A fast Monte Carlo scheme for additive processes and option pricing

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Outline

- Research Question & Motivation (RQM)
  - ✓ Research Question
  - ✓ Motivation
    1. Independent increments: from Levy to Additive
    2. Calibration precision

- The method
  - ✓ Two methods: literature in brief
  - ✓ Simulation via CDF
  - ✓ Lewis-FFT-S
  - ✓ Error optimization result

- Numerical results
  - ✓ Error components & Interpolation Error
  - ✓ Accuracy
  - ✓ Computational Time

- Conclusions

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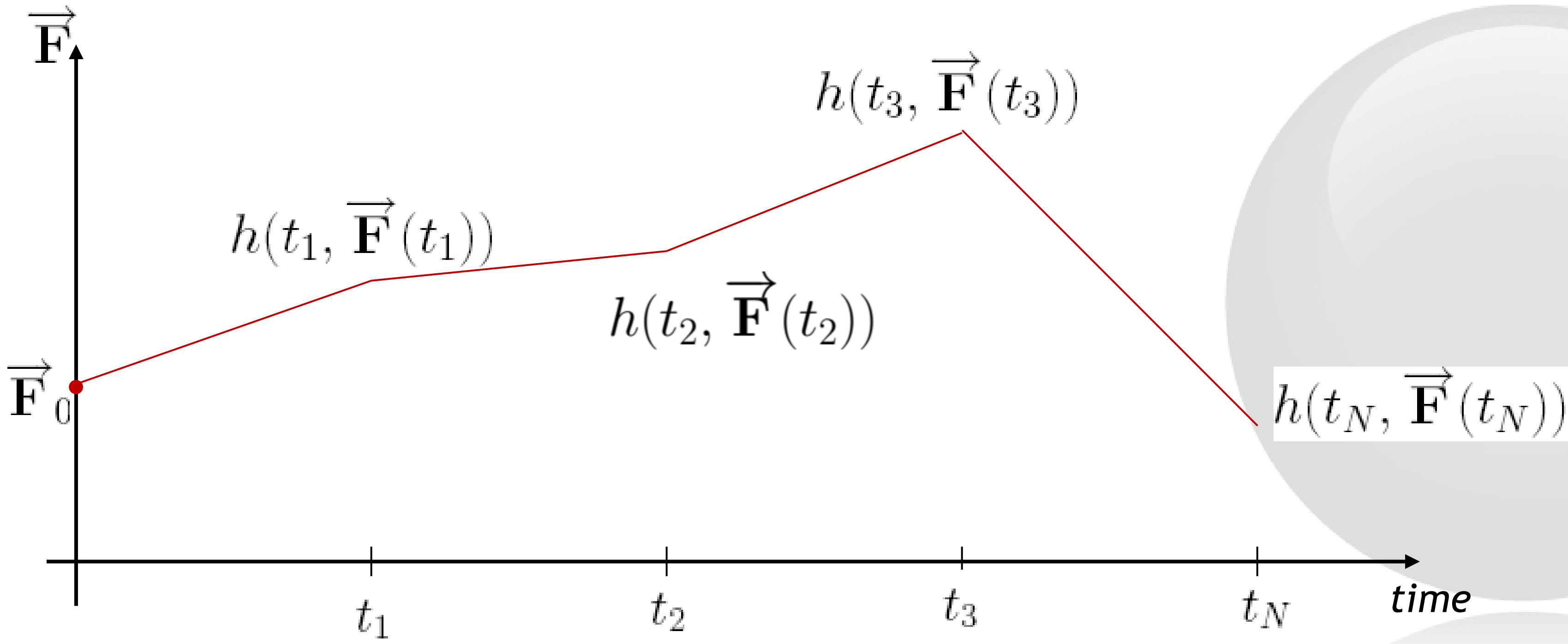
Research Question & Motivation

- RQ: Can we obtain a fast Monte Carlo for (additive) processes?

As fast as a GBM?

An application: discretely monitoring derivative



RQM (Motivation): The importance of being… additive processes

From Lévy

- stationary
- independent

$\Big\}$ increments

- stochastic continuity

to Additive

- ~~stationary~~
- independent

$\Big\}$ increments

- stochastic continuity

Additive processes (sometimes called *time-inhomogeneous* Lèvy processes)

- present infinitely divisible marginal probability distributions
- maintain several *analytical & numerical* Lévy properties (see e.g. Sato, 1999)

Additive processes in finance

Applications in quantitative finance are relatively few:

- Carr-Geman-Madan-Yor (2007) introduce self-similar (or Sato) processes in derivative pricing;
- Benth-Sgarra (2012) apply Additive processes in the electricity market;
- Li-Li-MendozaArriaga (2016) build a quite large class of Additive proc. via Levy subordination;

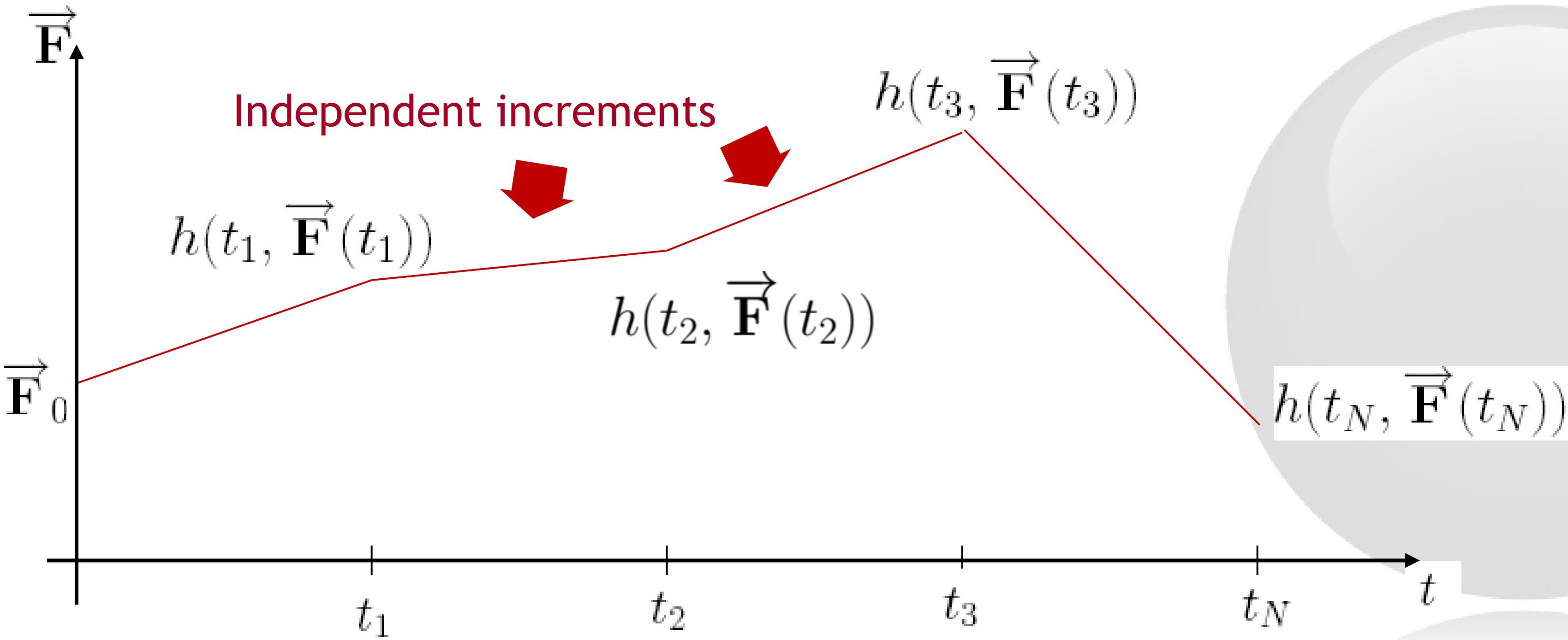
… but are the new frontier:

- Madan-Wang (2020) Additive bilateral VG;
- Carr-Torricelli (2021) European options (Black & Bachelier like) with a simple closed formula;
- Azzone-B. (2021, 2022) Excellent calibration, short-time properties and power law scaling.

- M: Why *additive*?
  with *additive* we can focus on ***finite-time increments*** in a simulation

independent increments

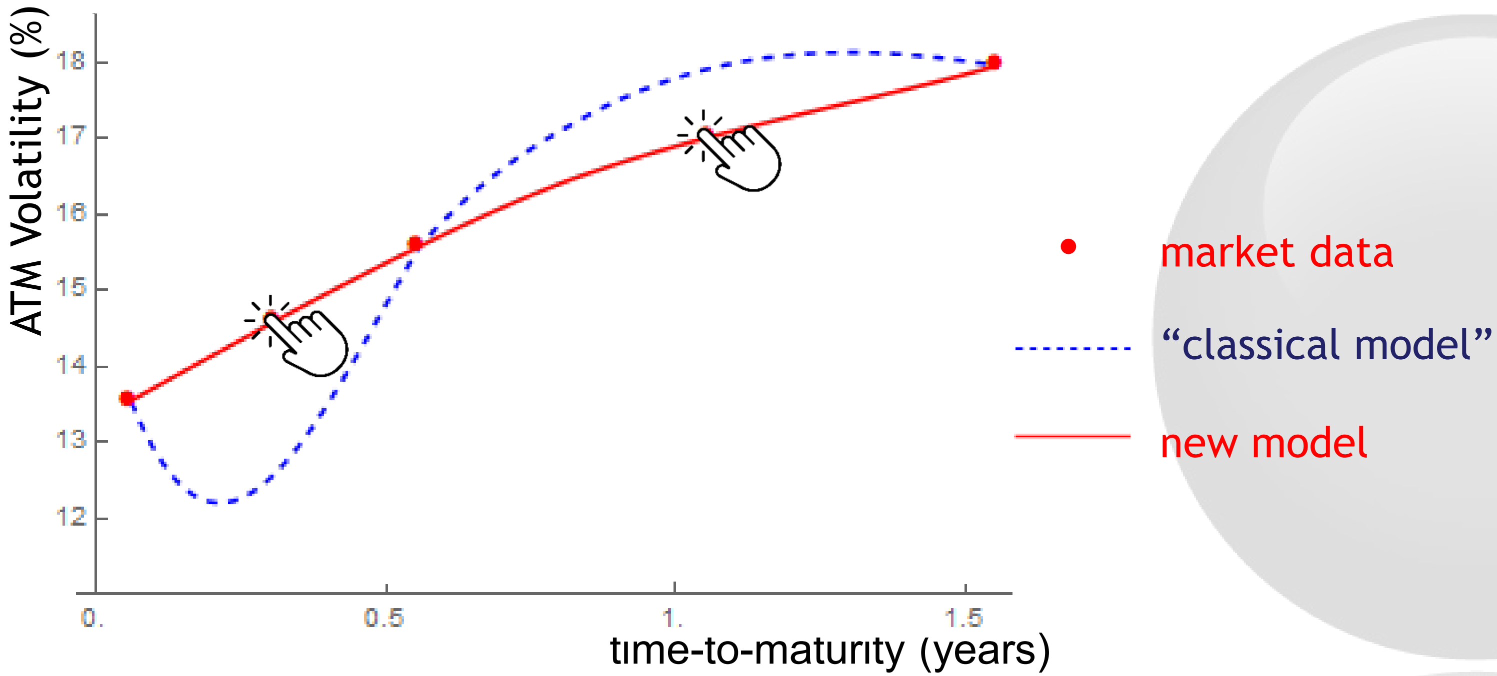
An application



- M: Why *additive*?
  with *additive* we impose the "***exact***" ATM vol term-structure: $\sigma(x=0,t)\,;\, x=\ln K/F_0$

calibration precision

An example: SP500 ATM Vol on the 30th of May 2013 11:00 am NYT

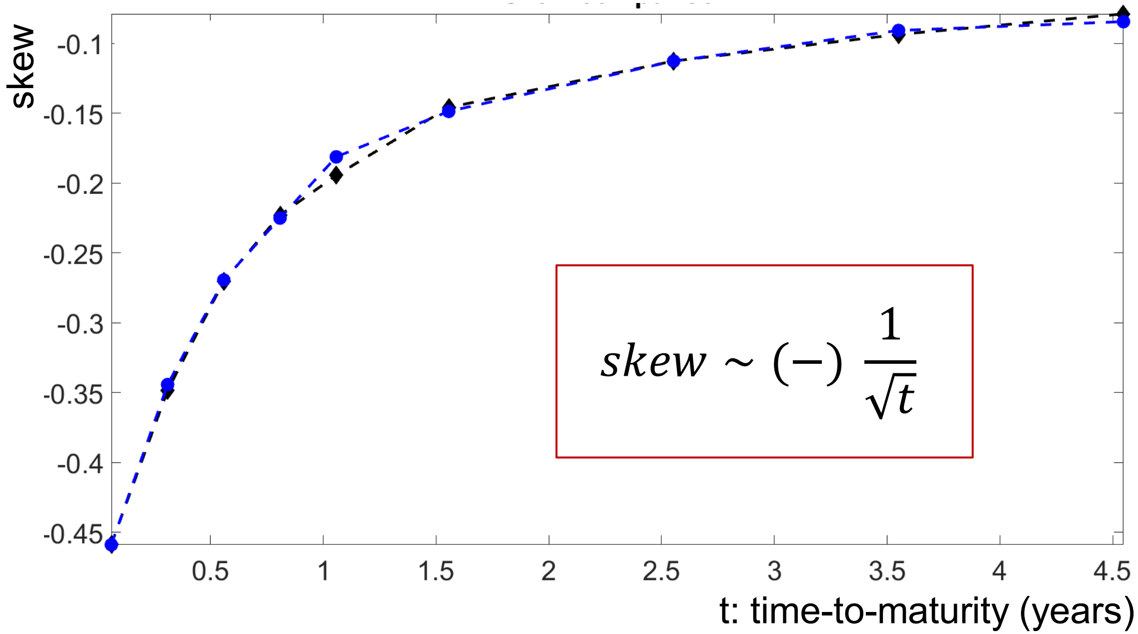


- M: Why *additive*?
  with *additive* we reproduce volatility ***skew***

$$skew := \left.\frac{\partial\, \sigma(x,t)}{\partial\, x}\right|_{x=0}$$

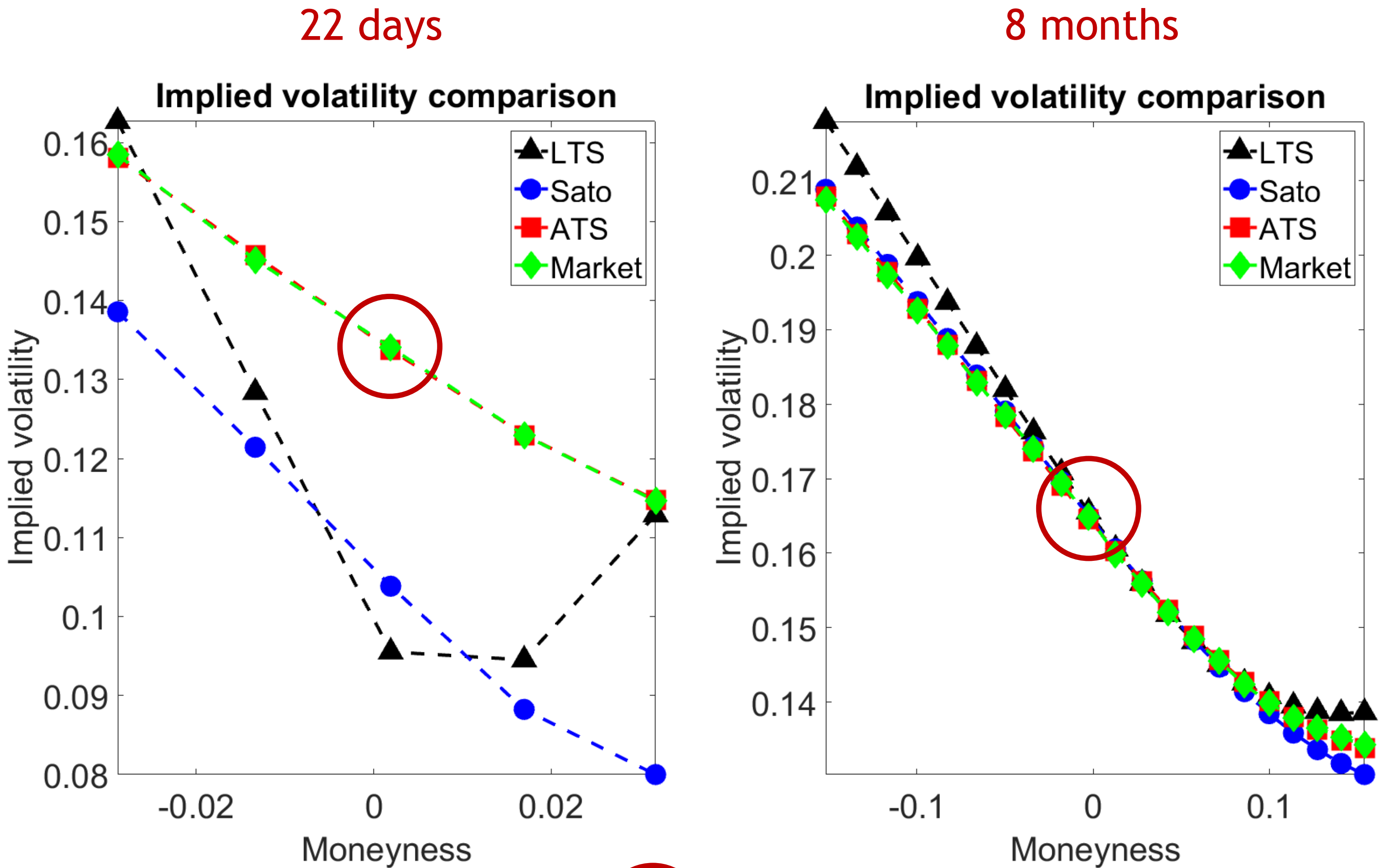
calibration precision

An example: EURO STOXX50 *skew*, same date



see e.g. Carr-Wu (2003), Fouque-Papanicolaou-Sircar-Solna (2004), Gatheral… Azzone-B. (2021)

RQM: Motivation 2 (calibration precision)



: "exact" ATM calibration

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Jump based approaches:

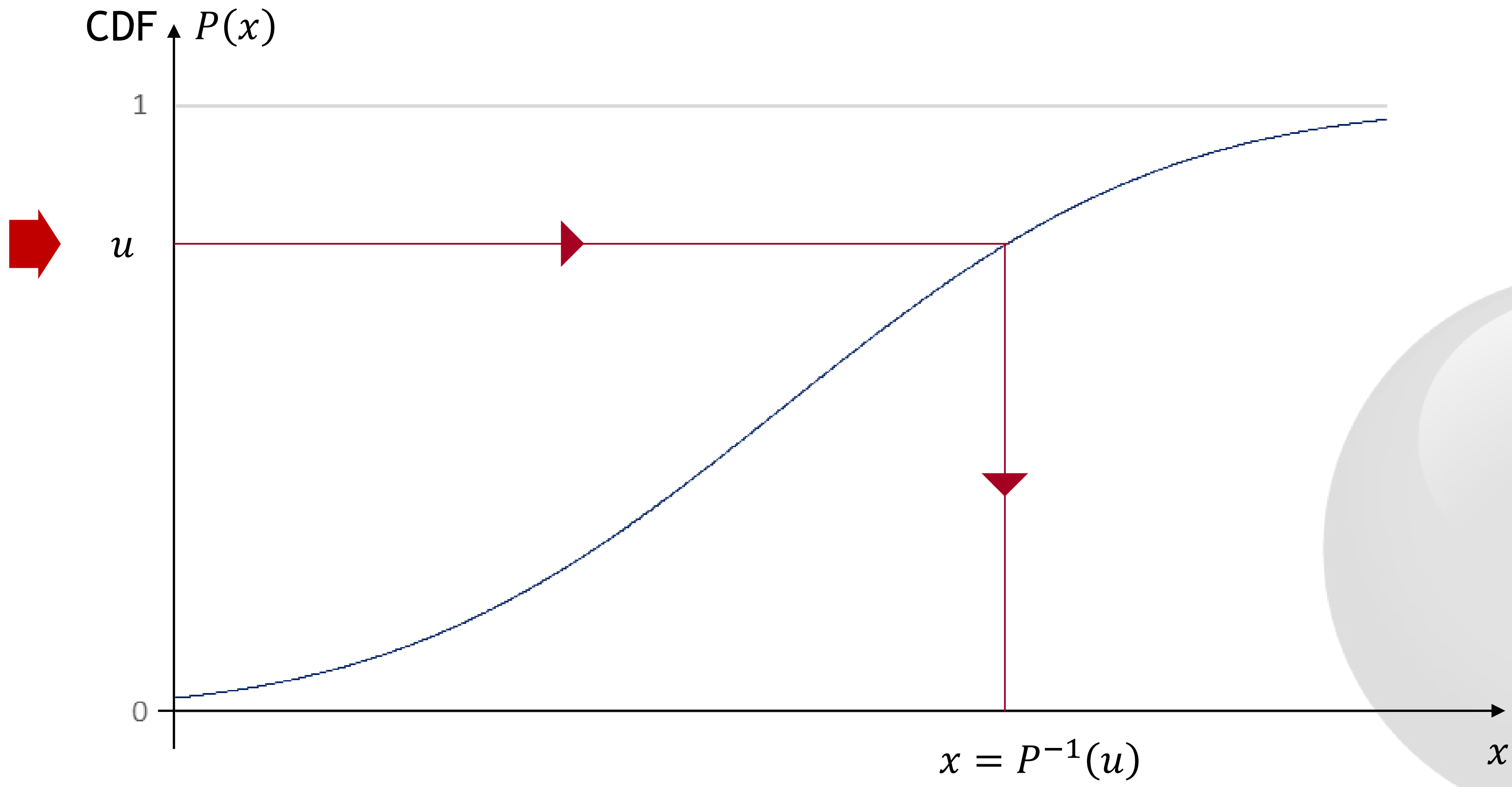
- Cont, R. & Tankov, P. (2003). And references therein (e.g. for Gaussian-Approximation)
- Eberlein, E. & Madan, D. B. (2009). Pricing of structured products.

CDF-FFT based approaches for Lèvy:

- Glasserman-Liu (2010). Error bounds estimation due to linear CDF interpolation;
- Chen-Feng-Lin (2012). Simulation method error;
- Ballotta-Kyriakou (2014) An FFT simulation technique with error bounds estimation.

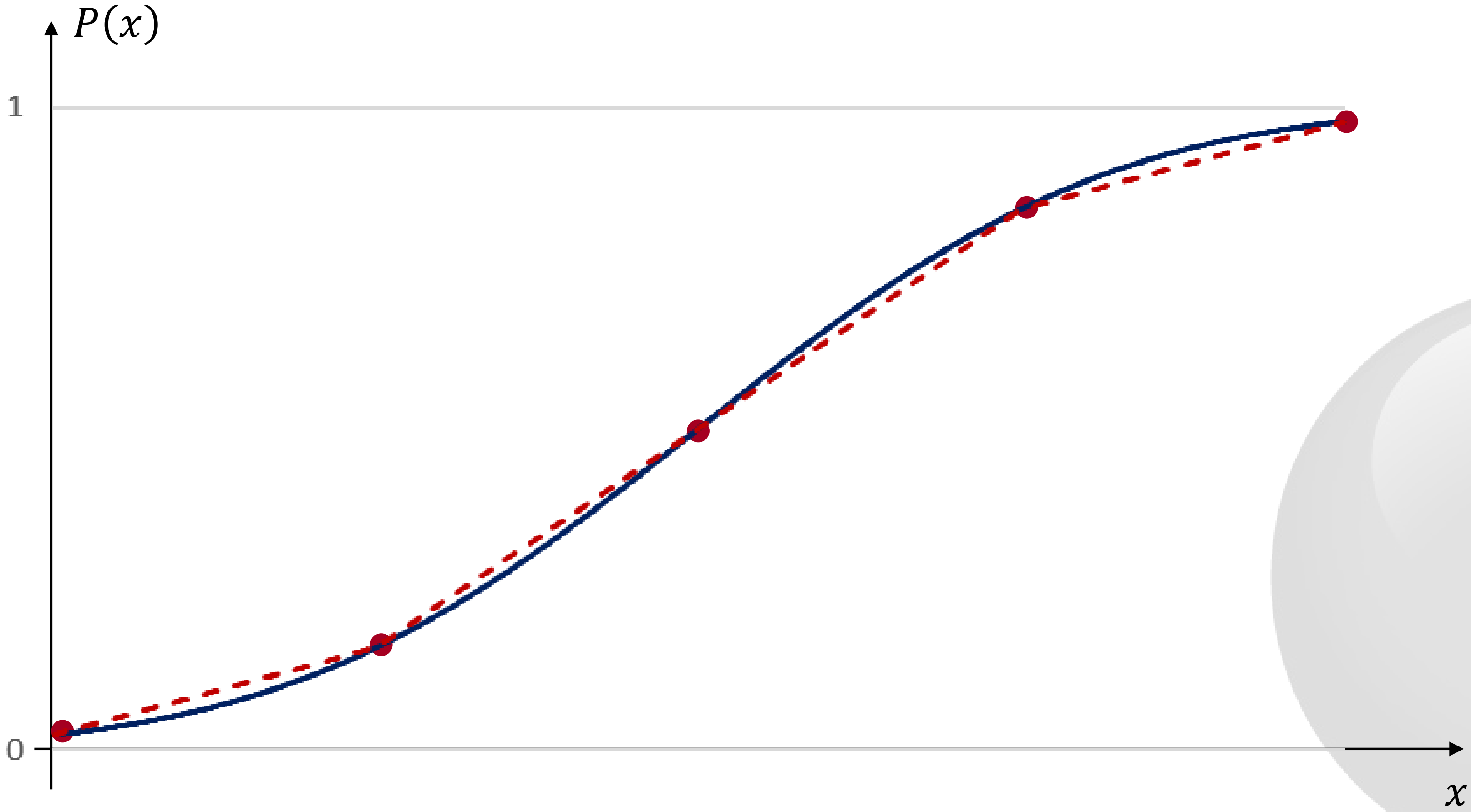
The idea (part A): Simulation via Cumulative Distribution Function (CDF)

Extract u uniform in $[0,1]$ ➡ Obtain x with CDF $P(x)$



The idea (part B): CDF not simple function

CDF not known in simple form ➡ Interpolation



The idea (part C): Simulation from CF

the CDF ... from the Characteristic Function (CF) of the time increment

$$\boxed{P(x) = \frac{1}{2} - \frac{1}{2\,\pi} \int_{-\infty}^{\infty} du \; \frac{e^{-i\,u\,x}\,\phi_{s,t}(u)}{i\,u}}$$

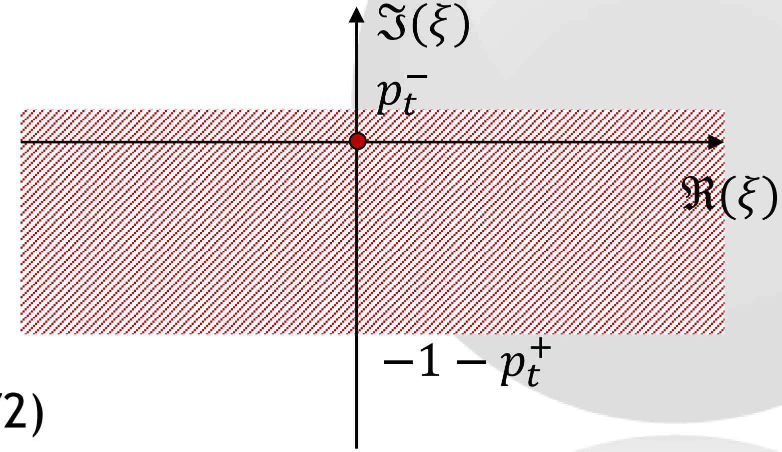
... and its FFT version

A brief recall: basic additive properties

- ✓ characteristic function $$\phi_t(\xi) \coloneqq E\left[e^{i\,X_t\,\xi}\right]$$

- ✓ characteristic function for increments $$\phi_{s,t}(u) = \frac{\phi_t(u)}{\phi_s(u)}$$

- ✓ bounds of analyticity strip for $\phi_t(\xi)$



Lukacs (1972)

The method:

➢ CDF *via*

- Lewis $$P(x) = 1 - \frac{e^{-a\,x}}{2\,\pi} \int_{-\infty}^{\infty} du \; \frac{e^{-i\,u\,x}\,\phi_{s,t}(u - ia)}{i\,u + a}$$ $a \in (0, p_t^+ + 1)$
- FFT $$\hat{P}(x) := 1 - \frac{e^{-ax}}{\pi} \sum_{l=0}^{N-1} Re\left[\frac{e^{-i(l+1/2)hx}\phi_{s,t}((l+1/2)h - i\,a)}{i\,(l+1/2)h + a}\right]$$

Lewis (2001), Lee (2004)

➢ r.v. drawn *via* Spline interpolation

Our main contributions:

- ✓ Estending to additive processes
- ✓ Selecting optimal a
- ✓ Spline interpolation

- Assumption

  $\forall t > s \geq 0, \exists\, B, b, \omega > 0$ s.t. for sufficiently large u

$$|\phi_{s,t}(u - i\,a)| < B e^{-b\,u^{\omega}} \qquad \forall a \in (0, p_t^{+} + 1)$$

- Prop. 2.2.

  For all Additive processes that satisfy the Assumption

  1. CDF error bound (N number of grid points)

$$\mathcal{E}_N^{CDF}(x) = O(N^{-\omega/(1+\omega)})\boxed{\exp(-bN^{\omega/(1+\omega)})}$$

  2. optimal bound for $a = (p_t^{+} + 1)/2$

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Example: Additive (Normal) Tempered Stable (ATS)

Forward with expiry T is modeled as an exponential Additive

$$F_t(T) := F_0(T)\ \exp(f_t)$$

with f_t an ATS, whose **characteristic function** is

$$\boxed{\mathbb{E}\left[e^{iuf_t}\right] = \mathcal{L}_t\left(iu\left(\frac{1}{2}+\eta_t\right)\sigma_t{}^2 + \frac{u^2\sigma_t^2}{2};\ k_t,\ \alpha\right) e^{-iu\,t\ \ln\mathcal{L}_t\left(\sigma_t^2\,\eta_t;\, k_t\right)}}$$

with $\quad k_t = \bar{k}\ t^{\beta}, \qquad \eta_t = \bar{\eta}\ t^{\delta}, \qquad \sigma_t = \bar{\sigma}\ , \qquad \alpha = (0,1)$

Excellent calibration

OK assumption

Error contributions

When pricing a generic derivative with an approximated CDF

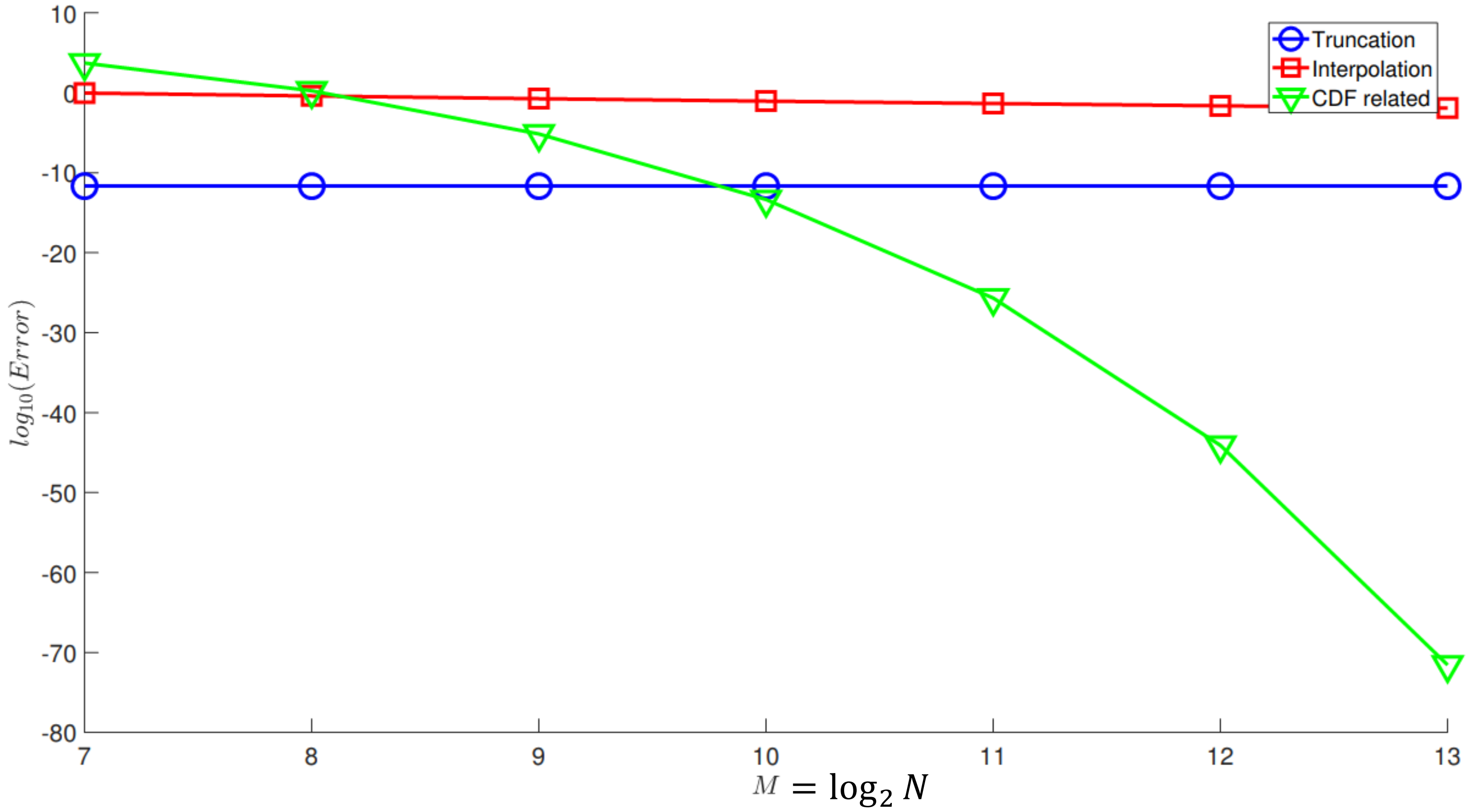
Error < CDF related Error +

Interpolation Error +

Truncation Error

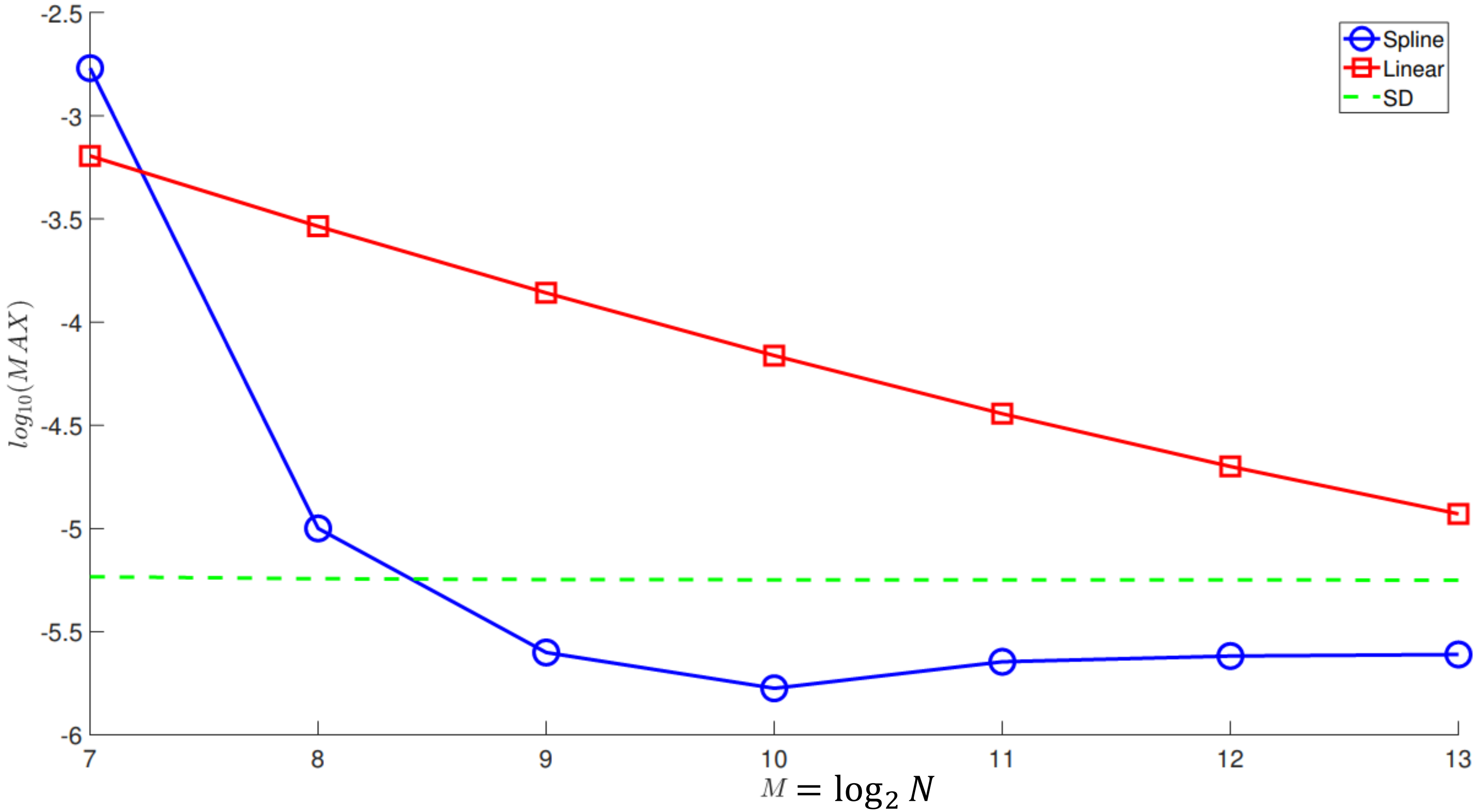
upper bounds can be 'controlled' analytically

Error contributions



One-month European call case with linear interpolation, $\alpha = 2/3$

Interpolation error: linear *vs* spline



Bias estimation: MAX error on 30 calls with moneyness degree in the range $(-0.2, 0.2)$
'Variance' estimation: SD error with 10^7 simulations

Accuracy

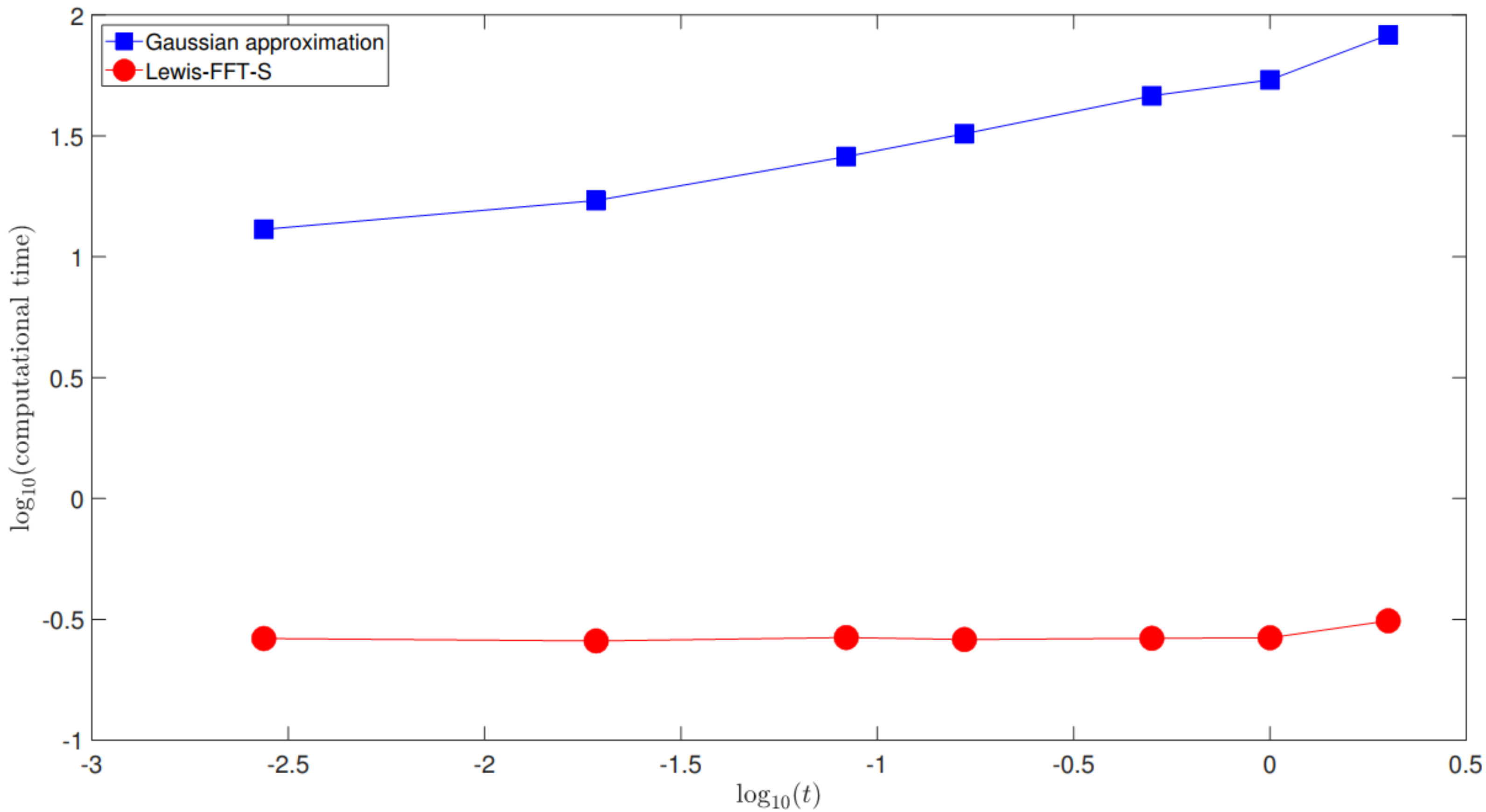
- Plain Vanilla calls (30 calls, $ttm = 1m$, moneyness degree in the range $(-0.2, 0.2)$, $N_{sim} = 10^7$)

	M	7	8	9	10	11	12	13
$\alpha = 1/3$	MAX [bp]	7.08	0.32	0.02	0.03	0.03	0.03	0.03
	RMSE [bp]	3.49	0.29	0.01	0.02	0.02	0.02	0.02
	MAPE [%]	2.30	0.21	0.01	0.01	0.01	0.01	0.01
	SD [bp]	0.12	0.12	0.12	0.12	0.12	0.12	0.12
$\alpha = 2/3$	MAX [bp]	317.29	0.19	0.04	0.01	0.02	0.03	0.03
	RMSE [bp]	282.99	0.16	0.03	0.01	0.02	0.02	0.02
	MAPE [%]	185.64	0.11	0.02	0.01	0.01	0.01	0.01
	SD [bp]	0.25	0.11	0.11	0.11	0.11	0.11	0.11

- Discretely monitored options (5y, Q/Q)

Moneyness	Asian [%]	SD [%]	Lookback [%]	SD [%]	Down-and-In [%]	SD [%]
-0.5	39.79	0.01	3.31	0.00	2.31	0.00
-0.25	24.36	0.01	8.72	0.00	3.98	0.00
0	10.04	0.01	23.07	0.01	6.15	0.01
0.25	2.57	0.00	50.53	0.01	8.95	0.01
0.5	0.55	0.00	86.98	0.01	12.55	0.01

Computational time: Lewis-FFT-S vs GA (with Ziggurat)



with $\boxed{\text{Lewis-FFT-S error} \underset{\text{always}}{\leq} \text{GA error}}$

Computational time: Lewis-FFT-S vs GBM

Simulation 10^7 trials

	M	7	8	9	10	11	12	13
$\alpha = 1/3$	Time [s]	0.23	0.27	0.28	0.28	0.28	0.28	0.29
$\alpha = 2/3$	Time [s]	0.25	0.27	0.28	0.28	0.28	0.28	0.28

$\simeq$ $\boxed{\textit{3 times GBM}}$

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- **Conclusions**

Conclusions: additive vs other model classes. Not only fast-simulation!

	Additive processes	**Stochastic volatility**	**Rough volatility**
simple derivatives	closed formula ✓	closed formula ✓	NO closed formula ✗
short time	$skew \sim t^{-\frac{1}{2}}$ ✓	$skew \sim const(t)$ ✗	$skew \sim t^{-\frac{1}{2}}$ ✓
calibration	• very low MSE/MAPE ✓ • skew ✓ • parsimony ✓	• high MSE/MAPE ✗ • wrong skew ✗ • parsimony (?) ✓ ✗	• (?) MSE/MAPE ✓ ✗ • skew ✓ • parsimony (?) ✓ ✗
simulation	$\boxed{\text{fast} \sim O(1)\ GBM}$ ✓	slow $\sim 10^2 - 10^3\ GBM$ ✗	very slow $\sim (\,?\,)\ GBM$ ✗

THM. Additive model: fast & precise

precise

fast

- excellent calibration
  - ✓ exact fit vol term-structure
  - ✓ reproduces market skew (being parsimonious)

- fast simulation;
  - ✓ via Lewis-FTT-S
  - ✓ as fast as GBM

Bibliography sketch (I)

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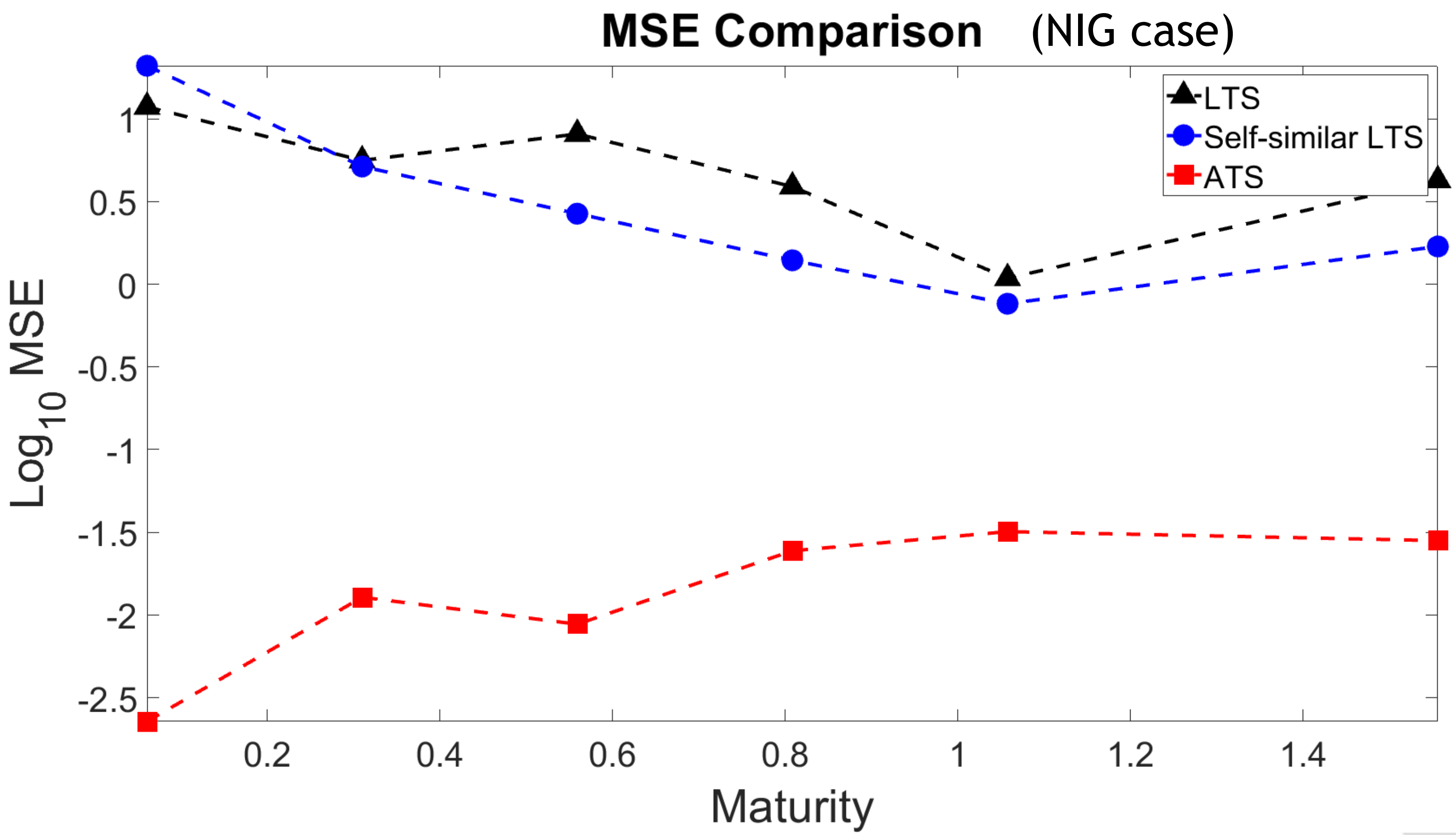
Appendix 1. Additive Calibration performance



Appendix 2. Ballotta-Kyriakou-Hughett Regularization.

- CDF

  - ✓ $F_r(x) = F(x) - \frac{1}{2}\big(F(x-D) + F(x+D)\big)$

with D sufficiently large

- Characteristic Function

  - ✓ $\frac{1}{iu}\,\phi_r(u) \coloneqq \frac{1}{iu}\int_{-\infty}^{\infty} e^{iux} dF_r(x) = \frac{1-\cos(uD)}{iu}\,\phi(u)$