# A New Paradigm of Mortality Modeling via Individual Vitality Dynamics


Zijia Wang

(with Xiaobai Zhu (CUHK) and Kenneth Q. Zhou (UWaterloo))

The Chinese University of Hong Kong

*zijiawang@cuhk.edu.hk*


WRIEC 2025

August 6, 2025

# Table of Contents

# Table of Contents

# Background and Motivation

Mortality modeling serves as a cornerstone for numerous research domains and practical applications,

- for life insurers: insurance products pricing, determine reserves, fulfill regulatory requirements;
- pension fund management, public health planning, optimal consumption and investment strategies.

# Background and Motivation – Existing mortality models

- Mortality laws: e.g., Gompertz-Makeham law,

$$\mu_x(t) = A + Bc^{x+t}.$$

- Intensity-based models: mortality intensities described by stochastic processes,

$$d\mu_x(t) = a\mu_x(t)dt + \sigma dB(t).$$

- Factor-based models: combine age-specific components and time-varying factors,

$$\ln m_{x,t} = \alpha_x + \beta_x \kappa_t.$$

- Spline-based models: fit smooth spline curves,

$$\ln m_{x,t} = \sum_{i=1}^{c_a} \alpha_i a_i(x) + \sum_{i=1}^{c_a} \sum_{j=1}^{c_y} \beta_{i,j} b_i^{(a)}(x) b_j^{(y)}(t).$$

# Background and Motivation

Traditional mortality models fall short in capturing the individual mortality dynamics:

- the heterogeneity in health conditions
- occurrence of accidental events

$\Rightarrow$ This highlights the need for a more individualized approach to better account for the variability and complexity of individual mortality.

# Background and Motivation (cont.)

A new approach of modeling mortality: the **Survival Energy Model (SEM)**, see, e.g., Shimizu et al. (2021) and Shimizu et al. (2023).

- each individual is assumed to have a survival energy level that depletes stochastically over time.
- an individual's death time: the first passage time that the survival energy process crosses zero.

$\Rightarrow$ closely connected to the first passage analysis.

# Background and Motivation (cont.)

Individual survival energy is also known as **vitality**, which refers to "the capacity of an individual organism to stay alive." (Strehler and Mildvan (1960)). Existing studies on vitality modeling includes:

- explain **individual death-time variation**, see, e.g., Li and Anderson (2009); Missov (2013); Cha and Finkelstein (2016).
- distinguish **intrinsic and extrinsic** contributions to longevity, see, e.g., Finkelstein (2012); Sharrow and Anderson (2016).
- link classic **hazard rate models** with first passage problems, see, e.g., Cha and Finkelstein (2016); Cheng et al. (2023).

In this paper, we introduce a mortality modeling framework centered on individual vitality dynamics.

- **Mortality modeling**: use a four-component vitality model to capture various aspects of mortality dynamics.
- **Establish links** with existing mortality models: complement and improve existing models in describing complex mortality dynamics at the individual level.
- **Applications**: incorporate individual vitality into calculating life expectancies, pricing insurance products, and deriving optimal consumption decisions, among others.

# Table of Contents

# Vitality Model

$V(t)$: the vitality of an individual who is age $x$ at time 0.

- The individual is considered deceased when the vitality level is exhausted, i.e., $V(t) \leq 0$.
- the death time is then

$$\tau = \inf \{t \geq 0 : V(t) \leq 0\} .$$

- the $T$-year survival probability for the individual is

$$\Pr(\tau > T).$$

# Vitality Model (cont.)

We assume that $V(t)$ is described by four specific components:

$$V(t) = V(0) - Y(t) - W(t) - J(t),$$

where

1. $V(0)$ is the initial level of vitality at time 0,
2. $Y(t)$ is the trend at which the vitality level is depleted,
3. $W(t)$ is the diffusion that captures the variation of vitality,
4. $J(t)$ is the jump process that may affect the vitality level.

# Vitality Model – $V(0)$

Vitality process

$$V(t) = V(0) - Y(t) - W(t) - J(t).$$

- $V(0)$: the initial vitality level of the individual who is age $x$ at time 0.
    - If $x = 0$, then $V(0)$ represents the vitality level of a newborn.
- Fixed $V(0)$: if the initial vitality level is known and is the same for all of the individuals in a cohort.
- Random $V(0)$: inherent variability in the initial vitality level of individuals.

# Vitality Model – $Y(t)$

Vitality process

$$V(t) = V(0) - Y(t) - W(t) - J(t).$$

- $Y(t)$: describes the decline in the individual's vitality due to the natural aging process.
- We assume that
$$\mathrm{d}Y(t) = \mu_x(t)\,\mathrm{d}t, \quad t \geq 0,$$
where $\mu_x(t)$ is the vitality depletion rate at time $t$ when the individual is aged $x + t$. $\mu_x(t)$ could be a constant, a deterministic/stochastic function.

# Vitality Model – $W(t)$

Vitality process

$$V(t) = V(0) - Y(t) - W(t) - J(t).$$

- $W(t)$: a diffusion component to reflect the fact that the vitality level $V(t)$ can fluctuate around its depletion trend $Y(t)$.
- We assume that

$$\mathrm{d}W(t) = \sigma(t)\mathrm{d}B(t), \quad t \geq 0,$$

  where $B(t)$ is a standard Brownian motion and $\sigma(t)$ is a diffusion parameter at time $t$.

# Vitality Model – $J(t)$

Vitality process

$$V(t) = V(0) - Y(t) - W(t) - J(t).$$

- $J(t)$: describe sudden and significant changes to $V(t)$ due to events such as an automobile incident or a medical surgery.
- We assume that

$$J(t) = \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0,$$

  - $N(t)$: the number of accidents that have occurred up to time $t$;
  - $Z_i$: the size of the vitality jump caused by the $i$-th accident.

# Connection With Mortality Laws

Model setup:

1. The initial vitality $V(0) \sim \text{EXP}(1)$,
2. The trend component follows
$$dY(t) = \mu_x(t)dt,$$
where $\mu_x(t) \geq 0$ (e.g., a mortality law $\mu_x(t) = bc^{x+t}$),
3. $W(t) = J(t) = 0$ for all $t$.

Then, the $T$-year survival probability is

$$\mathbb{P}(\tau > T) = \exp\left(-\int_0^T \mu_x(t)dt\right),$$

which is the same as the survival probability implied by the mortality law $\mu_x(t)$.

# A Jump-Diffusion Vitality Model

Suppose the dynamics of the vitality level is

$$V(t) = V(0) - \int_0^t \mu_x(s)\mathrm{d}s - \sigma B(t) - \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0,$$

where $V(0) \sim F_0$, $N(t)$ is a Poisson process with intensity rate $\lambda(t)$, and $\{Z_i\}_{i\geq 1}$ is a sequence of iid positive random variables. The resulting $T$-year survival probability can be expressed as

$$\Pr(\tau > T) = \int_0^\infty \Pr\left(v - \sigma B(t) - \sum_{i=1}^{N(t)} Z_i > \int_0^t \mu_x(s)\mathrm{d}s,\ \forall t \leq T\right) \mathrm{d}F_0(v).$$

# A Jump-Diffusion Vitality Model (cont.)

If we further assume that $\mu_x(s) \equiv \delta$ and $Z_i = \infty$ for all $i \geq 1$, we have

**Proposition 1.**

In the case where $V(t) = V(0) - \delta t - \sigma B(t) - \sum_{i=1}^{N(t)} Z_i$ and $Z_i = \infty$ for all $i \geq 1$, the survival probability can be analytically solved as

$$\Pr(\tau > T) = e^{-\int_0^T \lambda(t)\mathrm{d}t} \cdot \Pr\left(V(0) - \delta t - \sigma B(t) > 0 \text{ for all } t \in [0, T]\right)$$

$$= e^{-\int_0^T \lambda(t)\mathrm{d}t} \cdot \int_0^\infty \Phi\left(\frac{v - \delta T}{\sigma\sqrt{T}}\right) - \exp\left(\frac{2\delta v}{\sigma^2}\right)\Phi\left(\frac{-v - \delta T}{\sigma\sqrt{T}}\right)\mathrm{d}F_0(v),$$

where $\Phi(\cdot)$ is the distribution function of a standard Gaussian distribution.

# Table of Contents

Suppose the vitality level of an individual has the following dynamics,

$$\mathrm{d}V(t) = -\delta \mathrm{d}t + \sigma_V \mathrm{d}B_V(t), \quad V(0) = V_0 > 0,$$

and the asset $A(t)$ of the individual is

$$\mathrm{d}A(t) = A(t)\left[r + \pi(t)\theta\sigma_S\right]\mathrm{d}t + \sigma_S \pi(t) A(t) \mathrm{d}B_S(t) - \zeta(t)\mathrm{d}t, \quad A(0) = A_0,$$

where $\pi(t)$ be the portfolio weight invested in the risky asset, $\zeta(t)$ is the consumption amount, $(B_S(t), B_V(t))$ is a two-dimensional standard Brownian Motion.

# Optimal Investment and Consumption Strategies (cont.)

Suppose the individual aims to maximize their lifetime utility by choosing the optimal investment and consumption strategies, that is,

$$\mathcal{H}(A(t) = a, V(t) = v) = \max_{\pi, \zeta \in \Pi} \mathbb{E}_{t,a,v} \left[ \int_t^\tau e^{-\beta(s-t)} \cdot u(\zeta(s)) \mathrm{d}s + \lambda \cdot u(A(\tau)) \right],$$

where $\tau$ is the death time, $\beta > 0$ is the discount rate, $u(\zeta) = \ln \zeta$ is the utility function, $\lambda \geq 0$ represents the bequest motive.

# Optimal Investment and Consumption Strategies (cont.)

**Theorem 1.**

The optimal investment and consumption strategies of the above problem are

$$\pi^*(t) = \frac{\theta}{\sigma_S}, \quad \zeta^*(t) = \frac{A(t)}{f(V(t))}.$$

where

$$f(v) = \left(\lambda - \frac{1}{\beta}\right) e^{k_1 v} + \frac{1}{\beta}, \quad \text{with} \quad k_1 = \begin{cases} \frac{\delta - \sqrt{\delta^2 + 2\sigma_V^2 \beta}}{\sigma_V^2} < 0, & \sigma_V > 0 \\ \frac{-\beta}{\delta}, & \sigma_V = 0 \end{cases}$$

# Disability Modeling

As a natural extension, we can incorporate disability risk into the vitality model: suppose there exists a threshold $\psi > 0$ such that when the vitality falls below $\psi$, the person becomes disabled.

A disability model with recovery:

- the initial vitality level $V(0) \sim F_0$;
- a constant vitality depletion rate $\mu(t) = \delta$;
- a diffusion component as a Brownian motion with volatility $\sigma$.

## Proposition 2.

For the above model, the $T$-year disabled probability for a healthy person aged $x$ is

$$\int_{\psi}^{\infty} \frac{1}{1-F_0(\psi)} \cdot \left( \int_0^{\frac{v}{\sigma}} \int_{\frac{v-\psi}{\sigma}}^{\frac{v}{\sigma}} f(m, w; T) \mathrm{d}w \mathrm{d}m \right) \mathrm{d}F_0(v),$$

and the recovery probability for a disabled person aged $x$ is

$$\int_{0}^{\psi} \frac{1}{F_0(\psi)} \cdot \left( \int_0^{\frac{v}{\sigma}} \int_{0}^{\frac{v-\psi}{\sigma}} f(m, w; T) \mathrm{d}w \mathrm{d}m \right) \mathrm{d}F_0(v),$$

where

$$f(m, w; T) = \begin{cases} \frac{2(2m-w)}{T\sqrt{2\pi T}} e^{\frac{\delta w}{\sigma} - \frac{\delta^2 T}{2\sigma^2} - \frac{(2m-w)^2}{2T}}, & w \leq m, m \geq 0 \\ 0, & \text{elsewhere.} \end{cases}$$

# Cause-of-Death Modeling

In the proposed vitality-based modeling framework, death can be triggered by

- the natural depletion of vitality: modeled by the depletion trend $Y(t)$ and fluctuation $W(t)$;
- a sudden jump in vitality level due to a fatal accident: modeled by the jump term $J(t)$.

$\Rightarrow$ It is of interest to distinguish the cause of death due to different components.

# Cause-of-Death Modeling (cont.)

Consider the vitality process

$$V(t) = V(0) - \frac{bc^x}{\ln c}\left(c^t - 1\right) - \sum_{i=1}^{N(t)} Z_i, \quad t \geq 0.$$

- Denote $\tau_J = \inf\{t \geq 0 : V(t) < 0\}$ and $\tau_Y = \inf\{t \geq 0 : V(t) = 0\}$.
- $\tau_J \cdot 1(\tau_J < \tau_Y)$: the death time at which death is caused by an accident.
- $\tau_Y \cdot 1(\tau_Y < \tau_J)$: the death time at which death is caused by natural decay.

$\Rightarrow$ the death time $\tau = \tau_Y \wedge \tau_J$.

**Theorem 2.**

For $q \geq 0$, the Laplace transforms of the time of death due to an accident and the time of death due to natural decay in vitality level are given by the following expressions:

$$\mathbb{E}_v\left[\mathrm{e}^{-q\tau_J} 1\left(\tau_J < \tau_Y\right)\right] = \int_0^{t_v^*} \lambda \mathrm{e}^{-(q+\lambda)t - \alpha b_v^*(t)} \cdot I_0\left(2\sqrt{\alpha\lambda t b_v^*(t)}\right) \mathrm{d}t,$$

and

$$\begin{aligned} &\mathbb{E}_v\left[\mathrm{e}^{-q\tau_Y} 1\left(\tau_Y < \tau_J\right)\right] \\ &= \int_0^{t_v^*} \mathrm{e}^{-(q+\lambda)t - \alpha b_v^*(t)} \cdot bc^{x+t} \sqrt{\frac{\alpha\lambda t}{b_v^*(t)}} I_1\left(2\sqrt{\alpha\lambda t b_v^*(t)}\right) \mathrm{d}t + \mathrm{e}^{-(q+\lambda)t_v^*}, \end{aligned}$$

where $t_v^* = \frac{\ln(bc^x + v\ln c) - \ln b}{\ln c} - x$ (which refers to the remaining lifetime without any accidents/jumps), $b_v^*(t) = v - bc^x\left(c^t - 1\right)/\ln c$, and $I_k(x)$ is the modified Bessel function of the first kind.

# Cause-of-Death Modeling (cont.)

## Theorem 2. Cont.

Moreover, by the uniqueness of the Laplace transform, it follows that

$$\mathrm{Pr}_v\left(\tau_J \in \mathrm{d}t, \tau_J < \tau_Y\right) = 1(t < t^*)\lambda \mathrm{e}^{-\alpha b_v^*(t) - \lambda t} I_0\left(2\sqrt{\lambda\alpha t b_v^*(t)}\right)\mathrm{d}t,$$

and

$$\begin{aligned}&\mathrm{Pr}_v\left(\tau_Y \in \mathrm{d}t, \tau_Y < \tau_J\right)\\ =&1(t < t^*)\mathrm{e}^{-\lambda t - \alpha b_v^*(t)} \cdot bc^{x+t}\sqrt{\frac{\alpha\lambda t}{b_v^*(t)}} I_1\left(2\sqrt{\alpha\lambda t b_v^*(t)}\right)\mathrm{d}t + \mathrm{e}^{-\lambda t}\delta_{t^*}(\mathrm{d}t),\end{aligned}$$

where $\delta_{t_v^*}(\cdot)$ is the Dirac mass at $t_v^*$.

# Thank You!

# Reference

Cha, J. H. and Finkelstein, M. (2016). Justifying the Gompertz curve of mortality via the generalized Polya process of shocks. *Theoretical Population Biology*, 109:54–62.

Cheng, C., Hilpert, C., Szimayer, A., and Zweifel, P. (2023). Reference health and investment decisions. *Available at SSRN 4523775*.

Finkelstein, M. (2012). Discussing the Strehler-Mildvan model of mortality. *Demographic Research*, 26:191–206.

Li, T. and Anderson, J. J. (2009). The vitality model: A way to understand population survival and demographic heterogeneity. *Theoretical Population Biology*, 76(2):118–131.

Missov, T. I. (2013). Gamma-Gompertz life expectancy at birth. *Demographic Research*, 28:259–270.

# Reference (cont.)

Sharrow, D. J. and Anderson, J. J. (2016). Quantifying intrinsic and extrinsic contributions to human longevity: Application of a two-process vitality model to the human mortality database. *Demography*, 53(6):2105–2119.

Shimizu, Y., Minami, Y., and Ito, R. (2021). Why does a human die? A structural approach to cohort-wise mortality prediction under survival energy hypothesis. *ASTIN Bulletin: The Journal of the IAA*, 51(1):191–219.

Shimizu, Y., Shirai, K., Kojima, Y., Mitsuda, D., and Inoue, M. (2023). Survival energy models for mortality prediction and future prospects. *ASTIN Bulletin: The Journal of the IAA*, 53(2):377–391.

Strehler, B. L. and Mildvan, A. S. (1960). General theory of mortality and aging: A stochastic model relates observations on aging, physiologic decline, mortality, and radiation. *Science*, 132(3418):14–21.