Optimal Life-Contingent Payoffs: A Peer-to-Peer Solution

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L20: The Twentieth International Longevity Risk and Capital Markets Solutions Conference

An Chen | Ulm University

Singapore, September 8-9, 2025

Optimal payoff/asset allocation problem

- Determining the optimal payoff for an investor with a fixed time horizon has been a fundamental problem in modern financial economics (pioneering work by Merton (1969))
- Extended to settings with *state-dependent* preferences (see, e.g., Browne (1999), Cvitanić & Spivak (1999), Bernard *et al.* (2015), and Steffensen & Søe (2023)).
  - In this paper, we consider investors whose preferences are contingent on their mortality in that they assign different utility to survival and death (bequest) state.

Typical optimal financial payoff

Solution to Optimal Financial Payoff under State-Dependent Utility

- Typical optimal financial payoff (see e.g. Steffensen & Søe (2023), and Chen *et al.* (2024))

$$C_T^* = f(\xi_T^F)$$

- Frequently also expressed as $f(S_T)$.
- The payoff is not life-contingent.

Increasing trend of life-contingent contracts

In 2021, life-contingent contracts accounted for approximately

40% of **market volume**

95% of **market growth**

in the life insurance sector in Europe.[1]

[1] in terms of Gross Written Premiums (EIO22, Figure 20).

What we do in our paper

Given a setting with

- an **agent** with general state-dependent utility in that she assigns **different utility to survival and bequest**,
- a **market** $(\Omega, \mathcal{F}, \mathbb{P})$ with financial risk and mortality risk,

we

1. **derive** the **optimal life-contingent payoff** C_T^* given that actuarially fair premiums for mortality risk are charged
2. **propose** a **peer-to-peer solution** to approximate the optimal life-contingent payoff C_T^* (asymptotic convergence)

Contributions of our paper

Contribution 1: What type of general life-contingent payoff is optimal?

- not only financial payoffs (as in Chen *et al.* (2024))
- not pre-determined type of life-contingent payoffs

→ yielding higher expected utility than the optimal finacial payoff

Contribution 2: How to deliver it via a **peer-to-peer solution**?

- generalization of tontine-like products (such as *group self annuitization schemes* (Piggott *et al.*, 2005), *pooled annuity funds* (Stamos, 2008) or *unit-linked tontines* (Chen *et al.*, 2022))

Agent– State-dependent utility u_I

Definition 1: State-dependent utility u_I

The agent assigns different utility to **survival** and **bequest**, that is,

$$u_I(x) = \begin{cases} u_a(x), & \text{if } I_T = 1, \\ u_d(x), & \text{if } I_T = 0, \end{cases}$$

where $T \in (0, \infty)$ and $I_T = 1$ with probability p.

Assumption 1: Inada conditions

The utility function $u_i : (0, \infty) \to \mathbb{R}, i \in \{a, d\}$,

- is strictly increasing, i. e. $u_i'(x) > 0$, strictly concave, i. e. $u_i''(x) < 0$, and twice continuously differentiable on $[0, \infty)$.
- satisfies the Inada-conditions, i. e.,

$$\lim_{x \to 0} u_i'(x) = \infty \text{ and } \lim_{x \to \infty} u_i'(x) = 0.$$

Market – Product space $(\Omega, \mathcal{F}, \mathbb{P})$

Definition 2: Market

We consider the market

$$(\Omega, \mathcal{F}, \mathbb{P}) = \underbrace{(\Omega^F, \mathcal{F}^F, \mathbb{P}^F)}_{\text{financial risk}} \times \underbrace{(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)}_{\text{mortality risk}}$$

where $\mathcal{F}^F = \sigma(S_T)$ with S_T denoting the value of risky asset at time T (see also Møller (2003) and Bernard & Vanduffel (2014)).

Assumption 2: Independence

Financial market risk is **stochastically independent** of mortality risk.

Assumption 3: Financial market $(\Omega^F, \mathcal{F}^F, \mathbb{P}^F)$

The financial market $(\Omega^F, \mathcal{F}^F, \mathbb{P}^F)$ is arbitrage-free, perfectly liquid, and complete, and there are neither transaction costs nor trading constraints.

Life-contingent payoffs

Definition 3 – Life-contingent payoff

We call $C_T \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ a **life-contingent payoff** if

$$C_T = f(S_T, I_T)$$

where $f : \Omega \to (0, \infty)$ is an $\mathcal{F}$-measurable function.

$\rightarrow$ But how to compute the price of C_T as $(\Omega, \mathcal{F}, \mathbb{P})$ is incomplete?

Pricing in the incomplete market $(\Omega, \mathcal{F}, \mathbb{P})$

Assumption 4: Pricing Rule

The **price** of any life-contingent payoff C_T is given by

$$c(C_T) = \mathbb{E}[\xi_T^F \cdot C_T]$$

where ξ_T^F is the unique state-price density on $(\Omega^F, \mathcal{F}^F, \mathbb{P}^F)$.

- Intuition: No risk-loading for mortality risk
- $C_T = I_T$
  $\rightarrow c(C_T) = \mathbb{E}[\xi_T^F I_T] = \mathrm{e}^{-rT}\mathbb{E}[I_T]$
- $C_T = f(S_T, I_T) = I_T \cdot f_1(S_T) + (1 - I_T) \cdot f_2(S_T)$
  $\rightarrow c(C_T) = p \cdot c(f_1(S_T)) + (1 - p) \cdot c(f_2(S_T))$,

using Assumption 2 (Independence).

Maximizing the State-dependent Utility Delivers...

Problem 1

Let $x_0 > 0$ denote the agent's initial budget. The agent solves

$$\sup_{C_T = I_T Y_T + (1-I_T) Z_T \in \mathcal{X}_I(x_0)} \mathbb{E}[u_I(C_T)]$$

where

$$\mathcal{X}_I(x_0) := \{C_T : \Omega \to (0, \infty)\, \mathcal{F}\text{-measurable}, \mathbb{E}[\xi_T^F C_T] = x_0\}$$

denotes the set of all life-contingent payoffs with cost x_0.

... Optimal Life-contingent Payoff C_T^*

Proposition 1: Optimal life-contingent payoff

Let Assumptions 1, 2, 3, and 4 hold. The agent solves

$$\sup_{C_T \in \mathcal{X}_l(x_0)} \mathbb{E}[u_l(C_T)]. \tag{1}$$

Then, an optimal life-contingent payoff is given by

$$\mathbf{C_T^*} := \mathbf{I_T} \cdot \mathcal{I}_{\mathbf{a}}(\lambda \xi_{\mathbf{T}}^{\mathbf{F}}) + (\mathbf{1} - \mathbf{I_T}) \cdot \mathcal{I}_{\mathbf{d}}(\lambda \xi_{\mathbf{T}}^{\mathbf{F}}), \tag{2}$$

where $\mathcal{I}_i(\cdot)$ is the inverse function of $u_i'(\cdot)$, for $i \in \{a, d\}$, and $\lambda > 0$ is chosen such that $C_T^* \in \mathcal{X}_l(x_0)$.

Proof Sketch

- By independence of financial and mortality risk, rewrite Problem (1) as

$$\sup\ \mathbb{E}[p\, u_a(Y_T) + (1-p)\, u_d(Z_T)]$$
$$\text{s.t. } p\,\mathbb{E}[\xi_T^F Y_T] + (1-p)\,\mathbb{E}[\xi_T^F Z_T] = x_0.$$

- Pathwise optimization: consider the Lagrangian

$$L(y,z) = pu_a(y) + (1-p)u_d(z) + \lambda(x_0 - p\xi^F y - (1-p)\xi^F z).$$

- FOC $\Rightarrow$ optimal candidate payoffs

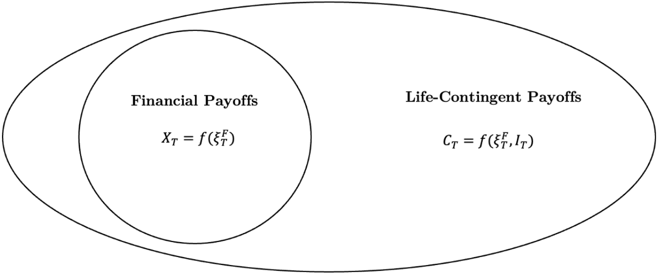
$$Y_T^* = I_a(\lambda\xi_T^F), \quad Z_T^* = I_d(\lambda\xi_T^F),$$

  hence

$$C_T^* = Y_T^* I_T + Z_T^*(1 - I_T).$$

- Existence and uniqueness of $\lambda > 0$ follow from Inada conditions and monotonicity: ensures budget constraint is satisfied.
- By concavity of utilities and FOC optimality, C_T^* dominates any other feasible payoff.

In theory, C_T^* delivers higher state-dependent utility...



$$\mathbb{E}[u_I(C_T^*)] \geq \mathbb{E}[u_I(X_T)],$$

The agent prefers the optimal life-contingent payoff C_T^* over any payoff $X_T = f(S_T)$ that is only contingent on financial risk and costs x_0.

... but, in practice, you may not be able to buy C_T^*

Proposition 2: Lower Bound for Ask Price

A risk-averse insurer's ask price p_a for the optimal life-contingent payoff in (2) satisfies

$$p_a \geq x_0.$$

(cf. Bernard & Vanduffel (2014))

$\Rightarrow$ If there exists no risk-neutral product provider that charges actuarially fair premiums, the life-contingent payoff C_T^* can no longer be purchased at the price of x_0.

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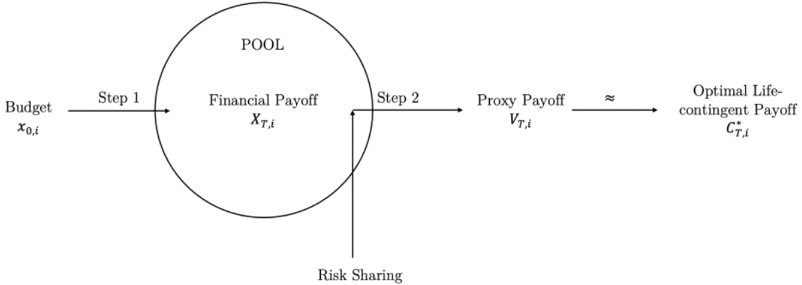


Basic idea to approximate the optimal life-contingent payoff $C^*_{T,i}$ for any member $i \in \{1, \ldots, n\}$ by leveraging the pool.

Step 1: Pool buys Financial Payoff

At $t = 0$, the pool **buys** at cost x_0 for every participant the **financial payoff**

$$X_{T,i} := \mathbb{E}[C_T^* | \xi_T^F] = \underbrace{\mathbf{p} \cdot \mathcal{I}_a(\lambda \xi_T^F)}_{=: X_{T,a,i}} + \underbrace{(\mathbf{1} - \mathbf{p}) \cdot \mathcal{I}_d(\lambda \xi_T^F)}_{:= X_{T,d,i}}, \tag{3}$$

where λ is chosen such that $c(X_{T,i}) = x_0$. We refer to $X_{T,a,i}$ and $X_{T,d,i}$ as the **survival financial payoff** and **death financial payoff**, respectively.

Note: $X_{T,i}$ is only contingent on **financial risk**.

Pool Setup – Homogeneous Mortality & Investment

Assumption 5: Homogeneous Mortality & Investment

The pool consists of n participants. Every participant

- survives with probability $p \in (0, 1)$ until time T, and
- has state-dependent utility u_I and budget $x_0 > 0$.

Further, we assume that the survival statuses $(I_i)_{i=1,\dots,n} \sim \mathrm{Ber}(p)$ of all participants are stochastically independent.

Risk-Sharing Rule $\mathcal{R}$ in a Homogeneous Pool

Risk Sharing Rule in Homogeneous Pool

Let the pool be homogeneous in the sense of Assumption 5.
Then we choose

$$\rho_{a,i} = \frac{1}{\sum_{j=1} I_j} = \frac{1}{\#\text{survivors}} \text{ and } \rho_{d,i} = \frac{1}{\sum_{j=1}(1 - I_j)} = \frac{1}{\#\text{decedents}}.$$

$$V_{T,i} := \begin{cases} X_{T,a,i} + \rho_{a,i} \cdot \sum_{j=1}^{n} (1 - I_j) \cdot X_{T,a,j}, & \text{if } I_i = 1, \\ X_{T,d,i} + \rho_{d,i} \cdot \sum_{j=1}^{n} I_j \cdot X_{T,d,j}, & \text{if } I_i = 0. \end{cases}$$

Goal

$$V_{T,i} \approx C^*_{T,i}$$

Pool Delivers C_T^* If Pool Is Sufficiently Large

Propositon 3: Convergence against $C_{T,i}^*$

Let Assumptions 1, 2, 3, and 5 hold with

$$\rho_{a,i} = \frac{1}{\sum_{j=1} I_j} \text{ and } \rho_{d,i} = \frac{1}{\sum_{j=1}(1 - I_j)}.$$

Consequently, we have for every participant $i = 1, \ldots, n$ that

$$V_{T,i} \overset{n\to\infty}{\to} C_{T,i}^* \quad \mathbb{P}\text{-a.s.},$$

where $C_{T,i}^*$ is the optimal life-contingent payoff of participant i defined in (2).

Sketch of the Proof

Note that we can write

$$V_{T,i} := \begin{cases} X_{T,a,i} + \rho_{a,i} \cdot \sum_{j=1}^{n} (1 - I_j) \cdot X_{T,a,j}, & \text{if } I_i = 1, \\ X_{T,d,i} + \rho_{d,i} \cdot \sum_{j=1}^{n} I_j \cdot X_{T,d,j}, & \text{if } I_i = 0. \end{cases}$$

$$= I_i \cdot \left(X_{T,a,i} + \rho_{a,i} \cdot \sum_{j=1}^{n} (1 - I_j) \cdot X_{T,a,j} \right) + (1 - I_i) \cdot \left(X_{T,d,i} + \rho_{d,i} \cdot \sum_{j=1}^{n} I_j \cdot X_{T,d,j} \right)$$

$$= I_i \cdot \left(p \cdot \mathcal{I}_a(\lambda \xi_T^F) + p \cdot \mathcal{I}_a(\lambda \xi_T^F) \cdot \frac{\frac{1}{n} \sum_{j=1}^{n} (1 - I_j)}{\frac{1}{n} \sum_{j=1}^{n} I_j} \right)$$

$$+ (1 - I_i) \cdot \left((1 - p) \cdot \mathcal{I}_d(\lambda \xi_T^F) + (1 - p) \cdot \mathcal{I}_d(\lambda \xi_T^F) \cdot \frac{\frac{1}{n} \sum_{j=1}^{n} I_j}{\frac{1}{n} \sum_{j=1}^{n} (1 - I_j)} \right)$$

$$\overset{n \to \infty}{\to} I_i \cdot \left(p \cdot \mathcal{I}_a(\lambda \xi_T^F) + p \cdot \mathcal{I}_a(\lambda \xi_T^F) \cdot \frac{(1 - p)}{p} \right)$$

$$+ (1 - I_i) \cdot \left((1 - p) \cdot \mathcal{I}_d(\lambda \xi_T^F) + (1 - p) \cdot \mathcal{I}_d(\lambda \xi_T^F) \cdot \frac{p}{(1 - p)} \right) = C_{T,i}^*, \ \mathbb{P}\text{-a.s.}$$

Consequence: Ensure sufficient poolsize

$B_T = B_0 \exp(rt)$, $B_0 > 0$, and $S_T = S_0 \exp((\mu - 1/2\sigma^2)T + \sigma W_T)$, $S_0 > 0$ with $\mu = 0.05, r = 0.02, \sigma = 0.2$ and $W_T \sim N(0, T)$ under $\mathbb{P}^F$. Investor with $\gamma_a = 2.2, \gamma_d = 2$ and $p = 0.92$.

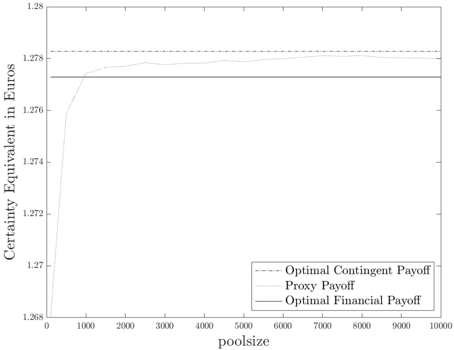


Figure: Certainty equivalent of investor for different payoffs

Further pools

Additional pools considered in the paper:

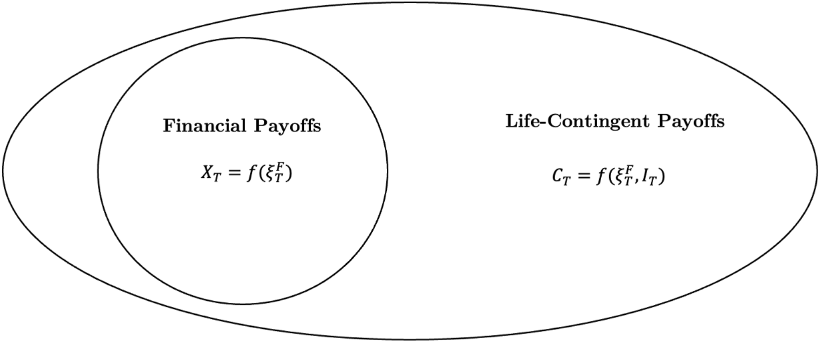
- a heterogeneous pool where members may have different survival probabilities but the same wealth and state-dependent preferences (Risk-sharing rule)

$$\rho_{a,i} = \frac{\frac{q_i}{p_i}}{\sum_{j=1}^{n} \frac{q_j}{p_j} \cdot I_j} \text{ and } \rho_{d,i} = \frac{\frac{p_i}{q_i}}{\sum_{j=1}^{n} \frac{p_j}{q_j} \cdot (1 - I_j)}, \quad \forall i \in \{1, \ldots, n\}.$$

- a heterogeneous pool where members have different survival probabilities, wealth, and state-dependent preferences.

Note: Similar results for pool with heterogeneous mortality & heterogeneous investment.

Stability – Which members are at risk to leave?



You may have an incentive to leave if $\mathbb{E}[u_I(C_T^*)] - \mathbb{E}[u_I(X_T^*)] \approx 0$

Stability – Which members are at risk to leave?

$$u_i(x) = x^{1-\gamma_i}/(1-\gamma_i), i \in \{a, d\}, \gamma_d = 2, x_0 = 1, T = 10 \text{ and}$$
$$\text{BS-market with } \mu = 0.05, r = 0.02, \sigma = 0.2$$

γ_a \ p	0.00	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95	1.00
2.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3.00	0.00	0.07	0.14	0.20	0.26	0.31	0.35	0.39	0.42	0.42	0.44	0.44	0.44	0.42	0.39	0.36	0.31	0.26	0.18	0.11	0.00
4.00	0.00	0.18	0.36	0.52	0.67	0.81	0.94	1.06	1.15	1.21	1.27	1.30	1.32	1.30	1.25	1.16	1.04	0.88	0.64	0.36	0.00
5.00	0.00	0.29	0.57	0.84	1.09	1.33	1.56	1.78	1.97	2.11	2.25	2.36	2.43	2.45	2.41	2.30	2.10	1.82	1.36	0.79	0.00
6.00	0.00	0.38	0.76	1.12	1.47	1.81	2.15	2.48	2.78	3.04	3.30	3.52	3.70	3.81	3.84	3.75	3.52	3.12	2.41	1.42	0.00
7.00	0.00	0.46	0.92	1.37	1.81	2.25	2.69	3.13	3.57	3.96	4.37	4.74	5.07	5.34	5.52	5.55	5.36	4.87	3.88	2.34	0.00
8.00	0.00	0.54	1.06	1.58	2.10	2.63	3.17	3.73	4.30	4.84	5.42	5.98	6.51	7.01	7.41	7.67	7.65	7.18	5.91	3.63	0.00
9.00	0.00	0.60	1.19	1.77	2.36	2.97	3.61	4.28	4.97	5.66	6.42	7.19	7.95	8.73	9.46	10.09	10.41	10.16	8.74	5.51	0.00
10.00	0.00	0.66	1.29	1.93	2.58	3.26	3.99	4.76	5.58	6.42	7.35	8.33	9.35	10.45	11.58	12.69	13.58	13.86	12.63	8.31	0.00

Table: Gain in certainty equivalent in percent between $C^*_{T,i}$ and $X^*_{T,i}$

Key-Take-Away: Members with similar risk-aversions may have an incentive to leave!

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Thank you very much for your attention!

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