Volatility Shape-Shifters:
arbitrage-free transformations of a volatility surface

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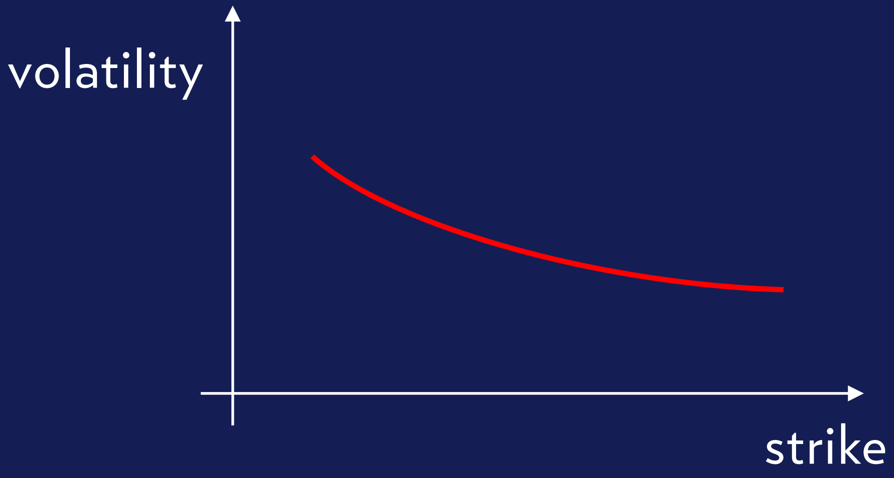
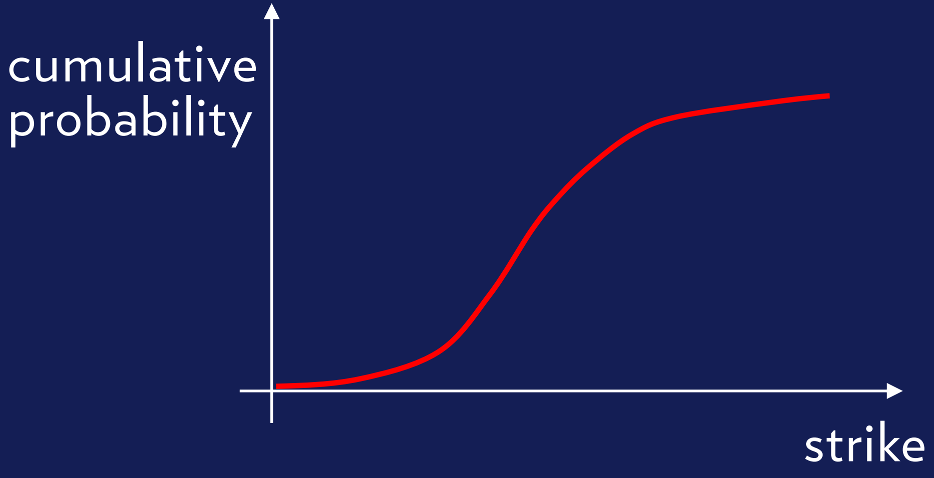
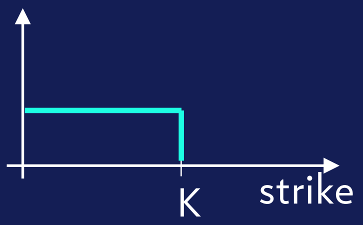
Outline

- No Arbitrage & Convex order
- Optimal transport
- One-X & Ohlin's theorem
- Convex Order generating/preserving transformations
- Time extrapolation:
  - No market information
  - Regeneration with *Memory*
- Conclusions

Volatility surface & arbitrage







$$\text{putSpread}(x) \approx \text{digital}(x) = E\{S < x\} = cdf(x)$$

$$cdf(x) = -\frac{d}{dx} call(x, \sigma(x), t) \qquad call(K) = \int_K^\infty \big(1 - cdf(x)\big) dx$$

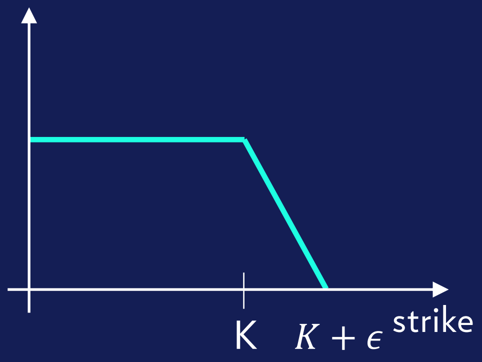
(*) undiscounted quantities

Arbitrage

- Strike arbitrage ⟷ • Inconsistent marginal distribution

- digital arbitrage:

$$put(x+\epsilon) < put(x)$$

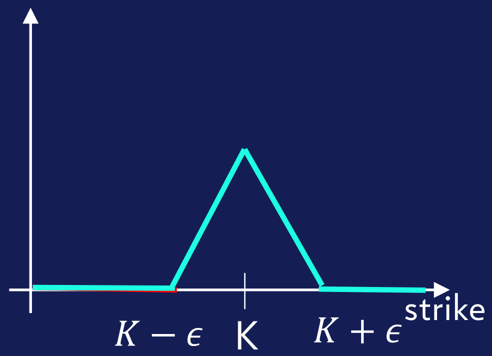


$$cdf(x) < 0 \quad or \quad cdf(x) > 1$$

- butterfly arbitrage:

$$call(K-\epsilon) - 2call(K) + call(K+\epsilon) < 0$$

$$\frac{d^2}{dx^2} call(x, \sigma(x), t) < 0$$



$$pdf(x) < 0$$

Arbitrage

- <u>Calendar non-arbitrage</u>

  Calls at $t_2 > t_1$ need to satisfy:

  $$\frac{E\left[\left(S_{t_2} - kF_{t_2}\right)^+\right]}{F_{t_2}} \geq \frac{E\left[\left(S_{t_1} - kF_{t_1}\right)^+\right]}{F_{t_1}}$$

$\Longleftrightarrow$

- <u>Convex order of marginals</u>: $\mu_1 \preceq \mu_2$

  any convex function $\varphi(x)$:

  $$\int \varphi(x) d\mu_1(x) \leq \int \varphi(x) d\mu_2(x)$$

- Goal of the Shape Shifters:

  transform a convex set of marginals into another convex set with desired property

Preliminaries

Our focus:

- Digital spread decreases to zero with diminishing strike difference:

$$\lim_{\epsilon \to 0} Digital(K+\epsilon) - Digital(K) = 0$$

$\Longleftrightarrow$

- lack of atoms, i.e. **continuous CDF**
  no accumulation of probability

- Digital spreads have strictly positive value:

$$Digital(K+\epsilon) - Digital(K) > 0$$

- support = R^+, i.e. **strictly increasing CDF**
  no 'dry patches' of zero probability density

We will focus in continuous, strictly increasing CDFs on R^+.

First goal: find a transformation operator for CDFs – a 'Shape Shifter'.

Link to optimal transport

Monge 's problem: moving a heap of sand optimally

Minimizes the total cost $c(x, y)$:

$$\int c\big(x, T(x)\big)\mu(dx)$$

Unique solution:

$$T^*(x) = (\nu^{-1} \circ \mu)(x) = \nu^{-1}\big(\mu(x)\big)$$

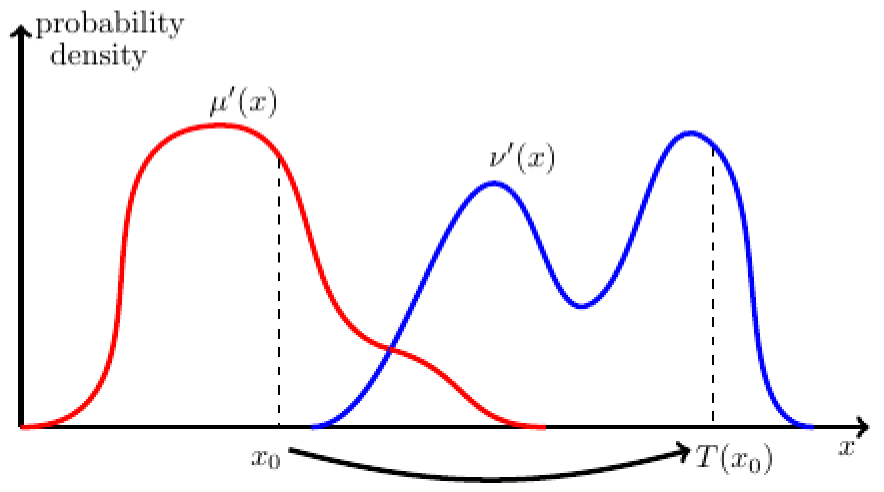


Figure 1: Optimal transport map between two distributions.

Link to optimal transport

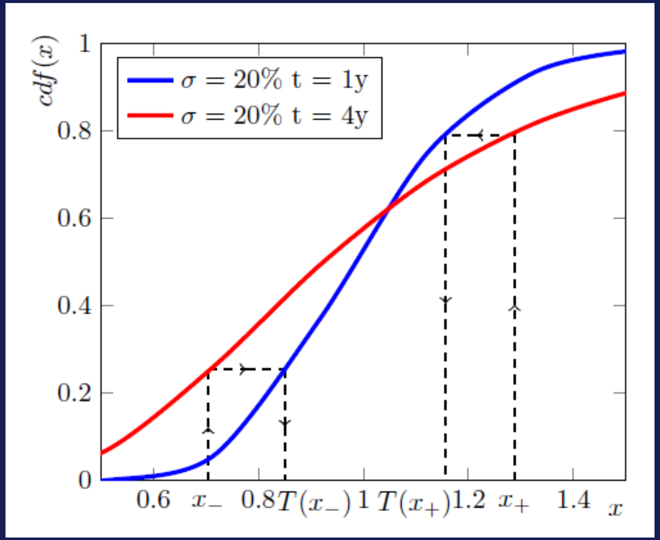
Example: lognormal distributions

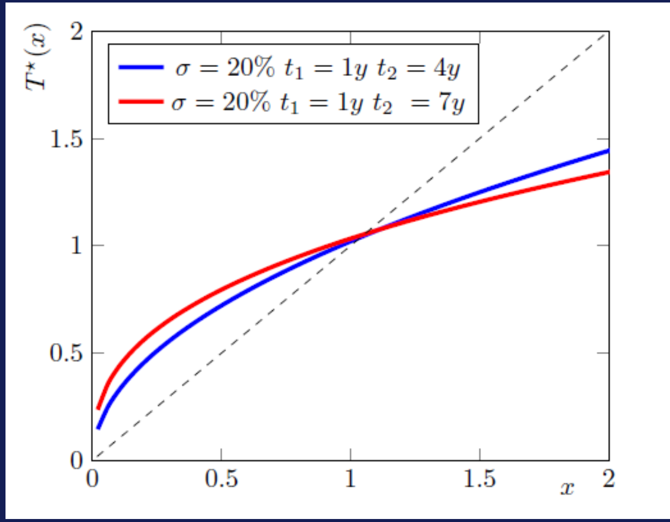
$$T^*_{\Phi_1,\Phi_2} = \Phi_1^{-1} \circ \Phi_2$$

$$\Phi_{F_t,\sigma,t}(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\ln\left(\frac{x}{F_t}\right)+\frac{1}{2}\sigma^2 t}{\sigma\sqrt{2t}}\right)\right]$$

$$T_{\Phi_{F_{t_1},\sigma_1,t_1};\Phi_{F_{t_2},\sigma_2,t_2}} = e^{m_1 - \frac{m_2\sigma_1\sqrt{t_1}}{\sigma_2\sqrt{t_2}}} x^{\frac{\sigma_1\sqrt{t_1}}{\sigma_2\sqrt{t_2}}}$$

where $m_i = \ln(F_i) - \frac{1}{2}\sigma_i^2 t_i$





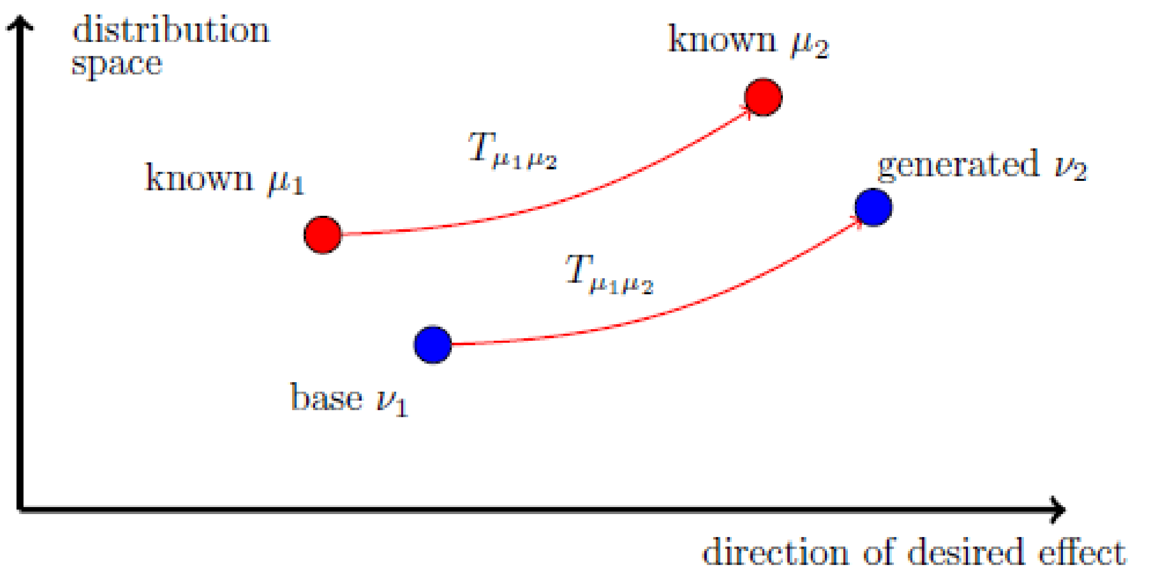
Link to optimal transport

- Monge's problem:

  how to move the sand

- Our problem:

  where to move it



Schema of the approach.

Solution: $\boldsymbol{\nu_2 = \nu_1 \circ T_{\mu_1\mu_2}}$ for suitable μ_1, μ_2

Examples

- Re-centering of distribution:

  Let μ denote a distribution with mean $\bar{x}$. Then the distribution $\nu(x) \ = \ \mu(\frac{\bar{x}}{\bar{y}}x)$ has mean $\bar{y}$.

  $\mathrm{T}(x) = \frac{\bar{x}}{\bar{y}}x$

- Widening of distribution:

  e.g. use lognormal distributions with the same forward and time, different variance:

  $\mathrm{T}(x) = \ \mathrm{T}_{\Phi_{F,\sigma_1,t}\,;\Phi_{F,\sigma_2,t}}(x)$

- Skewing of distribution:

  e.g. use lognormal and Merton distributions with the same mean and variance:

  $\mathrm{T}(x) = \ \mathrm{T}_{\Phi_{F,\sigma_1,t}\,;Merton}(x)$

Ohlin's theorem & One-X

Next: **full surface** transformation? Need convex order preservation.

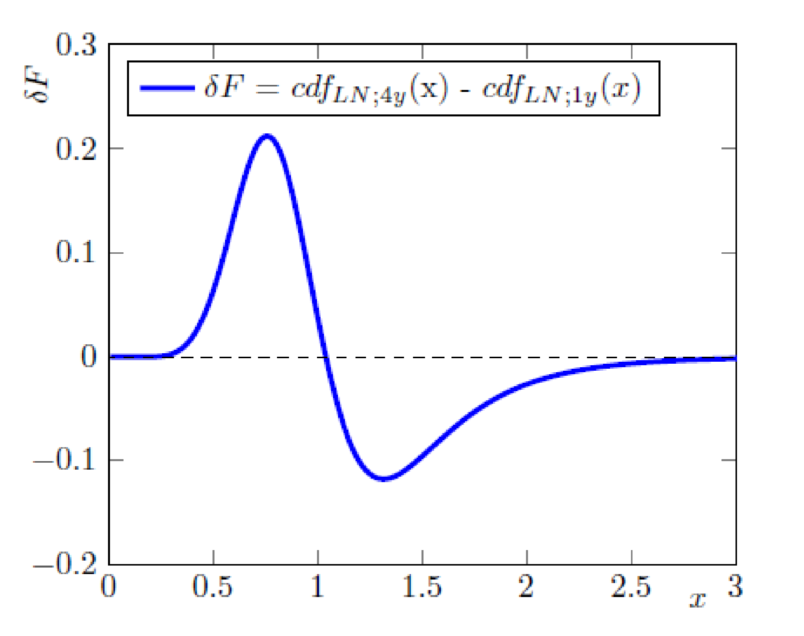
- Ohlin's theorem: sufficiency

Let x_1 and x_2 be two random variables with the same finite mean: $E[x_1] = E[x_2]$,
and μ_1 and μ_2 their cumulative distribution functions.
If there exists an x_0 such that,

$$\mu_1(x) < \mu_2(x) \qquad for \qquad x < x_0$$
$$\mu_1(x) > \mu_2(x) \qquad for \qquad x > x_0$$

Then $E[f(x_1)] \leq E[f(x_2)]$ for any continuous convex function f.

- One-X property: $\delta F = \mu_2 - \mu_1$



Convex Order generating/preserving transformations

If $\mu_1 \preceq \mu_2$ satisfy One-X, then, after a re-centering transformation(*):

- convex order preservation: for any transport map T :

$$\mu_1 \circ T \preceq \mu_2 \circ T$$

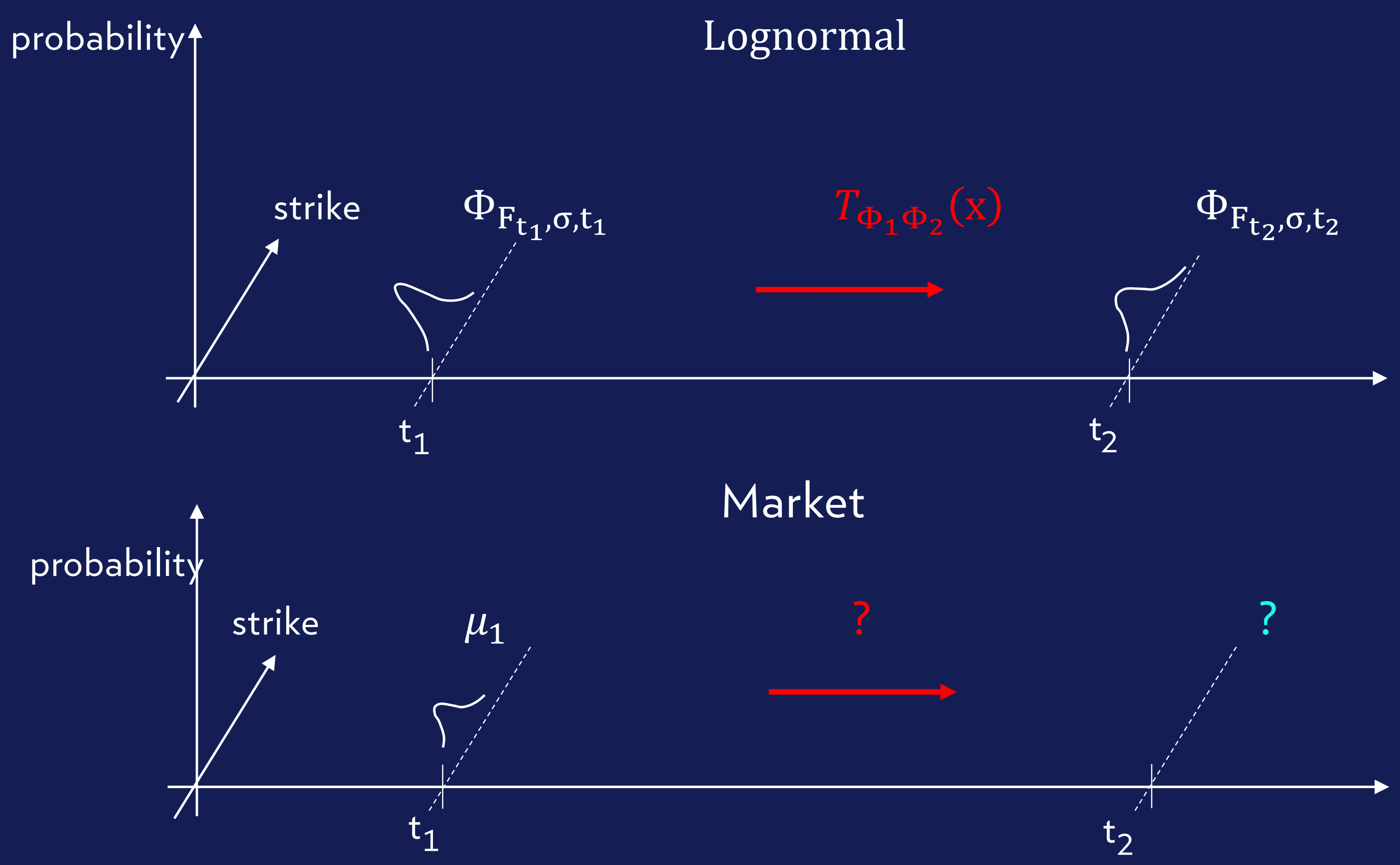
- convex order generation:

$$\nu \preceq \nu \circ \mu_1^{-1} \circ \mu_2$$

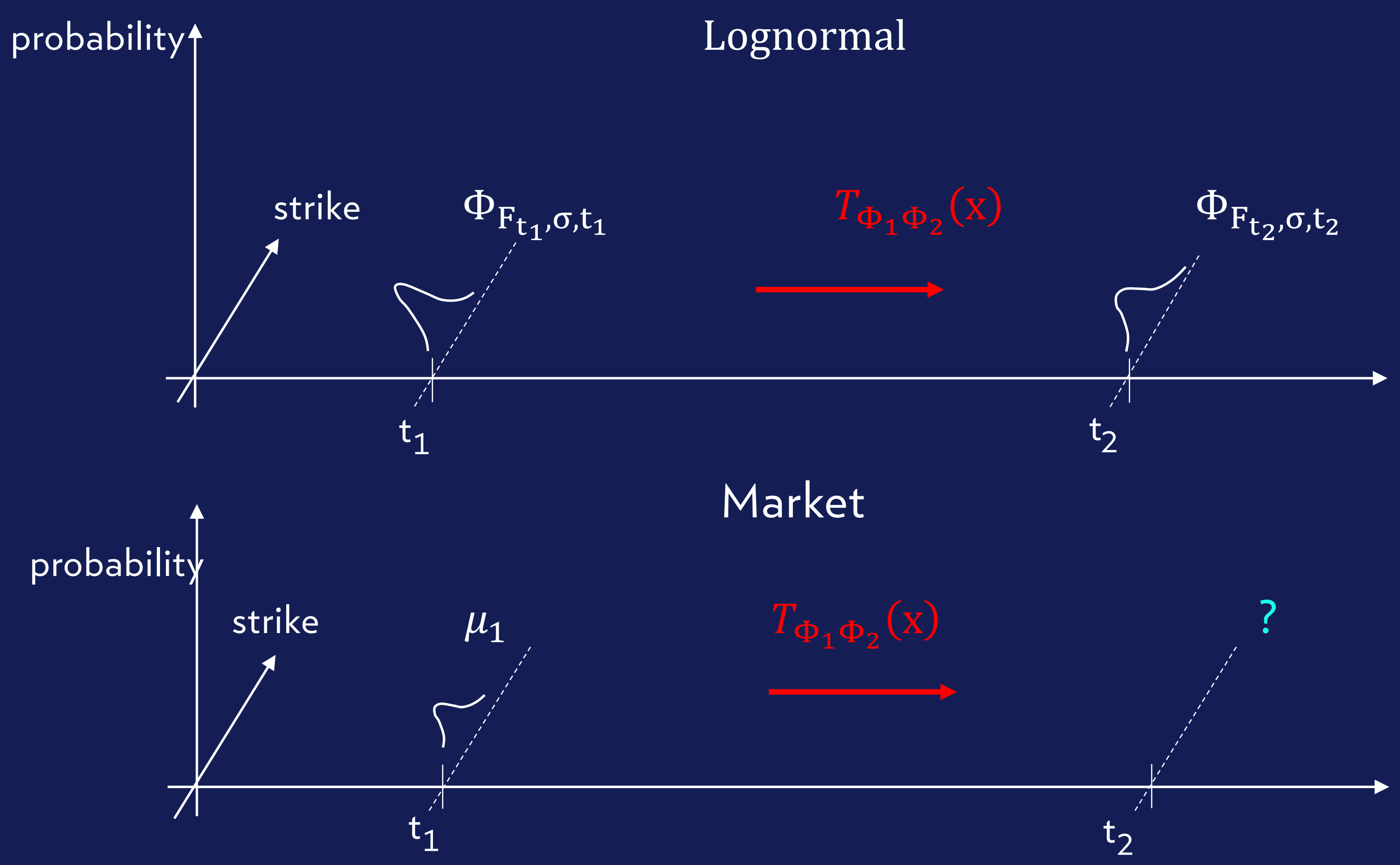
(*) = recentering + possible arb-correction

Extrapolation in time

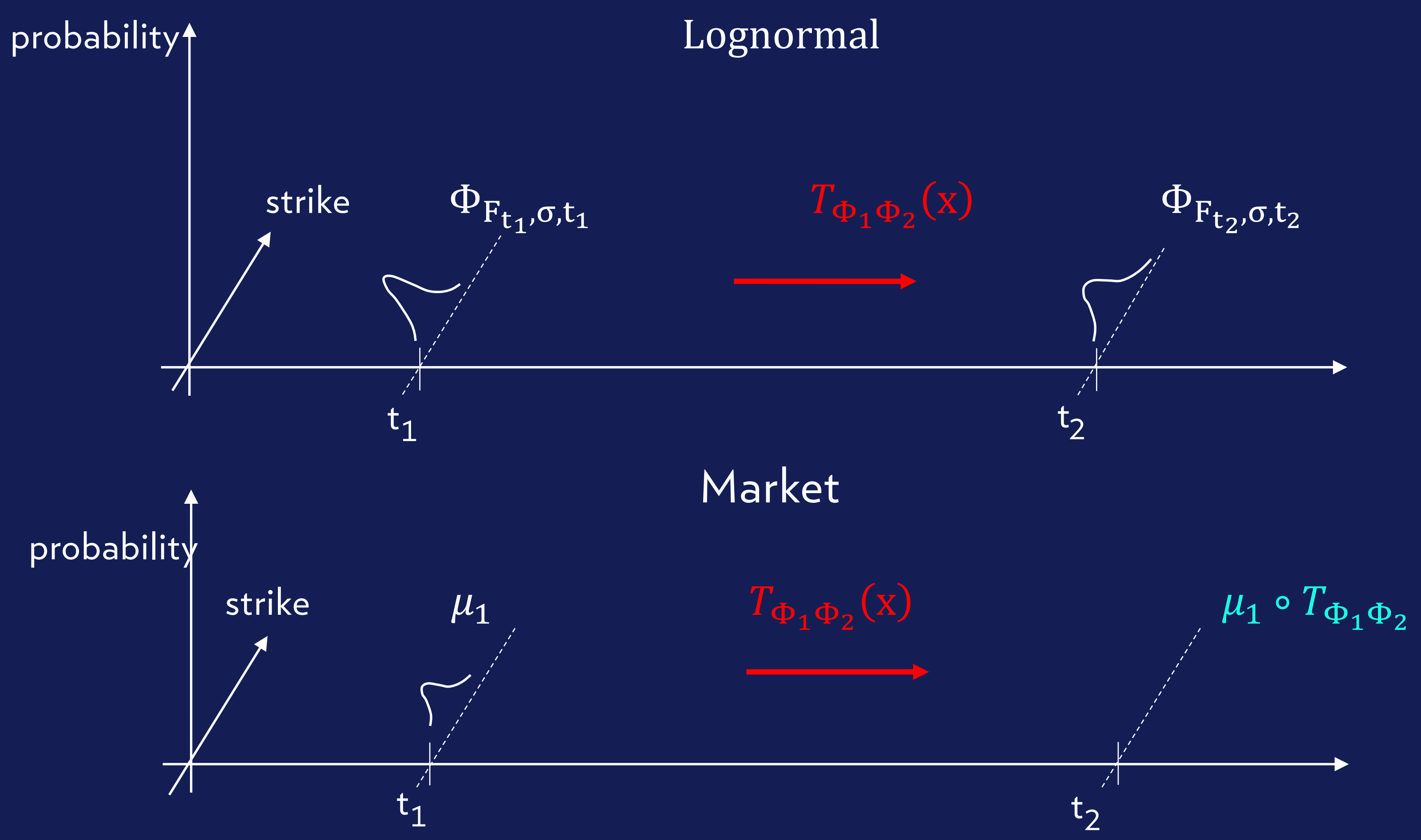
Extrapolation in time



Extrapolation in time

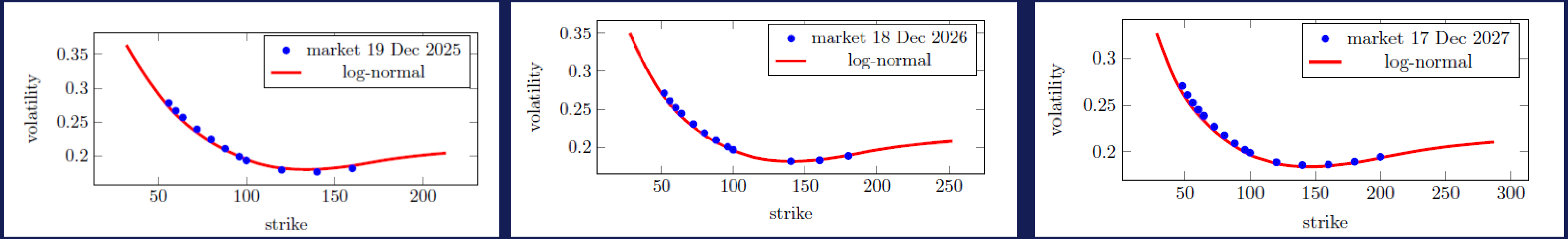


Extrapolation in time



Extrapolation: low volatility stocks

- Lognormal process time-extrapolation works very well



Implied volatility as a function of strike for HEIA NA stock for Dec 2025, 2026 and 2027 maturities as of 4th of Jul 2023 for market data and log-normal extrapolation. The extrapolation for all three maturities used the option market data for 20 Dec 2024 as the base.

Extrapolation: high volatility stocks

- Lognormal process:
  - generates skews with higher convexity and lower skew compared to the market

Extrapolation: high volatility stocks

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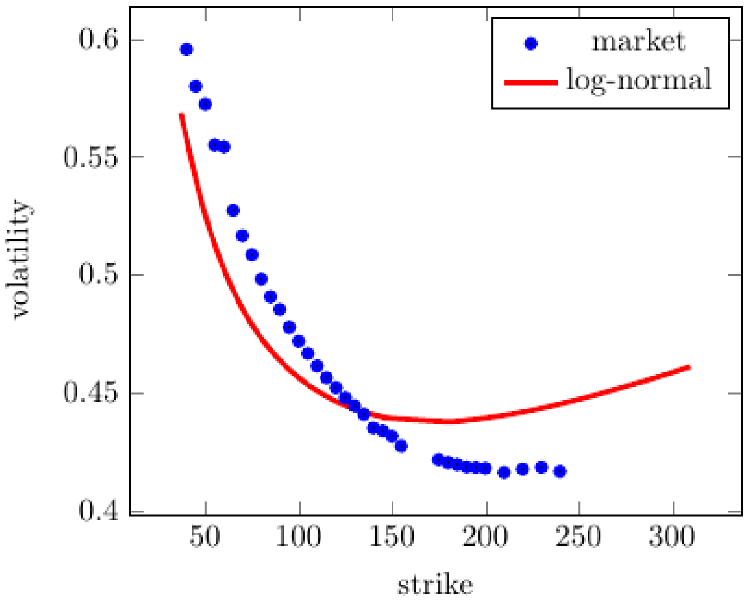
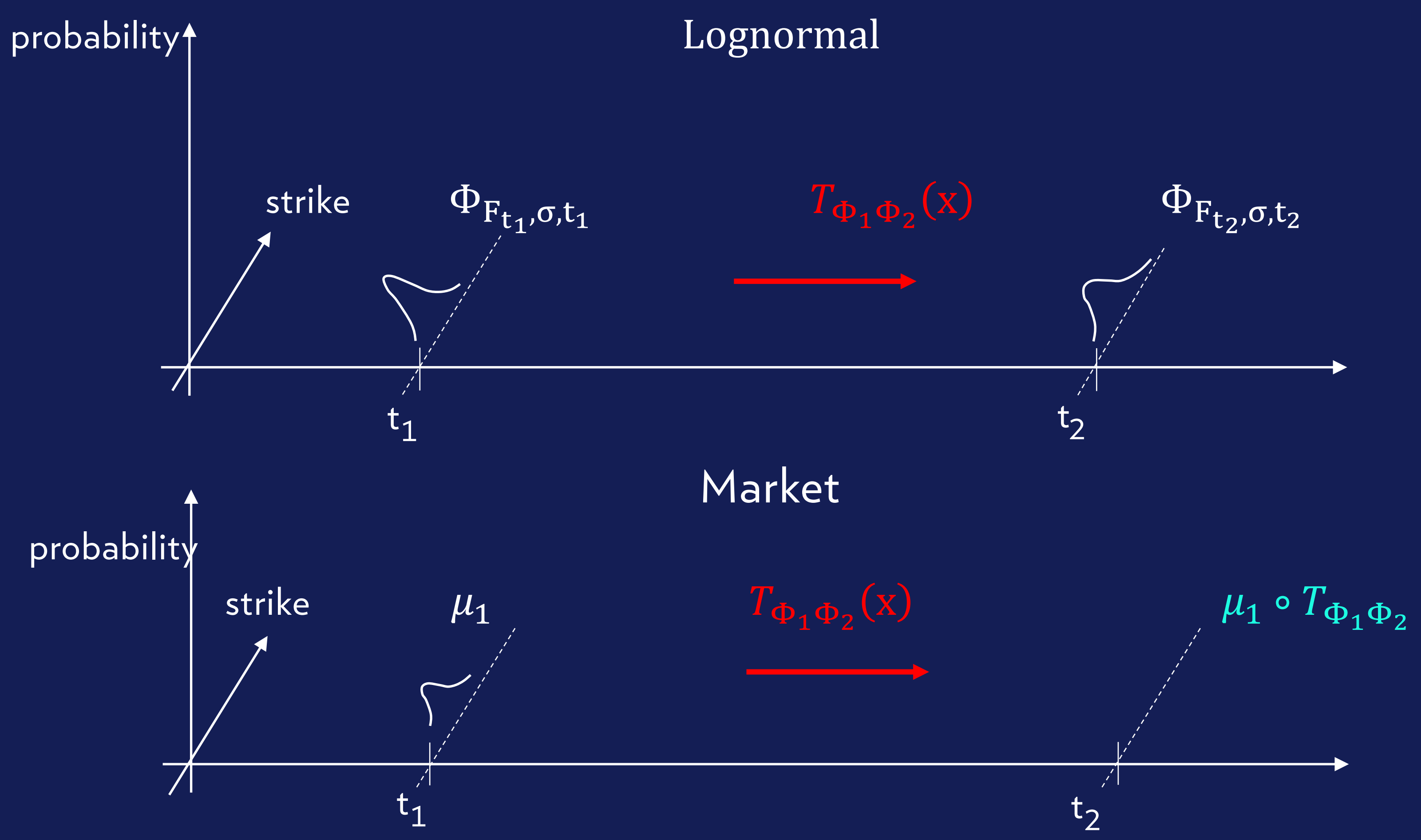
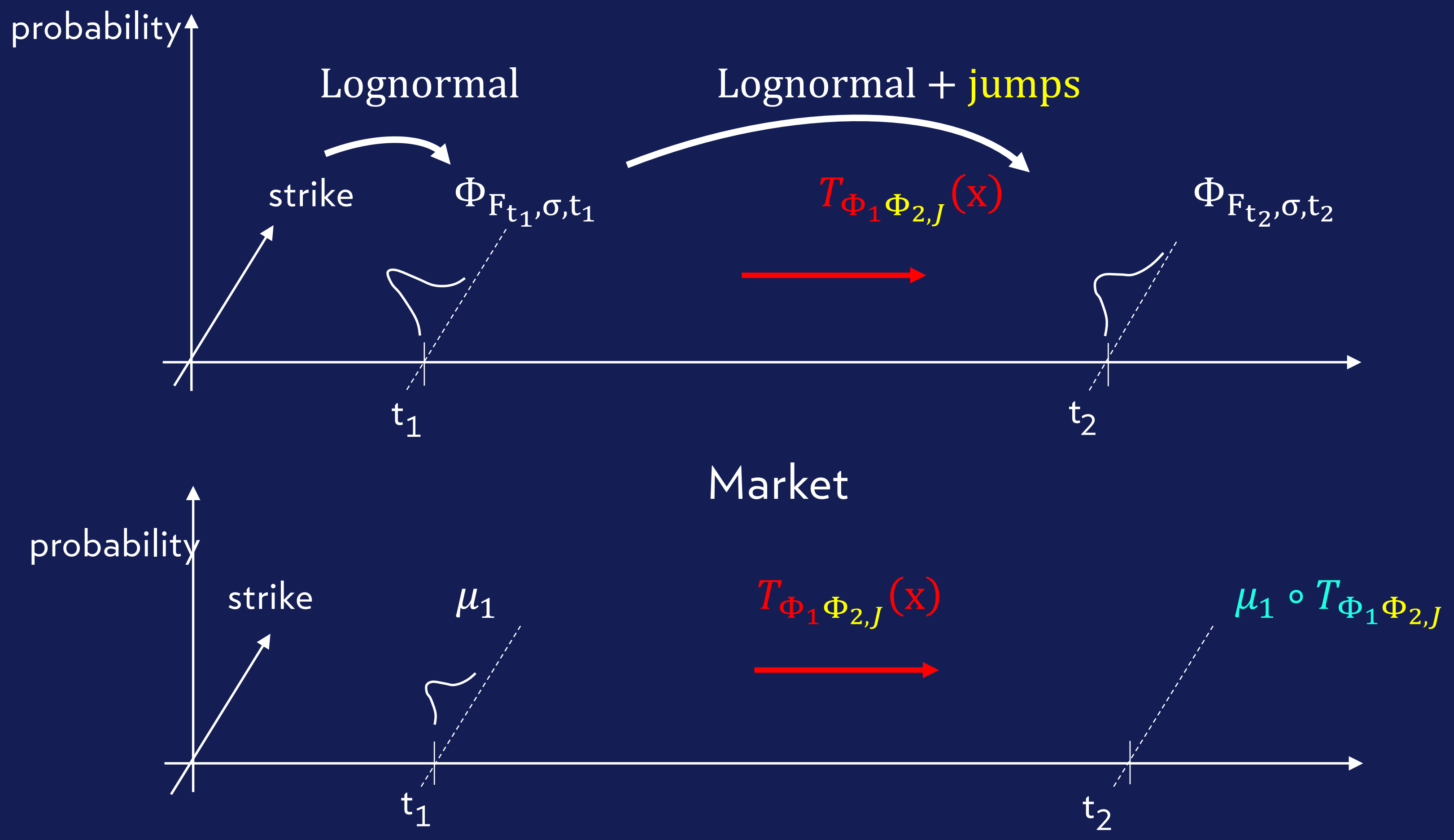


Figure 6: Implied volatility as a function of strike for BIDU US stock for 17 Jan 2025 maturity as of 30 Jun 2023 for the following sources: market data, log-normal extrapolation. Extrapolation used the market data for maturity 19 Jan 2024 as the base.

Extrapolation in time

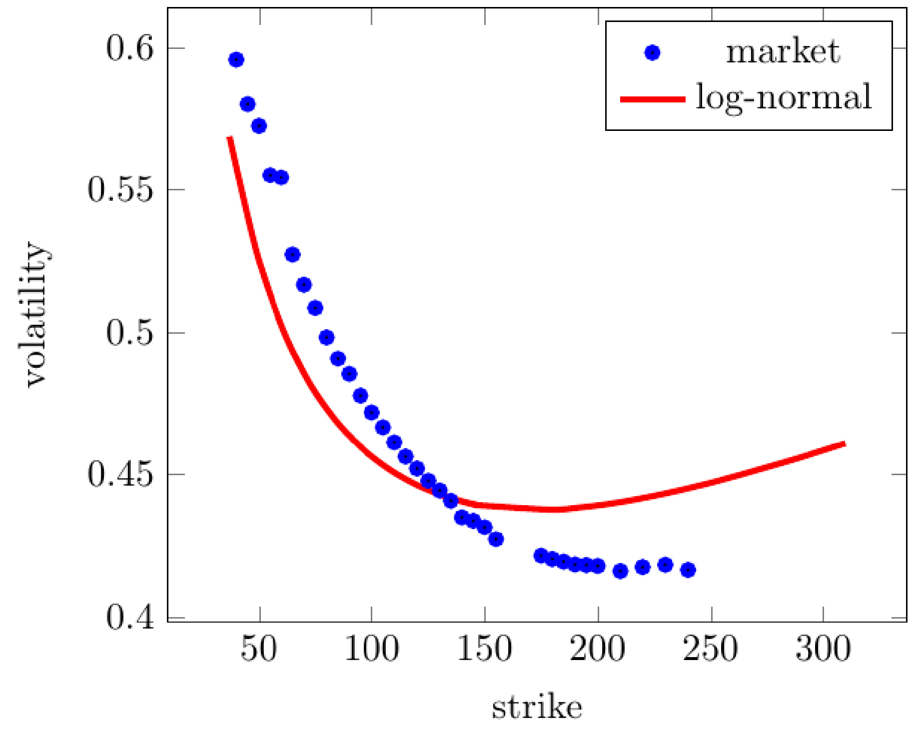


Extrapolation in time



Extrapolation: high volatility stocks

- Lognormal process:
  - skews with higher convexity and lower skew

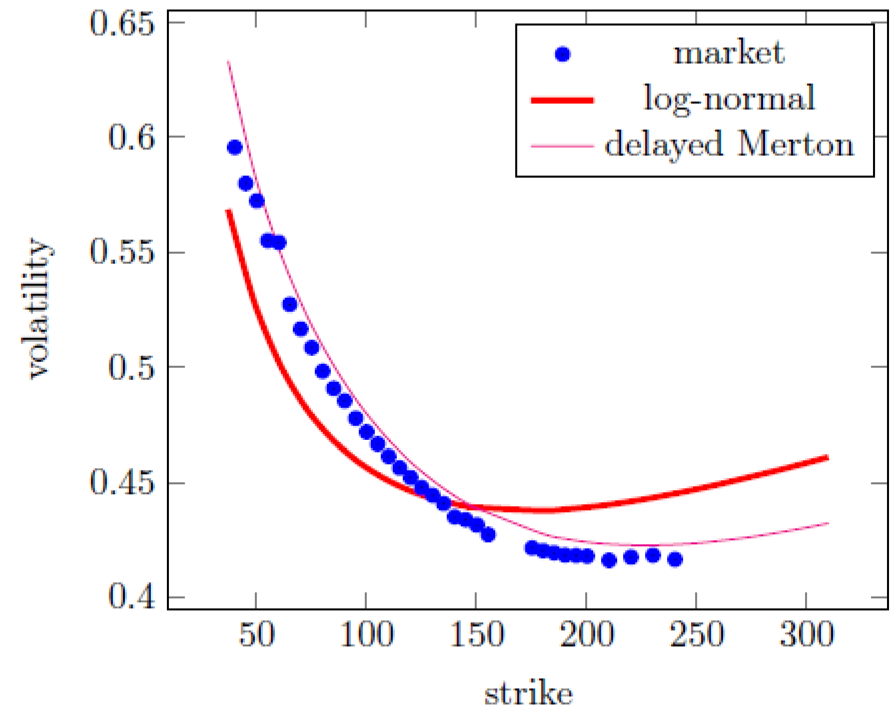


Implied volatility as a function of strike for BIDU US stock for 17 Jan 2025 maturity as of 30 Jun 2023 for the following sources: market data, log-normal extrapolation. Extrapolation used the market data for maturity 19 Jan 2024 as the base.

Extrapolation: high volatility stocks

- Lognormal process:
  - skews with higher convexity and lower skew
- Delayed-Merton process:
  - skews can be tuned to produce market-matching skews

Delayed-Merton:
lognormal to t_1 + jumps between (t_1, t_2)

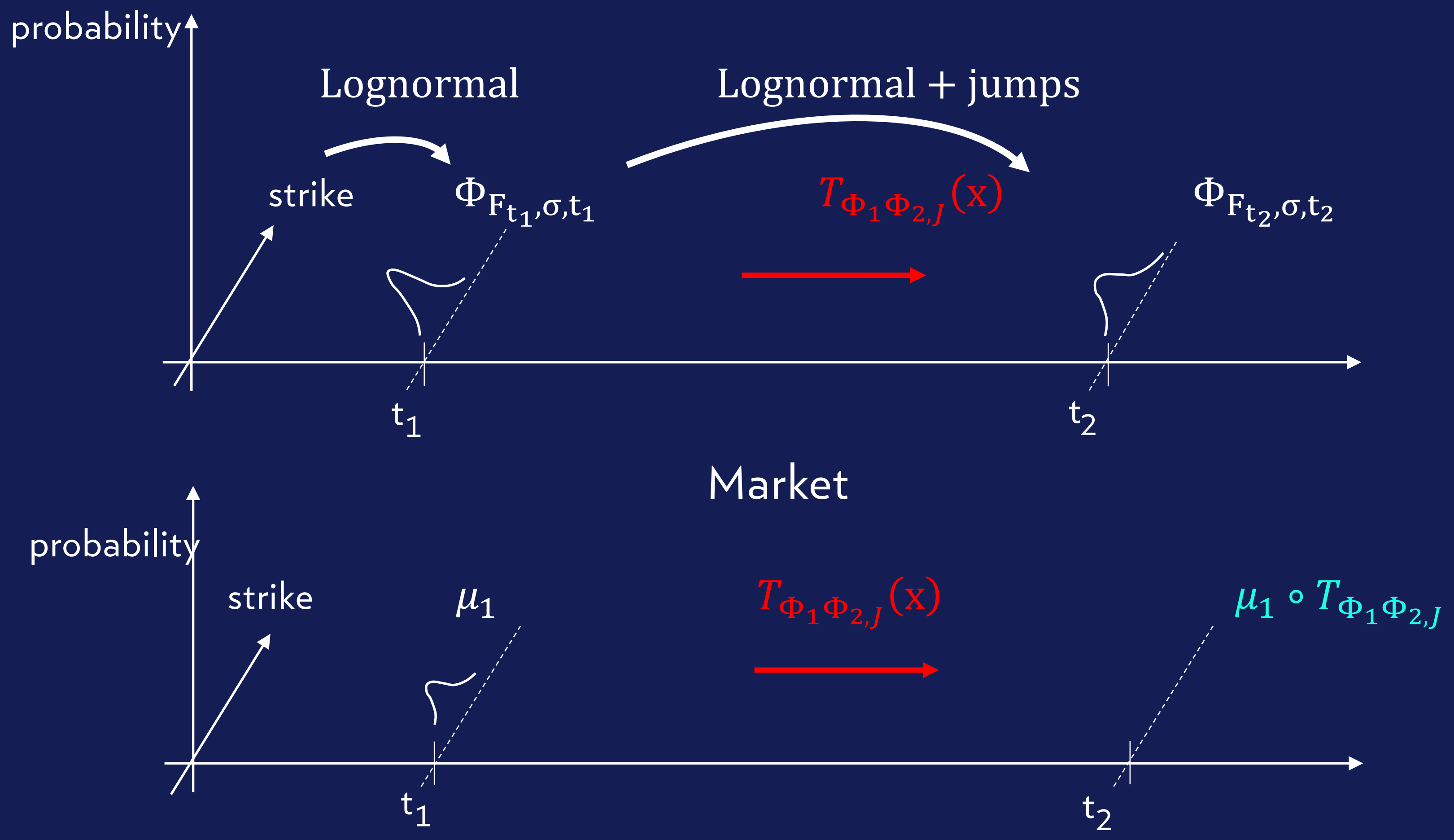


Implied volatility as a function of strike for BIDU US stock for 17 Jan 2025 maturity as of 30 Jun 2023 for the following sources: market data, log-normal extrapolation, delayed Merton extrapolation. Both extrapolations used the market data for maturity 19 Jan 2024 as the base.

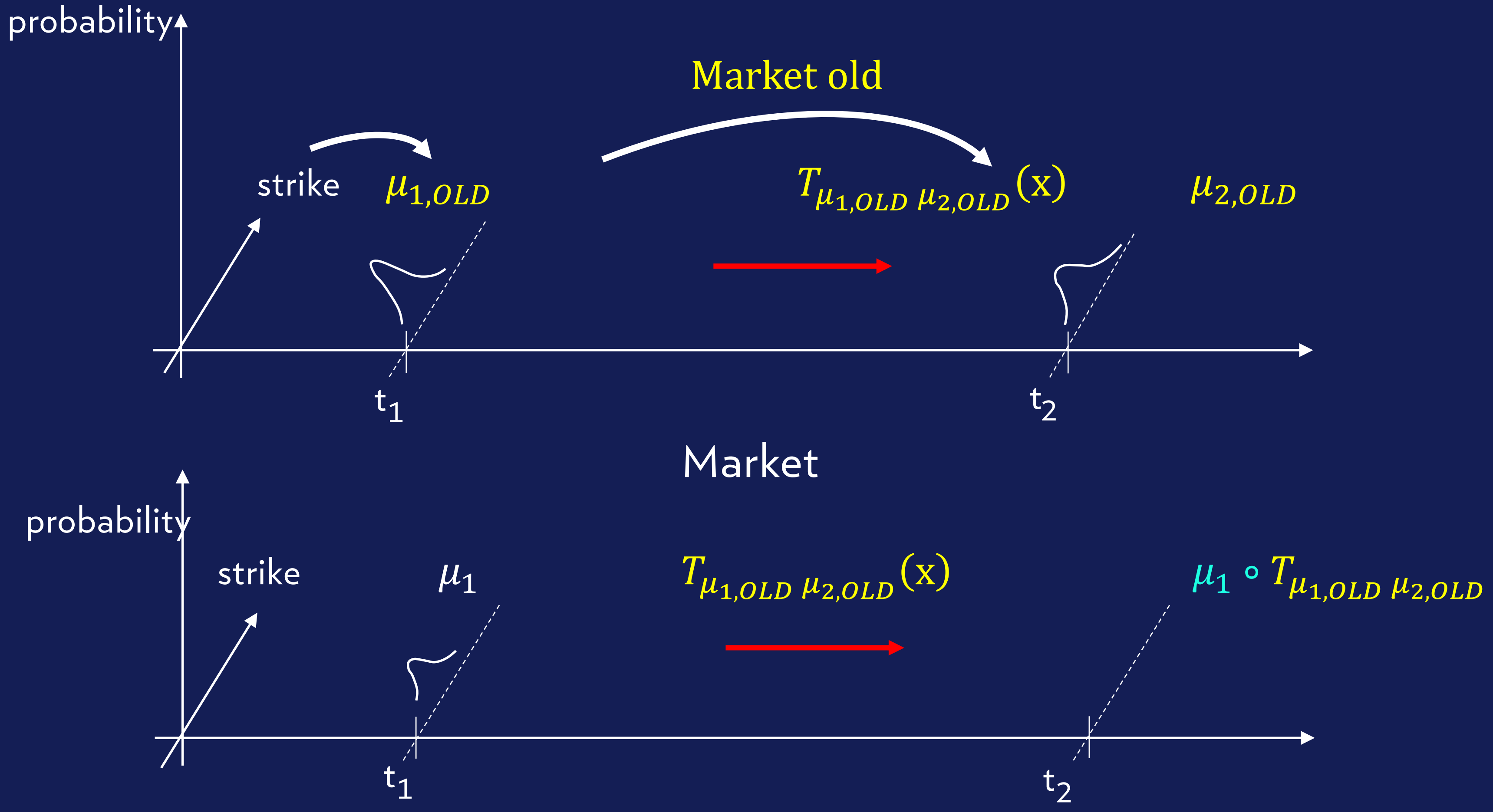
Extrapolation: with Memory

- Maturity drop:
  - often a long maturity is not quoted, but ..
  - historical data is available
  - Instead of Lognormal/delayed Merton we can use previously quoted market distributions!

Extrapolation in time



Extrapolation in time with memory



Extrapolation: with Memory

- Algorithm:
  - calculate and store the transport map between the implied distribution at maturities t_m and t_{m+1} at the time both of them were still quoted:

$$T_{\mu_m,\mu_{m+1}} = \mu_m^{-1} \circ \mu_{m+1}$$

  - recenter the transport map to reflect the new forward:

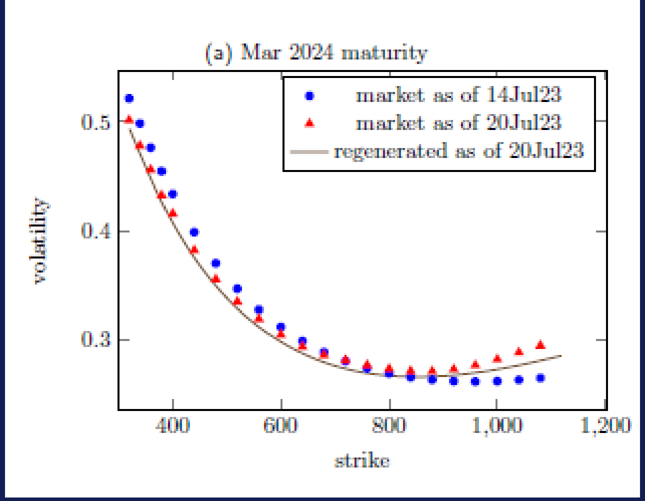
$$T_{center}(x;\alpha) = \frac{1}{\alpha} T(\alpha x)$$

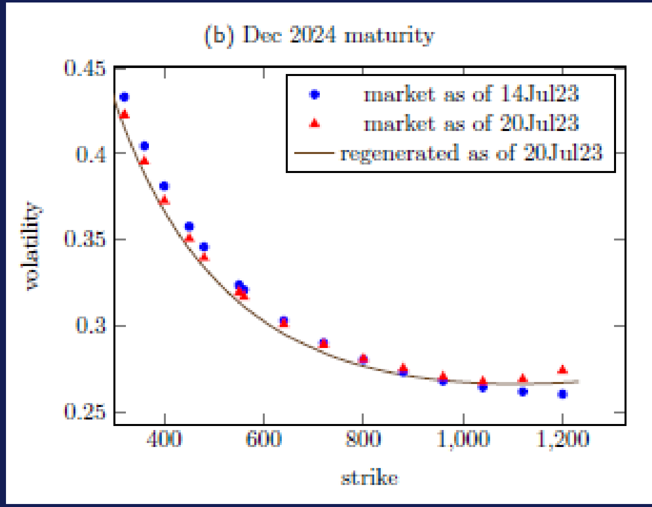
    where $\alpha = \frac{F_{original}}{F_{current}}$ is the ratio of forwards.
  - when the drop happens, we apply the same transport map to the distribution $\mu_{m;current}$ implied from the latest quotes of t_m:

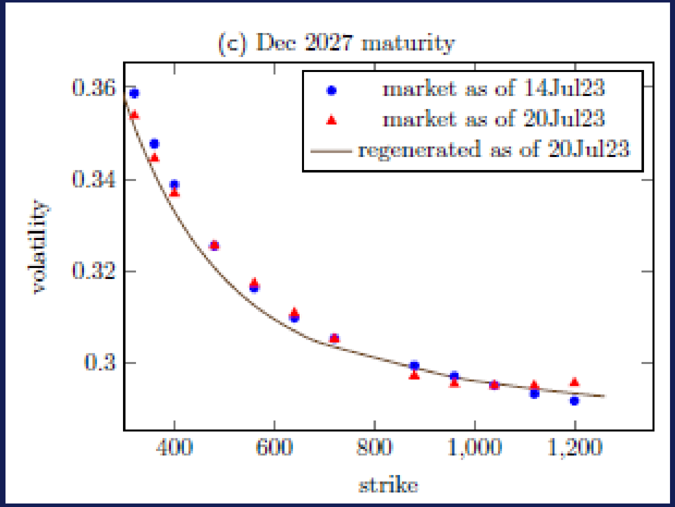
$$\mu_{m+1;regenerated} = \mu_{m;current} \circ \mu_{m;last}^{-1} \circ \mu_{m+1;last}$$

  - apply distribution widening to maintain the same forward volatility between the maturities
  - recenter the distribution to match the forward at t_{m+1}

Extrapolation with Memory: 1 week old



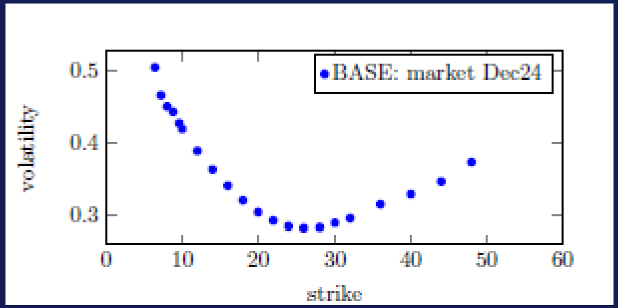


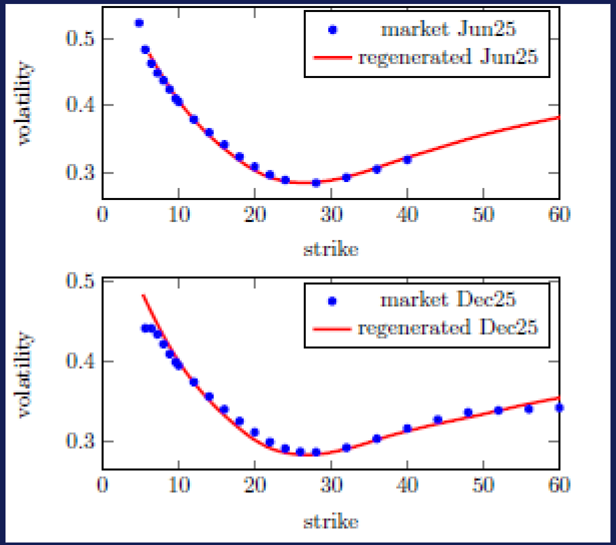


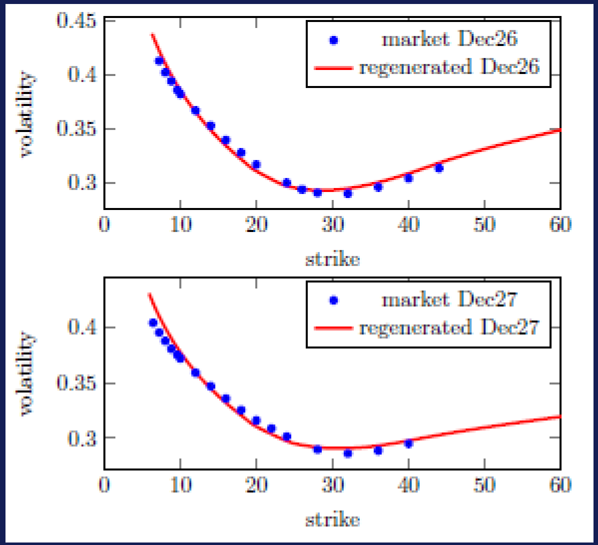
Regenerated volatility skews of ASML NA as of 20th of Jul 2023. The regeneration for all three maturities used the option market data for Dec 21st, 2023 as the base for $\mu_{m;current}$ and the corresponding transport maps extracted from the market as of 14th of Jul 2023.

The spot price has moved from 672.03 on Jul 14 to 626.3 on Jul 20th, which represents a drop of approximately 7%.

Extrapolation with Memory: 1 month old







Regeneration of simulated dropped maturities for PHIA NA over a period of one month. Drop date: Oct 26, 2023 at 10:20am. Regeneration date: Nov 28 2023 at market close (17:30). The base for regeneration was chosen to be t_m = Dec24 maturity. All graphs show data as of the regeneration date Nov 28 2023. The left graph: market implied volatilities for the base maturity Dec24 used for regeneration of the rest of shown maturities. The rest of the graphs show market implied volatilities (blue dots) vs regenerated implied volatility (red curve) for various maturities. Bid-offer spreads vary between 1.8% and 3.6%. Spot increased by 13% during the simulation window.

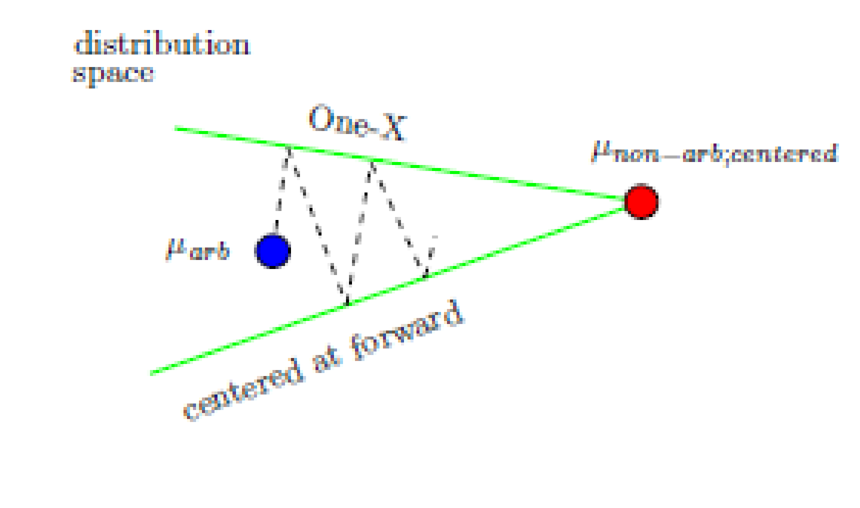
Arbitrage correction

- Alternating projections algorithm

 - Cap and floor the distribution(s) until One-X property is achieved.

 - Recenter the distribution(s) to have the correct forward.

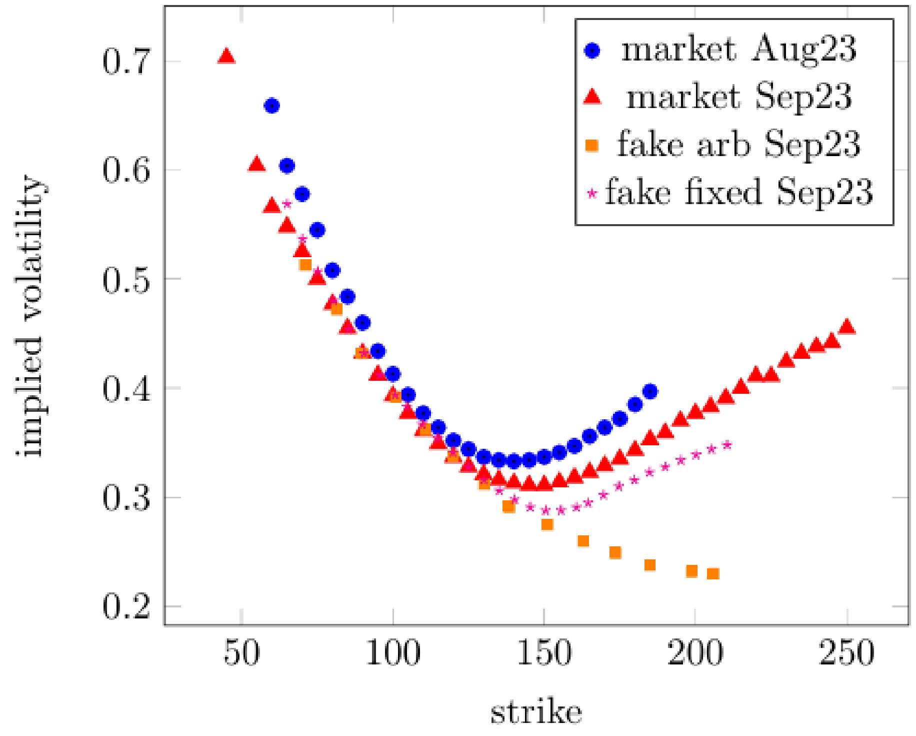
 - Repeat the previous steps until the final distributions have both the correct forward and satisfy the One-X property



Alternative projections algorithm.

Arbitrage correction

- Alternating projections algorithm: example



Volatility as a function of strike for AMZN US stock as of 14 Jun 2023 for the following sources and maturities: market data for 18 Aug 2023, market data for 15 Sep 2023, fake arbitrageable data for 15 Sep 2023 and the previous fake data with fixed calendar arbitrage issue.

Conclusions

- No Arbitrage & Convex order
- One-X & Ohlin's theorem
- Optimal transport
- Convex Order generating/preserving transformations
- Time extrapolation:
  - No market information
  - Regeneration with *Memory*
- Arbitrage correction

References

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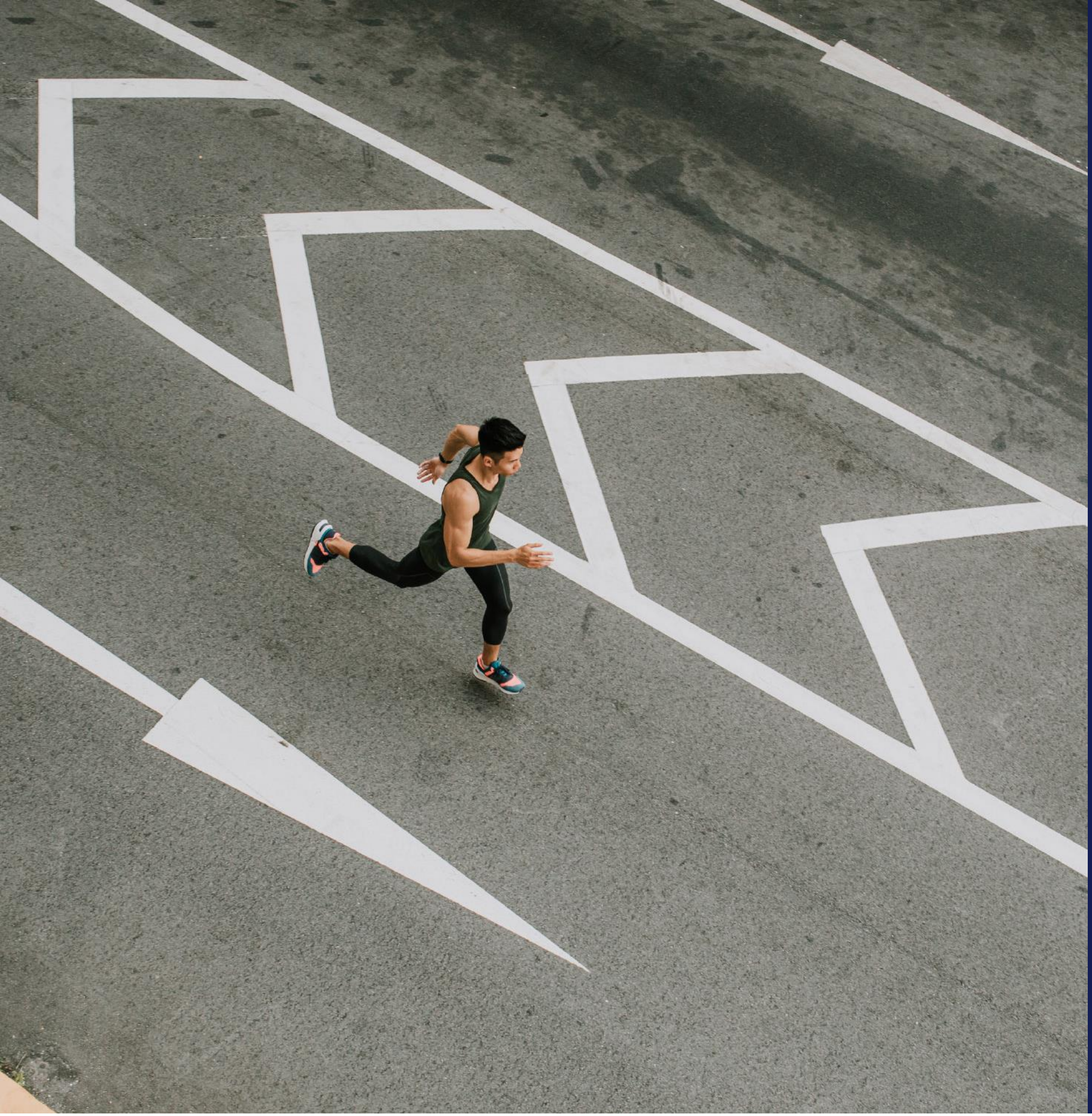

Thank you for your time!

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