{\omega(s,v)}{s} - r\Big)\pi\Big]xf_x + \frac{1}{2}\frac{\eta^2(v)\Gamma^2(s)}{s^2}\pi^2x^2f_{xx}\\ &+ \omega(s,v)f_s + \frac{1}{2}\eta^2(v)\Gamma^2(s)f_{ss} + \beta(v)f_v + \frac{1}{2}\zeta(v)f_{vv}\\ &+ \eta^2(v)\Gamma^2(s)\frac{\pi x}{s}f_{xs} + \zeta(v)\eta(v)\Gamma(s)\rho\Big(\frac{\pi x}{s}f_{xv} + f_{sv}\Big),\end{aligned} \tag{64}$$

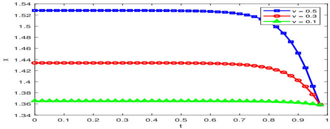
  with terminal condition

$$V(T,x,s,v) = G(x), \quad x\in[0,\infty).$$

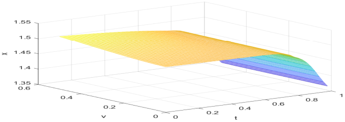
- We solve the optimal stopping investment problem with power utility.
- For Heston model (59), The values of model parameters are taken the same as before. The utility discount factor is $\gamma = 0.15$ and $L = 1.0$. For $4/2$ stochastic volatility model (60), the values of model parameters are taken from Grasselli (2017): $\rho = -0.7$, $\sigma = 0.2$, $\kappa = 1.8$, $\theta = 0.04$, $r = 0.02$, $T = 1$, $a = 0.5$, $b = 0.04$.
- In addition, the utility discount factor $\gamma = 0.05$ and $L = 1.0$. For α-Hypergeometric stochastic volatility model (61), we set $a = 0.01$ and the other parameters are taken as same as $4/2$ model.
- The 3D and 2D figures of optimal exercise boundaries are plotted below.



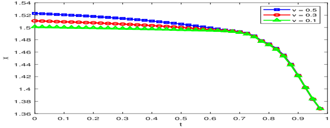
(a) Heston model



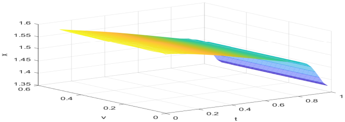
(b) Heston model



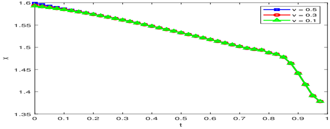
(c) 4/2 model



(d) 4/2 model



(e) α-Hypergeometric



(f) α-Hypergeometric

Further Research: Inversion via Integration

- Dirac Delta sifting property + change of variables:

$$f^{-1}(x) = \int_0^\infty f^{-1}(u)\delta(u-x)du$$
$$= \lim_{\varepsilon\to+0}\int_0^\infty \delta_\epsilon(f(s)-x)\cdot sdf(s). \tag{65}$$

- Simulate random variables[8]:
$X = F^{-1}(U) = \lim_{\varepsilon\to+0}\int_{-\infty}^{\infty} \delta_\epsilon(F(s)-U)\cdot sdF(s)$.
- Stochastic inverse: Let $A_t, t\geq 0$ denote a stochastic increasing process, and denote τ as its right-continuous inverse, i.e. $\tau_a := \inf\{u\geq 0, A_u\geq a\}$. Then we have $A_{\tau_a} = a$ almost surely, and $\tau_{A_u} = u$ almost surely. We have the following stochastic representation of τ:

$$\tau_u = \int_0^\infty \delta(s-u)\tau_s ds = \lim_{\varepsilon\to+0}\frac{1}{2\sqrt{\pi\varepsilon}}\int_0^\infty e^{-\frac{(s-u)^2}{4\varepsilon}}\tau_s ds$$
$$= \lim_{\varepsilon\to+0}\frac{1}{2\sqrt{\pi\varepsilon}}\int_0^\infty e^{-\frac{(A_t-u)^2}{4\varepsilon}}\tau_{A_t}dA_t$$
$$= \lim_{\varepsilon\to+0}\frac{1}{2\sqrt{\pi\varepsilon}}\int_0^\infty e^{-\frac{(A_t-u)^2}{4\varepsilon}}\cdot tdA_t, \tag{66}$$

[8] Cui, Z., Kirkby, J., Nguyen, D. and Taylor, S.M., 2021. A Novel Sampling Method based on Orthogonal Polynomial Expansions and Applications. Available at SSRN 3862760.

- Risk measures: Value-at-Risk (VaR): let $F_L(\cdot)$ be the CDF of loss distribution

$$VaR_\alpha = F_L^{-1}(\alpha) = \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \delta_\epsilon(F_L(s) - \alpha) \cdot s dF_L(s) \quad (67)$$

- Expected shortfall: $ES_\alpha = CVaR_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 VaR_u du$.
- Range value at risk (RVaR):

$$\mathrm{RVaR}_{\alpha,\beta} = \begin{cases} \frac{1}{\beta} \int_\alpha^{\alpha+\beta} \mathrm{VaR}_\gamma \, \mathrm{d}\gamma & \text{if } \beta > 0 \\ \mathrm{VaR}_\alpha & \text{if } \beta = 0 \end{cases}$$

$$= \begin{cases} \lim_{\varepsilon \to +0} \int_{-\infty}^{\infty} \left(\frac{1}{\beta} \int_\alpha^{\alpha+\beta} \delta_\epsilon(F_L(s) - \gamma) \mathrm{d}\gamma \right) \cdot s dF_L(s) & \text{if } \beta > 0 \\ \mathrm{VaR}_\alpha & \text{if } \beta = 0 \end{cases}$$

Questions?

Thank you!