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The Fractional Merton Model: A New Approach to Credit Risk Pricing (Revised)

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The Fractional Merton Model: A New Approach to Credit Risk Pricing

Abstract: Merton model is known to underestimate credit spreads. In this paper we develop the theoretical framework of the fractional Merton model, which allows to embed long memory properties of spreads in a straightforward manner in a credit risk pricing model. We carry out an extensive sensitivity analysis exercise and compute the spread sensitivities to the long memory parameter, firm leverage, firm volatility and variance, and risky debt time to maturity. We also compute sensitivities of the equity, risky debt, risk-neutral default probability and option to default to long memory. We show that theoretical spreads of the fractional Merton model are larger than the spreads predicted by the Merton model and hence closer to market spreads.

Keywords: C14, C22, G13.

J.E.L. Classification Numbers: Credit Risk, Structural Models, Credit Spreads, Fractional Integration.

I Introduction

Credit spread of corporate bonds has been shown to depend on a number of factors such as the issuer probability of default, the loss given default, the tax regime of corporate bonds relative to government bonds, and the systematic risk of corporate bonds. See for instance Elton, Gruber, Agrawal and Mann (2001).

Models of credit spread combine these factors in different ways. From Merton (1974) original idea, structural models use an option-pricing approach to define a stochastic process for the firm value and treat risky debt as a combination of a riskless bond and a short put option on the firm's assets. The put option is an "option to default", giving shareholders the opportunity to sell to bondholders the firm value at the debt nominal value. The value of the put takes into account systematic risk, probability of loss and recovery rate. On the other hand, reduced-form models (Jarrow and Turnbull, 1995, and Duffie and Singleton, 1999) assume that exogenous hazard rates (often based on transition probabilities of ratings) and the loss given default drive the spread.

Despite reduced-form models are sufficiently flexible to accommodate relevant market information in the default rate process (e.g., credit rating, firm specific or macroeconomic variables) and to be calibrated to market data, they have a key disadvantage over structural models in that the default rate process is not linked to the value of the firm's assets. This is a strong limitation in order to understand the determinants and dynamics of credit risk. In addition, models based on a direct assumption of the credit spread dynamics usually lack a rigorous treatment of the derivation of a risk-neutral probability from the fundamental assumptions about the underlying process.

In this paper we focus on structural models for pricing credit risk. Structural models, although intuitively appealing, are also prone to a number of limits which have made them difficult to implement empirically. First, as pointed out for instance by Jarrow, Lando and Turnbull (1997), firm's assets are not tradable or easily observable. This is inconsistent with the assumptions that the firm value follows a diffusion process and that firm's assets can be traded in continuous time. In addition, it is difficult to estimate the volatility of the firm's assets, which is a key piece of information for risky debt valuation. Second, the capital structure of the firm, which may generate complex priority rules of the payoffs to the firm's liabilities, needs to be fully specified and included in the valuation procedure. This is a particularly difficult task and may even become intractable when applied to coupon-bearing bonds, callable bonds or complex capital structures. Finally, since this approach does not use credit rating information, it cannot be employed to price credit derivatives whose payoffs depend on credit ratings.

Empirical studies on credit spreads are relatively scarce. Studies carried out to date show that structural models seem to generate credit spreads that are not consistent with the spreads observed in practice. Jones, Mason and Rosenfeld (1984), Ogden (1987) and Eom, Helwege and Huang (2004) all find that spreads predicted by Merton (1974) are smaller than market spreads. In addition, Eom, Helwege and Huang (2004) show that the alternative structural model proposed by Geske (1977) also predicts smaller spreads than observed, whilst the models of Longstaff and Schwartz (1995) and Leland and Toft (1996) generate larger spreads but also larger errors. Anderson and Sundaresan (2000) test to what extent structural credit risk models can explain credit spread and find that these models capture most of the time series variation of corporate yields. The performance of the models is significantly improved if one allows for endogenous default barriers. Some authors show that taxation and transaction costs may explain the difference between market and model spread. Elton, Gruber, Agrawal and Mann (2001) note that the taxation of corporate bonds in the US is higher than government bonds and this should increase the spread. Similarly, Ericsson and Renault (2006) find that there is a liquidity premium in the observed credit spread which increases the size of the spreads. This finding is in line with the result in De Jong and Driessen (2007) who estimate credit spreads in two different ways, using the Leland and Toft (1996) model and Moody's historical default and recovery rates.

To summarize, the main explanation of the discrepancy between model and observed spreads is that the firm value is modelled as a diffusion process and hence the time of default should always be predictable since, under a diffusion process, a sudden drop in the firm value is impossible. As a result, the firm's probability of default on very short-term debt should be zero, which implies that very short-term debt is risk-free and should have zero credit spread. This implication is clearly rejected by the empirical evidence and contributes to explain why structural models predict particularly small spreads for bonds that are near to maturity. However, a recent paper by Gemmill (2002) suggested that structural models may predict credit spreads correctly. Using for the first time a dataset of zero-coupon bonds issued by firms with simple capital structures, the author finds that spreads predicted by

Merton (1974) are consistent with market spreads.

This paper is motivated by the findings reported in two recent papers, where long memory properties of credit spreads are investigated. Della Ratta and Urga (2007) use 30-year Historical Treasury Constant Maturity Yields and Moody's Aaa, Aa, A and Baa Long-Term Corporate Bond Yield Averages, covering the period from December 1992 to November 2003, for 2703 observations. Authors found clear evidence that spreads are long memory nonstationary processes, with long memory parameter d not statistically different from unity in most cases. This conclusion is supported by the results of standard unit root (DF, ADF and PP) and stationarity (KPSS, R/S and modified R/S) tests, as well as results of parametric and semi-parametric estimation techniques for long-range dependence (GPH, Robinson, ARFIMA). The results are robust to heteroskedasticity, choice of sample frequency and choice of sample period. Yields and spreads are found to be long memory nonstationary processes also in Leccadito and Urga (2007), who uses Lehman Brothers Eurodollar Aaa, Aa, A and Baa Indices and U.S. Global Treasury Index, covering the period from June 1996 to July 2006, for 2613 observations.

These findings have profound implications for modelling credit spreads in a way which accommodates their long memory characteristics. In the structural form approach, this can be done by modelling the evolution of the firm value via a Geometric fractional Brownian motion:

$$dV = \mu V dt + \sigma V dB_H \tag{1}$$

where μ is the instantaneous expected rate of return on the firm, σ is the instantaneous standard deviation of return on the firm and B_H is a fractional Brownian motion with Hurst exponent H.

Merton's model assumes that the firm value follows a Geometric Brownian motion. This implies that the firm value is a lognormally distributed random variable and log returns of the firm value are normally distributed. Log returns are also assumed independent over time. The independence property is only satisfied in (1) when $H = 1/2$, in which case returns do not have long memory properties. By using a fractional Brownian motion instead of a Brownian motion, a dependence structure in returns can be accounted for.

The main aim of this paper is to propose a structural form approach for risky debt pricing which takes into account the long memory properties of credit spread, as the empirical findings suggest. Our work aims at exploring an alternative formulation of structural models to better explain the empirical credit spread. Specifically, we embed fractional integration in the Merton model by generalising the Brownian motion in the stochastic differential equation for the firm value to a fractional Brownian motion. This is what we call *fractional Merton model*. In addition, we perform a comprehensive sensitivity analysis to assess whether the fractional Merton model is able to predict credit spreads which are closer to market spreads than those predicted by standard Merton's model. We evaluate spread sensitivity to long memory, firm leverage, firm volatility, firm variance, to risk debt time to maturity. We also evaluate the sensitivity of equity, risk debt, risk-neutral default probability and the option to default to long memory.

The remainder of the paper is organised as follows. In Section II we derive a fractional version of the Merton's model, while a detailed sensitivity analysis of the model is reported in Section III. The empirical relevance of our model is also evaluated by calculating implied values for the Hurst parameter, as reported in Section IV. Section V concludes.

II The Fractional Merton Model

The option theory based model of Merton (1973) is a seminal paper in risky debt valuation. Merton's intuition was that the equity, E, in a firm is equivalent to a call option on the firm's assets, V. Black and Scholes (1973) results were then used to price this call option. Since the value of a firm's assets is equal to the value of its debt, f, plus the value of its equity, E, $(V \equiv E + f)$, in the simplest case of a firm with a single issue of zero-coupon debt outstanding, the debt value is equal to the difference between the firm's assets value and the value of a call option on the firm's assets, with strike price equal to the debt nominal value and time to maturity equal to the debt maturity. Equivalently, from the put-call parity, the debt is the combination of a riskless bond and a short put option on firm's assets. Bondholders buy a safe bond and give shareholders the option to sell them the firm's assets at the debt value. Shareholders will exercise this option upon default.

Over the last few years the academic literature has extended the Black-Scholes model to account for dependency in returns. This literature has proposed to resolve the issue of dependency of returns assuming that the source of risk is represented by a fractional Brownian motion, identified by the value assumed by the Hurst parameter

H. For the fractional case, H may assume any value in $(0, 1)$ for stationary processes and $H > 1$ for non-stationary long memory processes.

It is worth noticing that by replacing the Brownian motion with the fractional Brownian motion B_H, which captures the long range dependency property measured by H we obtain an alternative option pricing model, called the fractional Black-Scholes model. Recent papers include Sottinen and Valkeila (2001, 2003), Biagini, Øksendal and Sulem (2002), Benth (2003), Hu and Øksendal (2003), Elliot and van der Hoek (2003) and Biagini and Øksendal (2004). Björk and Hult (2005) raise serious issues concerning the definition of a self-financing portfolio as in Elliot and van der Hoek (2003), and the definition of the value process as in Hu and Øksendal (2003). They provide simple examples showing that these definitions conflict with the usual economically intuitive definitions because of the Wick products. The no-arbitrage fractional Black-Scholes model therefore appears to be obtained by redefining standard accounting notions of the budget constraint and value. The Appendix reports a fractional Black-Scholes model.

Consider a firm with one outstanding issue of zero coupon bonds with total face value D and time to maturity T. Following Hu and Øksendal (2003), in the fractional Merton model the value E of the firm's equity is given by:

$$E = VN\left(d_1\left(H\right)\right) - De^{-rT}N\left(d_2\left(H\right)\right) \tag{2}$$

with

$$d_1\left(H\right) = \frac{\ln\frac{V}{D} + rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{3}$$

$$d_2(H) = \frac{\ln \frac{V}{D} + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} = d_1(H) - \sigma T^H \tag{4}$$

For $H = 1/2$ equations (2)-(4) give the Merton's solution to the firm's equity. This is equivalent to the Black-Scholes price of a call option on a non-dividend paying stock. From (2) and the equivalence $V \equiv E + f$, we can derive the value of the firm's debt as:

$$f = V[1 - N(d_1(H))] + De^{-rT} N(d_2(H)) \tag{5}$$

From (5) we can calculate the value of the spread:

$$s = -\frac{1}{T} \ln \left\{ \frac{V}{D\, e^{-rT}} [1 - N(d_1(H))] + N(d_2(H)) \right\} \tag{6}$$

The risk neutral probability of default is:

$$P_{DEF} = 1 - N(d_2(H)) \tag{7}$$

Finally, the value of the option to default is equal to:

$$\begin{aligned} O_{DEF} &= -V[1 - N(d_1(H))] + De^{-rT}[1 - N(d_2(H))] \\ &= De^{-rT} - f \end{aligned} \tag{8}$$

This is the value of the put option in the fractional Merton model and is equal to the difference between the risk-free debt value and the risky debt value.

III A Comparative Statics Analysis of Fractional Merton Model

One of the strongest critiques to the Merton model is that predicted credit spreads are lower than the spreads observed in the market. This is because the time of default is predictable with increasing accuracy and therefore very short term risky debt carries zero spread when $V > D$ (the put option is out of the money). In this section, we perform a sensitivity analysis exercise of the fractional Merton models. We want to assess whether the fractional Merton model, specifically equation (6), is able to generate credit spreads which are closer to market spreads than those predicted by Merton's model.

We design our sensitivity analysis as in Merton (1974). First, we re-write the fractional Merton model by defining the firm leverage $L = \frac{De^{-rT}}{V}$. According to the Modigliani-Miller theorem, the firm value is invariant to its capital structure (i.e. $V \equiv E + f$). As a consequence, the firm leverage is the only relevant variable.

Merton (1974) defines leverage as the "quasi" debt-to-firm value ratio and assesses, amongst others, how the spread moves to changes in leverage, firm variance and debt time to maturity. In our basic sensitivity analysis exercise we extend Merton's comparative static analysis to values of $H \neq 1/2$. This will allow us to assess the sensitivity of the spread to the long memory parameter and directly compare the spread predicted by the Merton model (with $H = 1/2$) with the spread predicted by the fractional Merton model for different values of $H \neq 1/2$.

In addition, we examine how the value of the equity (call option), the risky debt, the option to default (put option), and the risk-neutral probability of default respond to changes in H.

A Spread sensitivity to long memory

The spread sensitivity to long memory is equal to $\frac{\partial s}{\partial H} = \sigma T^{H-1} \ln T n\left(d_2\right) e^{sT}$ where $n\left(d_2\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}}$ is the standard normal probability distribution function. Thus the spread sensitivity to long memory is positive for $T > 1$, equal to zero for $T = 1$ and negative for $T < 1$.

Figures 1 and 2 show how the spread reacts to changes in the long memory parameter. For $T > 1$ (Figure 1), we notice that the spread monotonically increases with an increase in the long memory parameter. This is true regardless of the firm's leverage. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 0.0493 for $H = 0.5$ and to 0.1685 for $H = 1.5$. This is more than three times higher. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical spread is equal to 0.0207 for $H = 0.5$ and to 0.1338 for $H = 1.5$. This is six and a half times higher.

[Insert Figures 1-2 somewhere here]

This result is extremely interesting and shows that, when we take into account long memory in spreads, the theoretical spread is significantly higher than the spread predicted by Merton's model. The magnitude of the spread is an indicator of the relative riskiness of corporate bonds. In presence of long memory corporate bonds are riskier than in absence of long memory.

The results are in line with the empirical findings reported in Della Ratta and Urga (2007) and Leccadito and Urga (2007) where spreads were found being long memory processes with $H > 0.5$, with cases in which the long memory parameter $d = 1$ implying an Hurst coefficient $H = 1.5$. An obvious important implication of these results is that spreads predicted by Merton's model underestimate market spreads because Merton's model does not assume long memory in spreads. Our results show that long memory is a fundamental factor affecting the magnitude of spreads and spreads predicted by the fractional Merton model can potentially better explain market spreads.

Although this conclusion is true for $T > 1$, it does not hold for $T \leq 1$. For $T = 1$, the spread remains constant for changes in H. For $T = 1$, the theoretical value of the spread in equation (6) does not depend on H any longer. For $T < 1$ (Figure 2), the spread decreases for increases in H. For example, for $T = 6$ months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 0.1156 for $H = 0.5$ and to 0.0577 for $H = 1.5$, approximately half the value. For a firm leverage of 80%, the theoretical spread is equal to 0.0078 for $H = 0.5$ and to $3.98 \cdot 10^{-5}$ for $H = 1.5$. In this case the theoretical difference in spreads can be hardly observed in the market given that spreads are very close to zero.

This result can be explained by the fact that for short maturities the fractional Merton model is affected by the same fundamental issue of Merton's model: given that default can be predicted with increasing accuracy over time, as we approach the debt time to maturity theoretical spreads tend to zero and therefore systematically underestimate market spreads, despite that spreads are assumed to have long memory properties.

B Spread sensitivity to firm leverage

The spread sensitivity to leverage is positive as can be seen from $\frac{\partial s}{\partial L} = \frac{1-N(d_1)}{L^2 T} e^{sT} = \frac{1-N(d_2)e^{sT}}{LT}$. (Henceforth, we omit the dependency of the variables on H.)

Figures 3 and 4 show how the spread changes to changes in the firm's leverage. We notice that the spread monotonically increases with an increase in leverage for any value of $0 \leq H \leq 1.5$ and $T \geq 0$. This is an expected result and reflects, *ceteris paribus*, the firm's greater probability of default for higher levels of leverage, which affects the yield demanded by the market on the firm's debt in terms of higher spread. Consistently with our previous results on the spread sensitivity to long memory, we also notice that, for $T > 1$, the higher H, the higher the spread for any level of leverage. The relationship is reversed for $T < 1$. Figures 3 and 4 generalise Figure 1 in Merton (1974).

[**Insert Figures 3-4 somewhere here**]

C Spread sensitivity to firm volatility

The spread sensitivity to the firm volatility is equal to $\frac{\partial s}{\partial \sigma} = n\left(d_2\right) T^{H-1} e^{sT}$ which is positive. It interesting to notice that the spread vega and the spread sensitivity to long memory differ by a factor $\sigma \ln T$, $\frac{\partial s}{\partial H} = \sigma \ln T \frac{\partial s}{\partial \sigma}$.

Figures 5 and 6 show how the spread changes to change in the firm's volatility. As expected, the spread monotonically increases with an increase in volatility for any value of $0 \leq H \leq 1.5$ and $T \geq 0$, reflecting the firm's greater probability of default for higher levels of volatility. We also notice that, for $T > 1$, the higher H the higher the spread for any level of volatility. For example, for $T = 3$ years, a firm leverage of 100%, a risk-free interest rate of 3% and a firm volatility of 20%, the theoretical spread is equal to 0.049 for $H = 0.5$ and to 0.168 for $H = 1.5$, approximately three and a half times higher. If we change the firm volatility to 50% and keep the other variables constant, the theoretical spread is equal to 0.136 for $H = 0.5$ and to 0.547 for $H = 1.5$, about four times higher.

For $T \leq 1$, we draw the opposite conclusion. For $T = 1$, although the spread increases with volatility, its sensitivity to volatility does not change for changes in H. For $T < 1$, the spread still increases with volatility, however, the relationship between spread and H is reversed: the lower H, the higher the spread for any level of volatility.

[Insert Figures 5-6 somewhere here]

D Spread sensitivity to firm variance

The spread sensitivity to the firm variance is equal to $\frac{\partial s}{\partial \sigma^2} = n\left(d_2\right) \frac{T^{H-1}}{2\sigma} e^{sT}$ which is positive. Its second derivative is $\frac{\partial^2 s}{\left(\partial \sigma^2\right)^2} = \frac{T^{2H-1}}{4\sigma^2} n\left(d_2\right) e^{sT} \left[\frac{2 \ln L}{\sigma^3 T^{3H}} - \frac{\sigma T^H}{4} - \frac{1}{\sigma T^H} + n\left(d_2\right) e^{sT}\right]$. The theoretical value of the spread where the function changes from convex to concave is equal to $s = \frac{1}{T} \ln \left[\frac{1}{n(d_2)} \left(\frac{\sigma T^H}{4} + \frac{1}{\sigma T^H} - \frac{2 \ln L}{\sigma^3 T^{3H}}\right)\right]$ which can be computed by setting $\frac{\partial^2 s}{\left(\partial \sigma^2\right)^2} = 0$.

Merton (1974), Fig. 2, plots the changes in the spread against the changes in the firm variance (i.e. volatility squared) rather than volatility. We carry out the same exercise and show the results in Figures 7a and 8a. As expected, the spread increases with the firm variance for any value of H. More specifically, the spread is a monotonically increasing function of the firm variance. In addition, the function is initially convex and subsequently concave. This result is consistent with Figure 2 of Merton (1974). However, in our plots the spread function turns out to be convex only for very small values of the variance and therefore cannot be noticed. Figures 7b and 8b show magnified sections of Figures 7a and 8a for very small values of the variance (<0.00001). These instances are associated with extremely small values of the spread (<0.003) which cannot be empirically observed. Figure 2 of Merton (1974), although theoretically correct, cannot therefore always be empirically tested in its convex region.

[Insert Figures 7-8 somewhere here]

E Spread sensitivity to risky debt time to maturity

The spread sensitivity to time to maturity is equal to $\frac{\partial s}{\partial T} = \frac{1}{T^2}\left[H\sigma T^H n\left(d_2\right)e^{sT} - sT\right]$. Figures 9 to 12 show how the spread changes to changes in the debt time to maturity. Spread can increase or decrease with time to maturity. This depends on the specific combination of leverage L and Hurst parameter H.

[Insert Figures 9-12 somewhere here]

For $H = 0$ (Figure 9), the spread monotonically decreases with an increase in time to maturity, for any value of L. Also, the spread increases unbounded as time to maturity approaches zero ($s \to \infty$ as $T \to 0$). This result is in contrast with Merton's model (Figure 10), where for $L < 100$, $s \to 0$ as $T \to 0$ and only for $L \geq 100$. For $H = 0.5$, Figure 10 also confirms that Figure 3 in Merton (1974) is incorrect, as pointed out by Lee (1981) and Pitts and Selby (1983). As H increases, we notice two interesting trends. First, spreads do not decrease monotonically any longer with the increase in time to maturity. For $H = 0.5$, they are monotonically decreasing only for $L \geq 100$, for$H = 1$ and $H = 1.5$ they are monotonically increasing for $L \leq 100$, whilst for $L > 100$ spreads decrease first and increase subsequently. Second, spreads do not increase unbounded any longer as $T \to 0$. Specifically, while for $H = 0.5$ spreads increase unbounded when $L \geq 100$, for $H = 1.5$ spreads increase unbounded only for $L > 100$. Results shown have been computed for a firm volatility value of 10% and a risk-free rate of 3%. Our conclusions are robust to changes in these variables. We can visualise the same results by comparing how the spread changes with time to maturity for different value of H**.** We show this in Figures 13 to 17.

[**Insert Figures 13-17 somewhere here**]

First, we notice that the higher the leverage, at any maturity, the higher the spread. This reflects the market expectation that firms with high leverage are riskier than firms with low leverage and therefore have higher spread. Second, the spread is invariant to changes in H for a one-year maturity. Third, for any $T > 1$, spreads for higher values of H are greater than spreads for lower values of H. Again, this result confirms our claim that the fractional Merton model can predict theoretical spreads better than Merton's model when $T > 1$. Fourth, for any $T < 1$, spreads for higher values of H are smaller than spreads for lower values of H. Fifth, as $T \to 0$, all spreads tend to zero or are unbounded with the exception of the spread calculated with $L = 100$ and $H = 1$. In this case the spread has a minimum value of 0.04.

F Equity sensitivity to long memory

The equity sensitivity to long memory is equal to $\frac{\partial E}{\partial H} = V\sigma T^H (\ln T)\, n(d_1)$. Figures 18 and 19 show how the value of the firm equity changes with long memory. For $T > 1$ (Figure 18), the equity value increases with an increase in the long memory parameter, for any value of the leverage. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical equity value is equal to 12.57 for $H = 0.5$ and to 36.26 for $H = 1.5$. This is almost three times higher. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical equity value is equal to 28.36 for $H = 0.5$ and to 53.07 for $H = 1.5$, which is almost twice as much.

This result shows that in the presence of long memory, the theoretical equity value is significantly higher than the equity value predicted by Merton's model. This result is linked to the result for the risky debt sensitivity to long memory, which has an economic interpretation (see next paragraph).

This conclusion does not hold true for $T \leq 1$. For $T = 1$, the equity value as per equation (2) remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 19), the spread decreases for increases in H. For example, for T =6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 5.58 for $H = 0.5$ and to 2.81 for $H = 1.5$, approximately half the value. For a firm leverage of 80%, the theoretical spread is equal to 25.01 for $H = 0.5$ and 24.63 for $H = 1.5$.

[Insert Figures 18-19 somewhere here]

G Risky debt sensitivity to long memory

Given the identity $V \equiv E + f$, the risky debt sensitivity to long memory is equal to minus the equity sensitivity to long memory $\frac{\partial f}{\partial H} = -V\sigma T^{H}\left(\ln T\right) n\left(d_1\right) = -\frac{\partial E}{\partial H}$.

For T>1, the risky debt value decreases with an increase in the long memory parameter, for any value of leverage. This is illustrated in Figure 20. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical debt value is equal to 78.82 for $H = 0.5$ and to 55.13 for $H = 1.5$. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical debt value is equal to 85.88 for $H = 0.5$ and to 61.17 for $H = 1.5$. *Ceteris paribus*, long memory reduces the value of the risky debt. This is a direct consequence of the fact that long memory increases the value of the spread. In addition, as the value of risky debt is a decreasing function of long memory, the equity value increases with long memory.

Again, the same conclusion does not hold true for $T \leq 1$. For $T = 1$, the risky debt value as per equation (5) remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 21), the risky debt increases for increases in H. For example, for T =6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical debt value is equal to 92.92 for $H = 0.5$ and to 95.69 for $H = 1.5$. For a firm leverage of 80%, the theoretical debt value is equal to 25.01 for $H = 0.5$ and 24.63 for $H = 1.5$.

[Insert Figures 20-21 somewhere here]

H Sensitivity of the risk-neutral default probability to long memory

The sensitivity of the risk neutral default probability to long memory is equal to $\frac{\partial PD}{\partial H} = -d_2 (\ln T)\, n(d_2)$. Figures 22 and 23 show how the risk neutral default probability changes with long memory. For $T > 1$ (Figure 22), default probability increases with long memory for $L \leq 100$, whilst first decreases and then increases for $L > 100$. As one would expect, lower values of leverage are associated with lower probabilities of default. It is interesting to notice that, as the long memory parameter increases, probabilities of default for different values of leverage increase and converge to a common value. This suggests that the firm leverage is not as much relevant a variable as in absence of long memory. The increase in default probability with long memory is a direct consequence of the increase in the spread. For instance, for T=3 years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the probability of default is equal to 56.88% for $H = 0.5$ and to 69.84% for $H = 1.5$. When the firm leverage is changed to 80% and the other variables remain constant, the probability of default is equal to 31.89% for $H = 0.5$ and to 61.99% for $H = 1.5$. This conclusion does not hold true for $T \leq 1$. For $T = 1$, the default probability remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 23), the default probability decreases with long memory for $L \leq 100$ and increases for $L > 100$. As the long memory parameter increases, the probability of default only increases for highly leveraged firms. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the probability of default is equal to 52.83% for $H = 0.5$ and to 51.43% for $H = 1.5$. For a firm leverage of 80%, the probability of default is equal to 0.07% for $H = 0.5$ and 0.001% for $H = 1.5$.

[Insert Figures 22-23 somewhere here]

I Sensitivity of the option to default to long memory

The sensitivity of the option to default to long memory is the same as the sensitivity of the equity to long memory $\frac{\partial O_{DEF}}{\partial H} = V\sigma T^{H}(\ln T)\, n(d_1) = \frac{\partial E}{\partial H}$. Figures 24 and 25 show how the value of the option to default changes with long memory. For $T > 1$ (Figure 24), the option to default increases with long memory for any value of the leverage. For instance, for T=3 years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the value of the option to default is equal to 12.57 for $H = 0.5$ and to 36.26 for $H = 1.5$, almost three times larger. When the firm leverage is changed to 80% and the other variables remain constant, the value of the option to default is equal to 5.51 for $H = 0.5$ and to 30.22 for $H = 1.5$, almost six times larger.

The increase of the value of the option to default is a direct consequence of the increase in the spread, which reduces the value of the risky debt. The higher the value of the option to default the higher the likelihood of exercise, which is consistent with a higher probability of default.

This conclusion does not hold true for $T \leq 1$. For $T = 1$, the option to default remains constant for changes in H as it does not depend on H any longer. For $T < 1$

(Figure 25), the option to default decreases with long memory for any value of the leverage. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the value of the option to default is equal to 5.58 for $H = 0.5$ and to 2.81 for $H = 1.5$. For a firm leverage of 80%, the value of the option to default is equal to 0.39 for $H = 0.5$ and 0.002 for $H = 1.5$.

[**Insert Figures 24-25 somewhere here**]

IV Theoretical vs. Market Spreads: Insights from Implied H Values

The comparative statics analysis shows that the theoretical spreads for long term maturities are closer to the observed ones when $H > 1/2$. An interesting final stage of our analysis is to see how close the theoretical spreads are to market spreads. We investigate this issue by computing the implied H values from a sample of individual companies data. Implied H values are computed by setting the theoretical spread equal to the observed one and solving for H. Of course, obtaining implied H larger than 1/2 would be a way to confirm the validity of the fractional Merton model.

We have data for four companies (Ford, General Motors, Endesa and RWE), spanning over the period January 2002–December 2003. 490 daily observations are available for the first two companies and 359 for the latter two. For each day, our dataset includes:

1. The volatility of the equity, σ_E

2. The value of the company's equity, E_t

3. The amount of debt principal, D_t to be repaid after $T = 5$ years,

4. The risk-free rate for maturity T, r_t

5. The 5-year CDS composite spread, used as a proxy for the credit spread.

All the data has been collected from Bloomberg, except the CDS data, which has been provided by Markit Partners.

In order to compute the theoretical spread for the Merton fractional model, and hence the implied H value that makes it equal to the observed one, we need to know the value of the company's assets at time t (V_t) and its volatility (σ), which are both unobservable. However, these two quantities can be calculated using the procedure described in Hull (2002, p. 622).

Let us consider the value of the equity from Merton fractional model as described in (2), and by using Itô's lemma (i.e.$\sigma_E E_t = N(d_1(H))\sigma V_t$), we can derive a system of two equations in the unknowns V_t and σ. The system is rearranged as follows to obtain more numerically stable functions

$$\begin{cases} V_t = \frac{D_t e^{-r_t T} N(d_2(H)) + E_t}{N(d_1(H))} \\ \sigma = \frac{\sigma_E E_t}{D_t e^{-r_t T} N(d_2(H)) + E_t} \end{cases} \tag{9}$$

For a given value of the Hurst parameter, (9) is a fixed-point equation of the form

$$(V_t, \sigma) = \begin{pmatrix} G_1(V_t, \sigma) \\ G_2(V_t, \sigma) \end{pmatrix}.$$

The system is solved numerically by computing the iterations G^n of function $G = (G_1, G_2)$ from an arbitrary starting point, until the error function

$$|V_t^n - G_1(V_t^n, \sigma^n)| + |\sigma^n - G_2(V_t^n, \sigma^n)|$$

becomes less than a tolerance level ϵ. The starting values are $V_t^0 = E_t + D_t$ and $\sigma^0 = 0.1$. Furthermore, we choose $\epsilon = 10^{-6}$. Let $s_t^{th}(V_t^{n^*}(H), \sigma^{n^*}(H))$ be the theoretical spread from the Merton fractional model with Hurst parameter H, obtained after n^* iterations of the above algorithm. The implied Hurst parameter H^* is such that

$$s_t^{th}(V_t^{n^*}(H^*), \sigma^{n^*}(H^*)) = s_t^{obs}$$

where s_t^{obs} is the observed 5-years CDS spread at time t.

Descriptive statistics for the calculated implied Hurst parameters, H^*, the values of the company's assets, $V_t^{n^*}(H^*)$, and its volatility, $\sigma^{n^*}(H^*)$, are reported in Table 1. The empirical densities of implied H^* are reported in Figure 26.

[Table 1 and Figure 26 about here.]

There is a clear evidence of the validity of the fractional version of the Merton model. In particular, for General Motors, the number of days for which which $H^* \leq 1/2$ is 9, while for the remaining 481 days $H^* > 1/2$. As far as Ford is concerned, $H^* \leq 1/2$ only in 47 out of 490 cases. As far as Endesa and RWE are concerned, the number $H^* \leq 1/2$ in 8 and 14 cases respectively, while for the remaining days H^* is between 1/2 and 1.

Table 1: Descriptive statistics for implied Hurst parameter H^*, company's assets $V_t^{n^*}(H^*)$ and volatility of the company's assets, $\sigma^{n^*}(H^*)$.

Ford

	Mean	Median	std	min	max	95% Range	99% Range
H^*	0.6476	0.6634	0.1087	0.2742	0.9370	0.4061	0.5572
$V_t^{n^*}(H^*)$	1.33E+11	1.33E+11	1.62E+10	1.01E+11	1.65E+11	5.55E+10	6.12E+10
$\sigma^{n^*}(H^*)$	0.1468	0.1328	0.0530	0.0850	0.4790	0.2015	0.3587

General Motors

	Mean	Median	std	min	max	95% Range	99% Range
H^*	0.8269	0.8522	0.1578	0.4004	1.3118	0.5874	0.6980
$V_t^{n^*}(H^*)$	1.66E+11	1.67E+11	8.20E+09	1.45E+11	1.89E+11	3.27E+10	4.23E+10
$\sigma^{n^*}(H^*)$	0.0928	0.0860	0.0314	0.0489	0.2143	0.1181	0.1573

Endesa

	Mean	Median	std	min	max	95% Range	99% Range
H^*	0.7032	0.6902	0.1290	0.4130	0.9181	0.3944	0.4978
$V_t^{n^*}(H^*)$	3.23E+10	3.15E+10	2.66E+09	2.70E+10	3.93E+10	1.05E+10	1.18E+10
$\sigma^{n^*}(H^*)$	0.1455	0.1435	0.0400	0.0862	0.2497	0.1476	0.1604

RWE

	Mean	Median	std	min	max	95% Range	99% Range
H^*	0.6887	0.7363	0.0961	0.4283	0.8539	0.3295	0.4118
$V_t^{n^*}(H^*)$	3.38E+10	3.85E+10	1.03E+10	1.38E+10	4.42E+10	2.89E+10	3.01E+10
$\sigma^{n^*}(H^*)$	0.1971	0.1425	0.1233	0.0822	0.4904	0.3770	0.4036

The implied Hurst parameter is, on average, bigger for GM than it is for Ford. Also, all the measures of variability (i.e. standard deviation, 95% and 99% ranges) are smaller for Ford. For the volatility of the asset values, the opposite holds: Ford has a bigger average σ, but also bigger measures of dispersion.

V Conclusions

In this paper we propose a fractional version of the structural Merton model suitable to embed long memory characteristics of spreads in pricing credit risk.

We carry out a comparative statics analysis to evaluate spread sensitivities to the long memory parameter, firm leverage, firm volatility and variance, and risky debt time to maturity. We also compute sensitivities of the equity, risky debt, risk-neutral default probability and option to default to long memory.

We find that when time to maturity is bigger than 1 year ($T > 1$), the spread monotonically increases as the long memory parameter increases. This result is extremely interesting and shows that, when we take into account long memory in spreads, the theoretical spread is larger than the spread predicted by Merton's model.

Our results show that long memory is a fundamental factor affecting the magnitude of spreads and spreads predicted by the fractional Merton model can better explain market spreads. This result does not hold for $T \leq 1$, where the spread decreases with an increase in long memory. This result can be explained for short maturities: given that default can be predicted with increasing accuracy over time, as we approach the debt time to maturity theoretical spreads tend to zero and therefore systematically underestimate market spreads. We also find that the spread monotonically increases with firm leverage and firm volatility. As expected, this finding reflects the firm's greater probability of default for higher levels of leverage or volatility, which affects the yield demanded by the market on the firm's debt in terms of higher spread.

To validate empirically our proposed model, we used individual companies data of Ford, General Motors, Endesa and RWE, to derive implied H values. We computed the implied H values by setting the theoretical spread equal to the observed one and solving for H. We obtained implied values of H larger than $1/2$, confirming the validity of the fractional Merton model we propose.

There are several additional issues that we leave for future analysis. So far we only used a small dataset of companies to evaluate implied H values. It would be interesting to further validate our theoretical conclusions by using a larger dataset in terms of both companies and number of time observations. In addition it is possible to consider the fractional version of another widely used structural model, such as Black and Cox (1976), where the company issuing debt is assumed to default the first time its assets fall below some default barrier. Another possible extension could be the jump-diffusion version of the model, where the firm value could also be subject to sudden jumps.

Appendix. The Fractional Black-Scholes Model

Given that Merton's model is based on the Black-Scholes framework, we can easily exploit the notion of fractional Brownian motion to generalise Merton's model to the 'fractional Merton model'.

As for $H \neq 1/2$ the fractional Brownian motion is neither a Markov process nor a semimartingale, the conventional Itô theory cannot be used to define stochastic integrals with respect to the fractional Brownian motion.

This has important consequences in terms of arbitrage. Absence of arbitrage opportunities is the main axiom of mathematical finance. The fundamental theorem of asset pricing states that no arbitrage implies the existence of an equivalent martingale measure. As a consequence, non-semimartingales are ruled out as models for assets. Given that the fractional Brownian motion is not a semimartingale, the Geometric fractional Brownian motion cannot be a semimartingale. Rogers (1997), Dasgupta and Kallianpur (2000) and Shiryaev (1998) prove the existence of arbitrage with the Geometric fractional Brownian motion in continuous time. Cheridito (2001) proves the existence of arbitrage in discrete time.

Arbitrage opportunities therefore seem to rule out the Geometric fractional Brownian motion as a model in finance. Mathematically, these opportunities are based on the path-wise Riemann-Stieltjes stochastic integral with respect to the fractional Brownian motion. The Riemann-Stieltjes integral of a process u_t is defined as $\int_0^T u_t dB_H$. A path-wise integration theory for the fractional Brownian motion was developed by Lin (1995) and Decreusefond and Üstünel (1999).

Several authors have suggested the use of divergence integrals (also called Wick-

Itô-Skorohod integral) to avoid arbitrage opportunities (see Sottinen and Valkeida, 2003).

We therefore have two pricing approaches in the fractional Merton model. The firm value dynamics can be expressed as a fractional Geometric Brownian motion in terms of the Riemann-Stieltjes equation

$$dV = \mu V dt + \sigma V dB_H \tag{10}$$

or the Skorohod equation

$$\delta V = \nu V dt + \sigma V \delta B_H \tag{11}$$

As shown by Sottinen and Valkeila (2003) for the fractional Black-Scholes model, by using the appropriate Itô formulas the solutions to (10) and (11) are, respectively:

$$V_t = V_0 e^{\int_0^t \mu(s) ds + \sigma B_H} \tag{12}$$

and

$$V_t = V_0 e^{\int_0^t \left(\nu(s) - \sigma^2 H s^{2H-1}\right) ds + \sigma B_H} \tag{13}$$

We conclude that the stochastic differential equations (10) and (11) imply the same pricing model for the firm value if and only if:

$$\mu(t) = \nu(t) - \sigma^2 H t^{2H-1} \tag{14}$$

Although the fractional Merton model does not have an equivalent martingale measure, it can be shown that there is a unique measure such that the solution to (10) or (11) is given by the Geometric fractional Brownian motion

$$V_t = V_0 e^{\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_H} \tag{15}$$

This is called generalised solution. The path-wise Riemann-Stieltjes solution to $\int_0^T u_t dB_H$ is instead given by:

$$V_t = V_0 e^{\mu t + \sigma B_H} \tag{16}$$

Hu and Øksendal (2003) show that with the generalised solution (15) the fractional pricing model is arbitrage-free and complete, in the sense that there is a fractional analogue of Itô's formula. When using path-wise solutions and under continuous trading, arbitrage cannot however be ruled out, as shown, for example, by Shiryaev (1998) and Dasgupta (1998). Cheridito (2001) shows that the model is arbitrage-free if trading is restricted such that there is some non-random minimal time interval between successive transactions. However, this is a very restrictive condition.

Despite the lack of the equivalent martingale measure in the path-wise fractional model, it is possible to compute the solution to the fractional Merton model by using the so-called 'weak pricing principle'. This solution is the same as the one obtained in the generalised model of Hu and Øksendal (2003).

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Figure 1 – Spread Sensitivity to Long Memory (T>1)

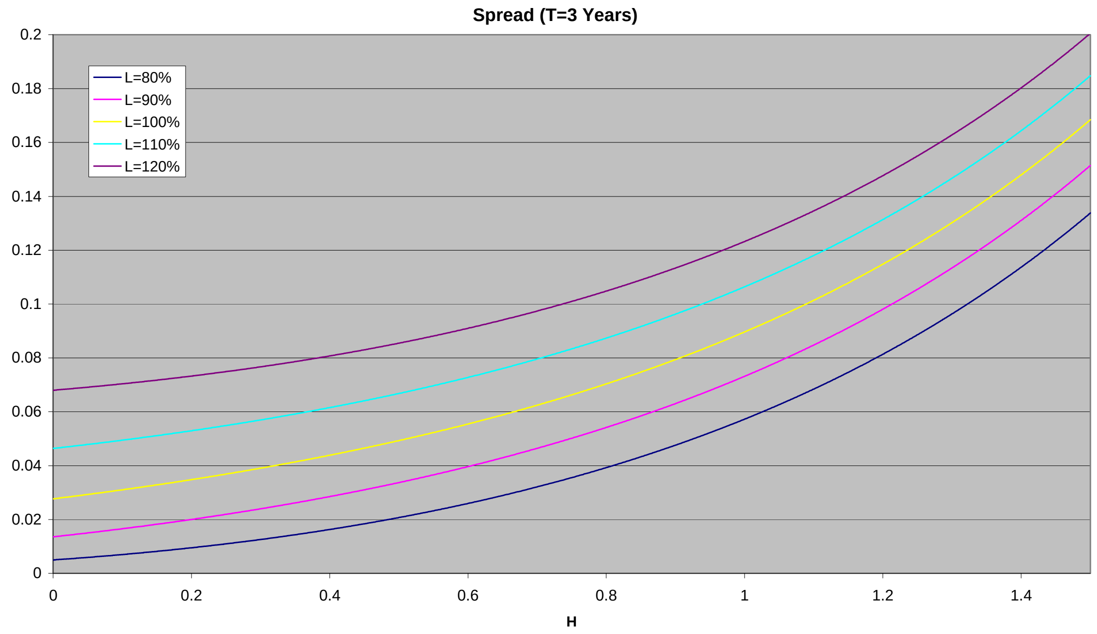

Figure 2 – Spread Sensitivity to Long Memory (T<1)

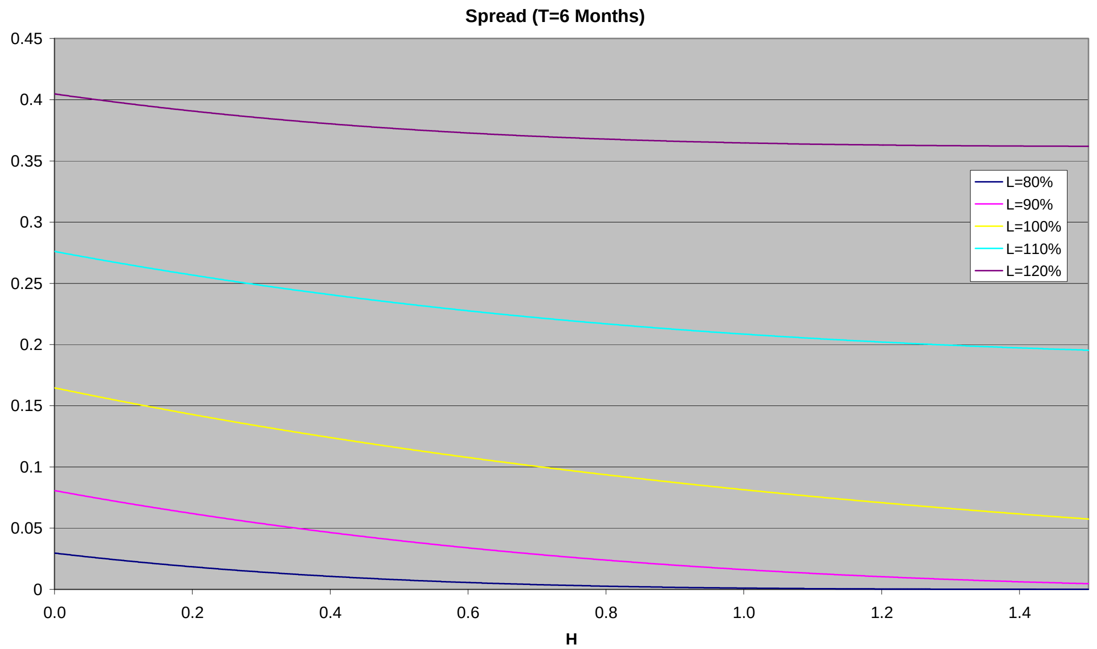

Figure 3 – Spread Sensitivity to Firm Leverage (T>1)

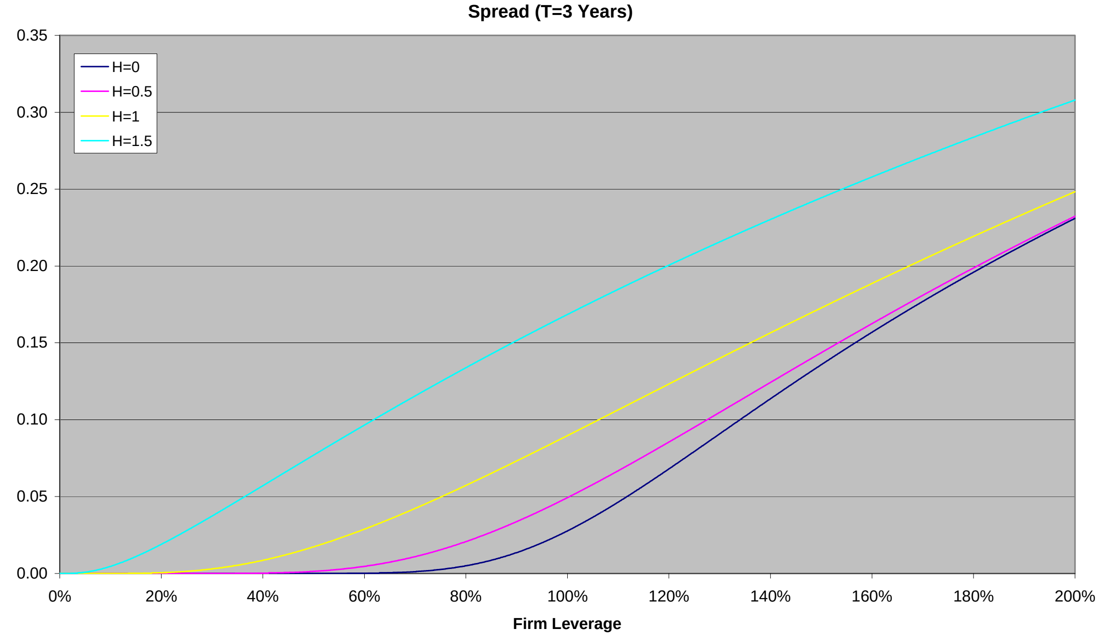

Figure 4 – Spread Sensitivity to Firm Leverage (T<1)

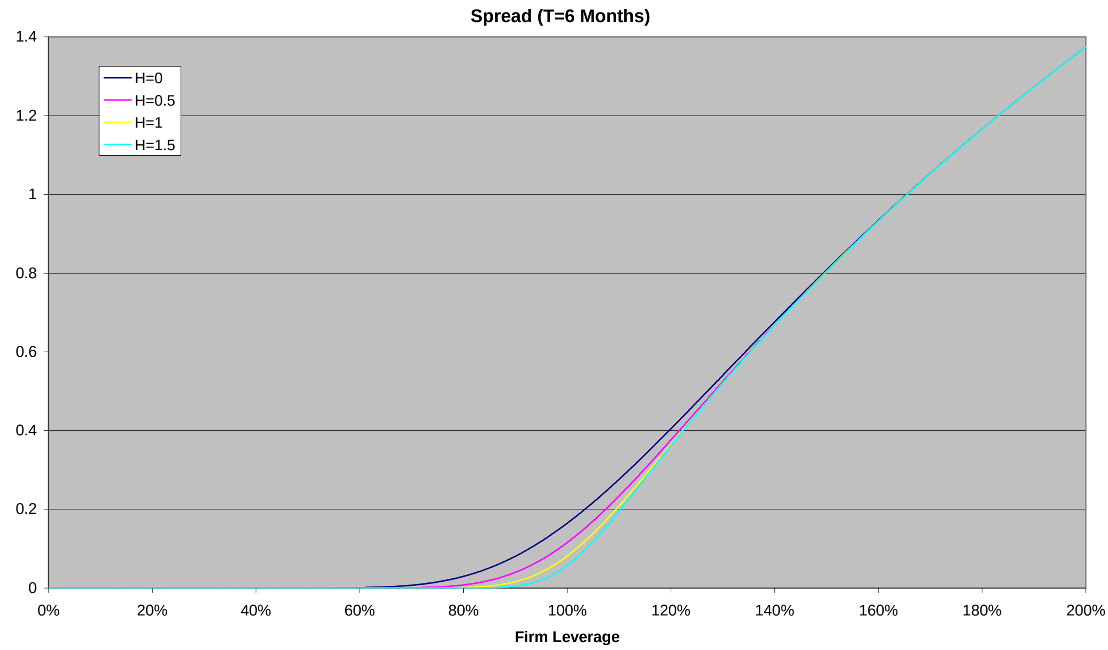

Figure 5 – Spread Sensitivity to Firm Volatility (T>1)

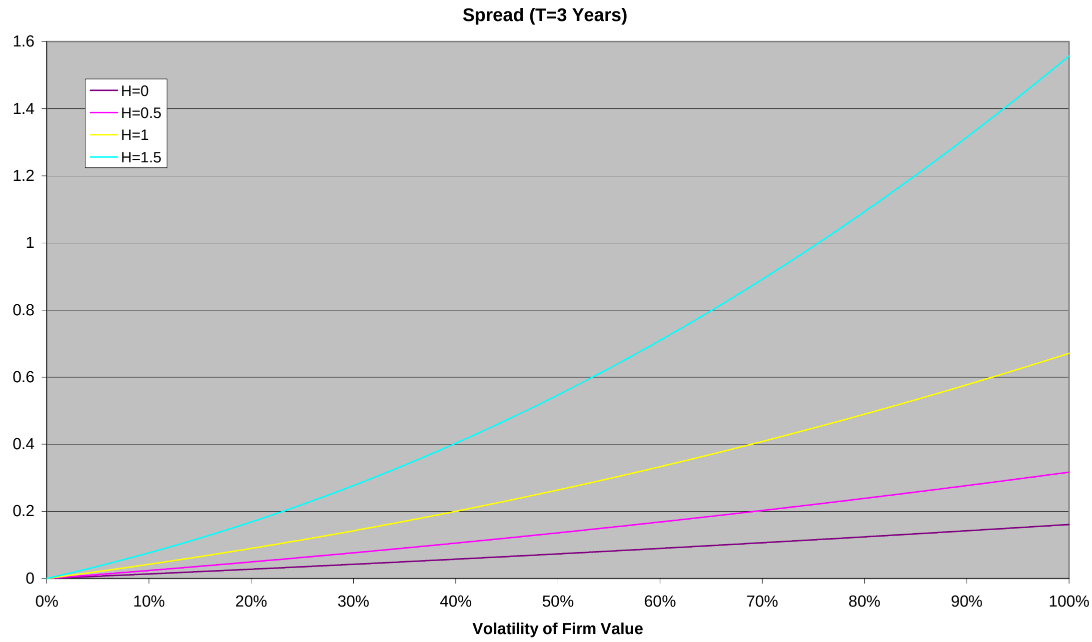

Figure 6 – Spread Sensitivity to Firm Volatility (T<1)

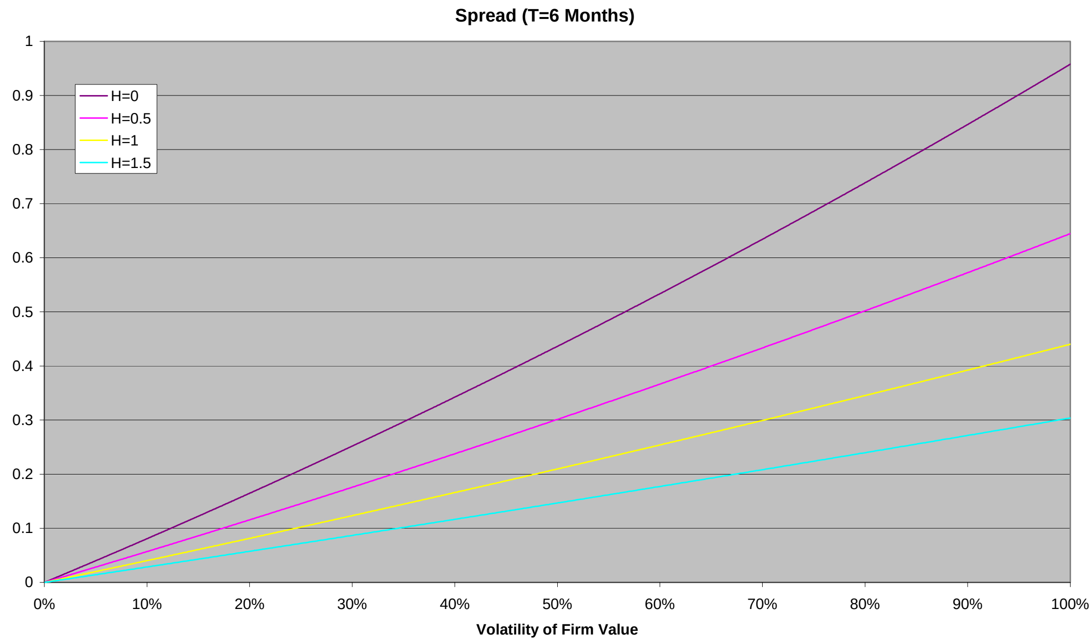

Figure 7a – Spread Sensitivity to Firm Variance (T>1)

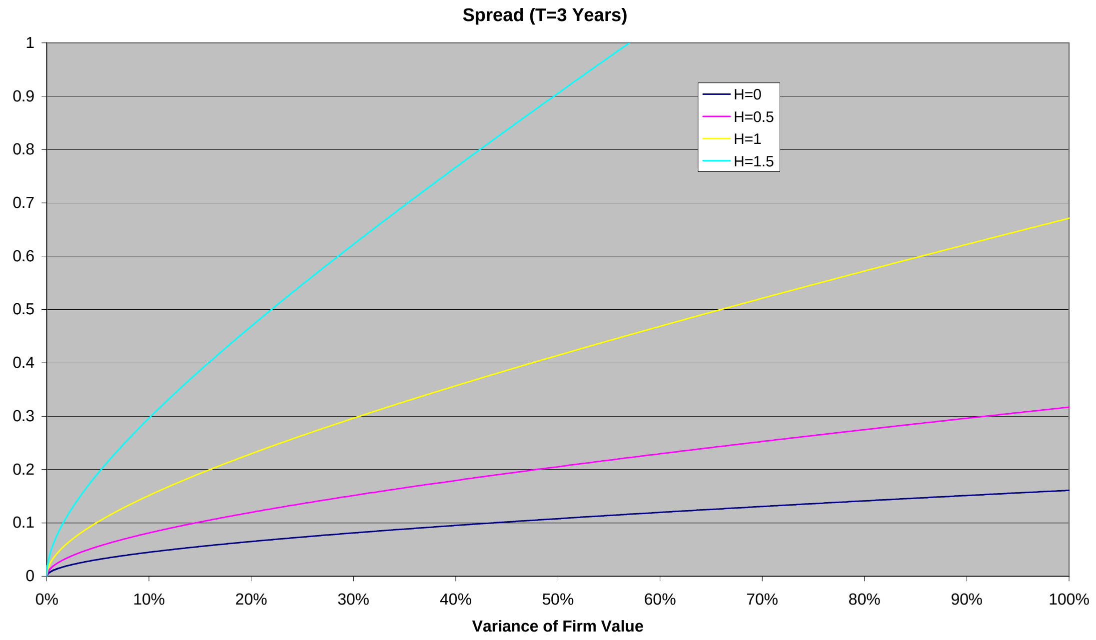

Figure 8a – Spread Sensitivity to Firm Variance (T<1)

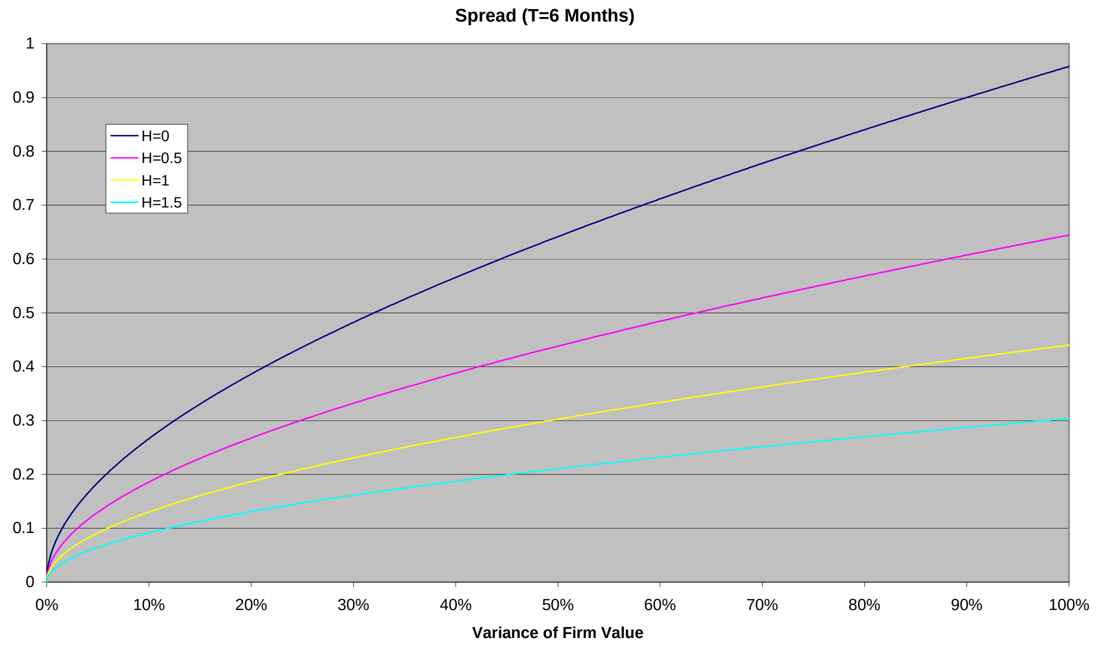

Figure 7b – Spread Sensitivity to Firm Variance (T>1)

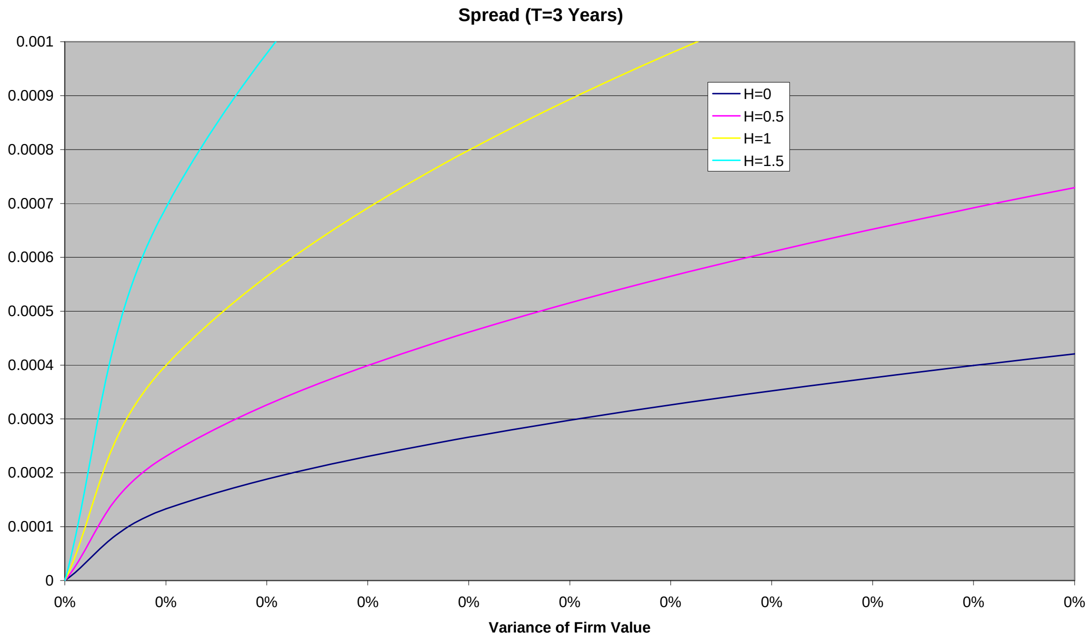

Figure 8b – Spread Sensitivity to Firm Variance (T<1)

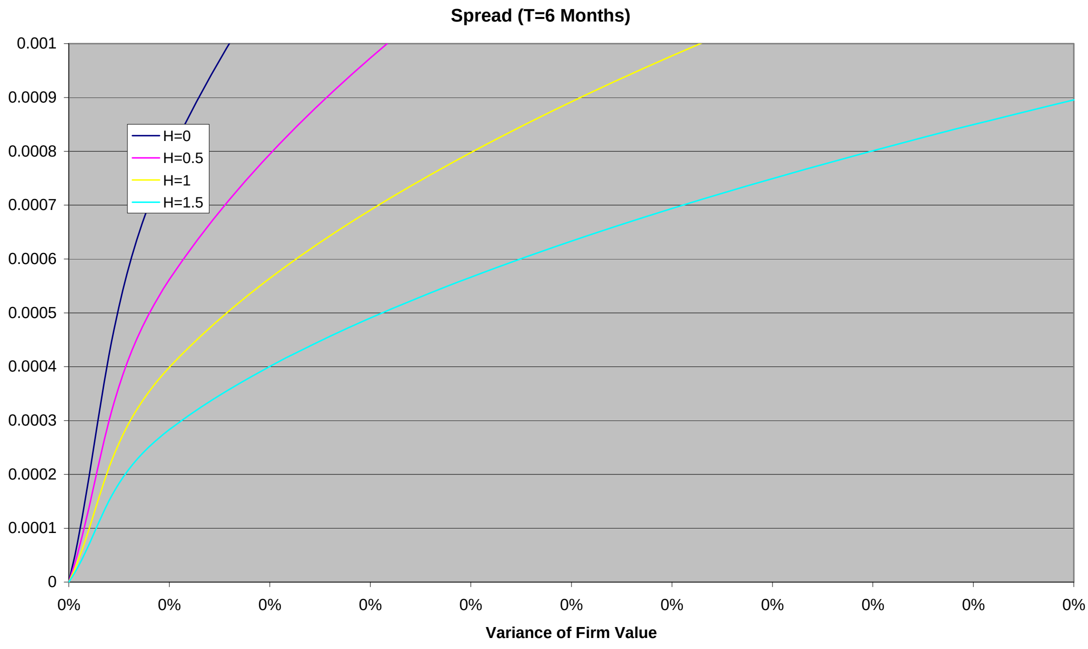

Figure 9 – Spread Sensitivity to Time to Maturity (H=0)

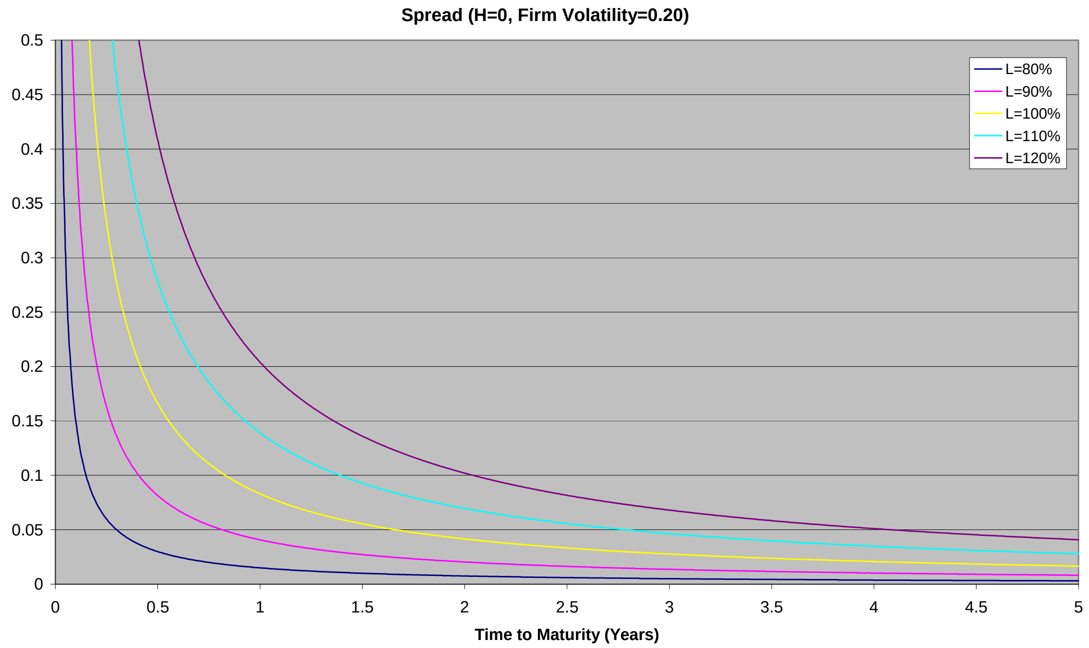

Figure 10 – Spread Sensitivity to Time to Maturity (H=0.5)

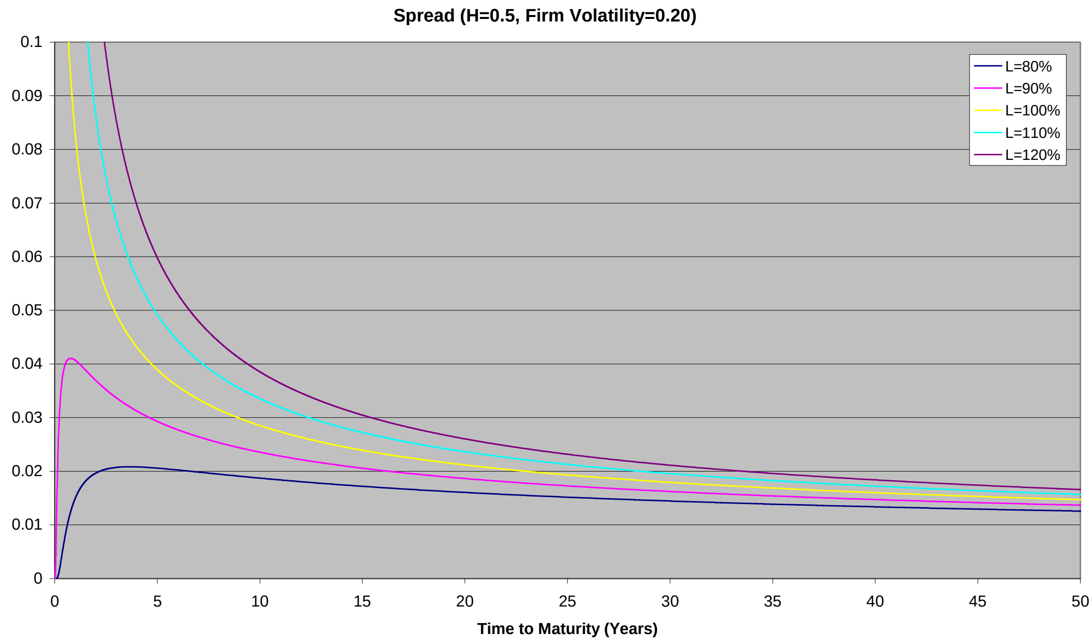

Figure 11 – Spread Sensitivity to Time to Maturity (H=1)

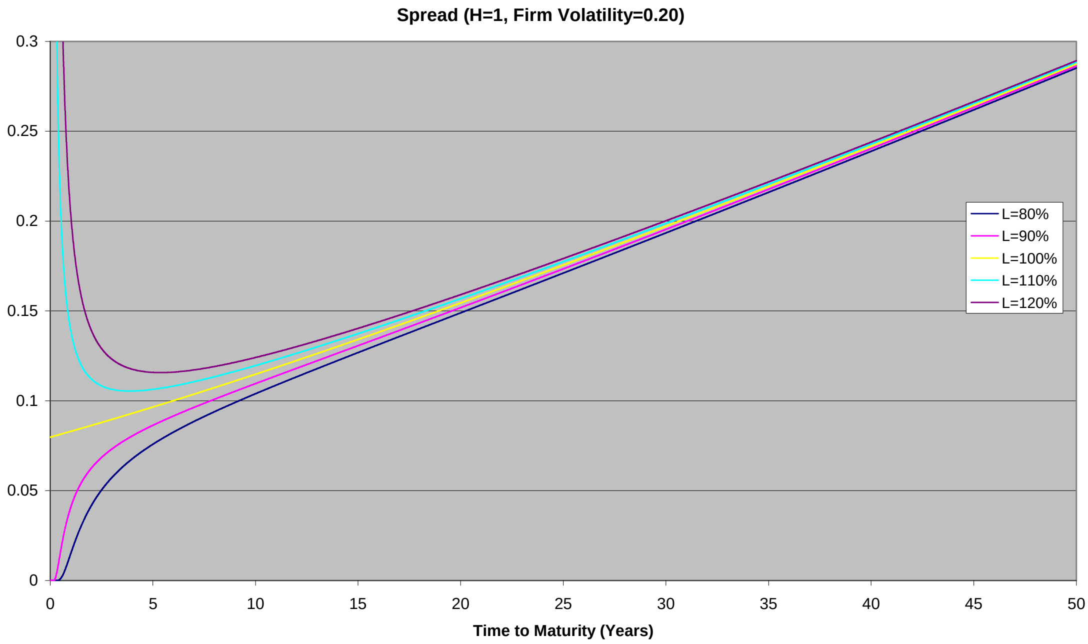

Figure 12 – Spread Sensitivity to Time to Maturity (H=1.5)

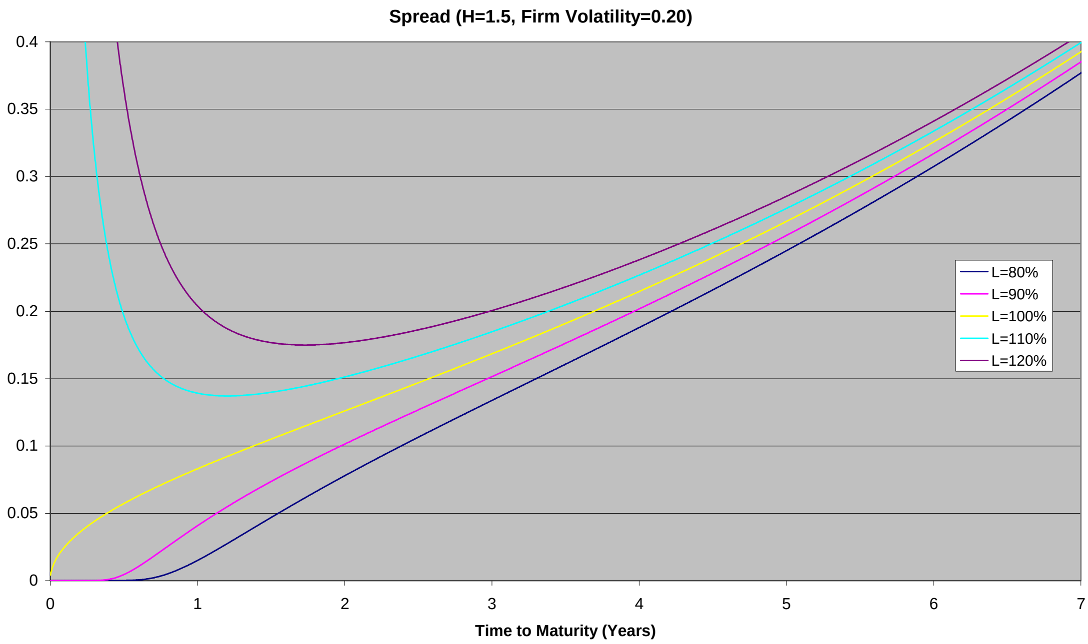

Figure 13 – Spread Sensitivity to Time to Maturity (L=80%)

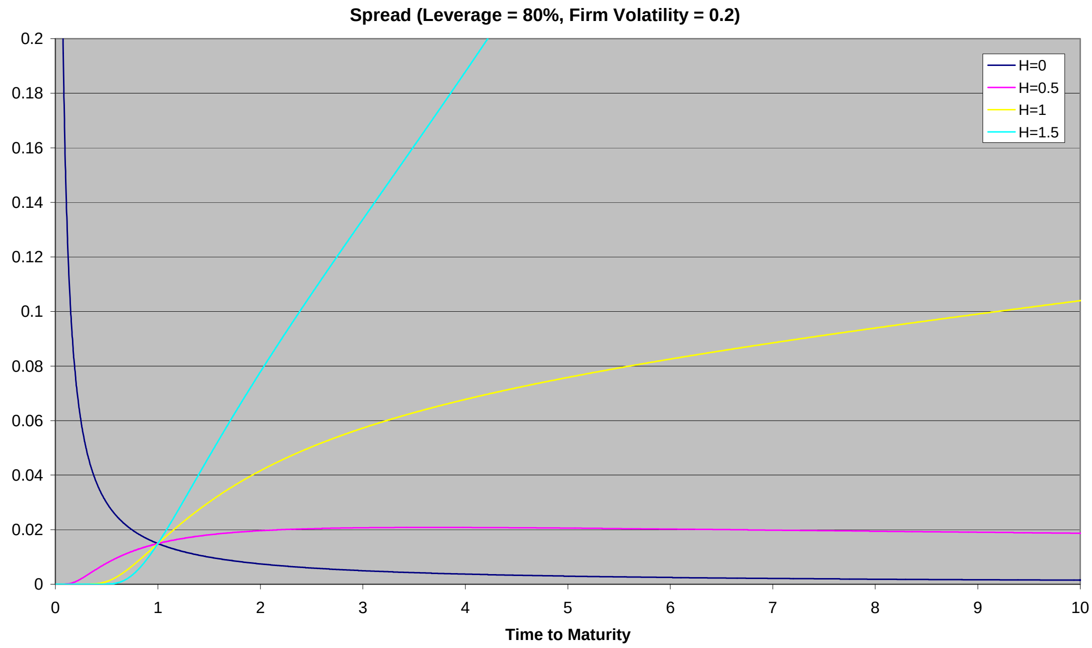

Figure 14 – Spread Sensitivity to Time to Maturity (L=90%)

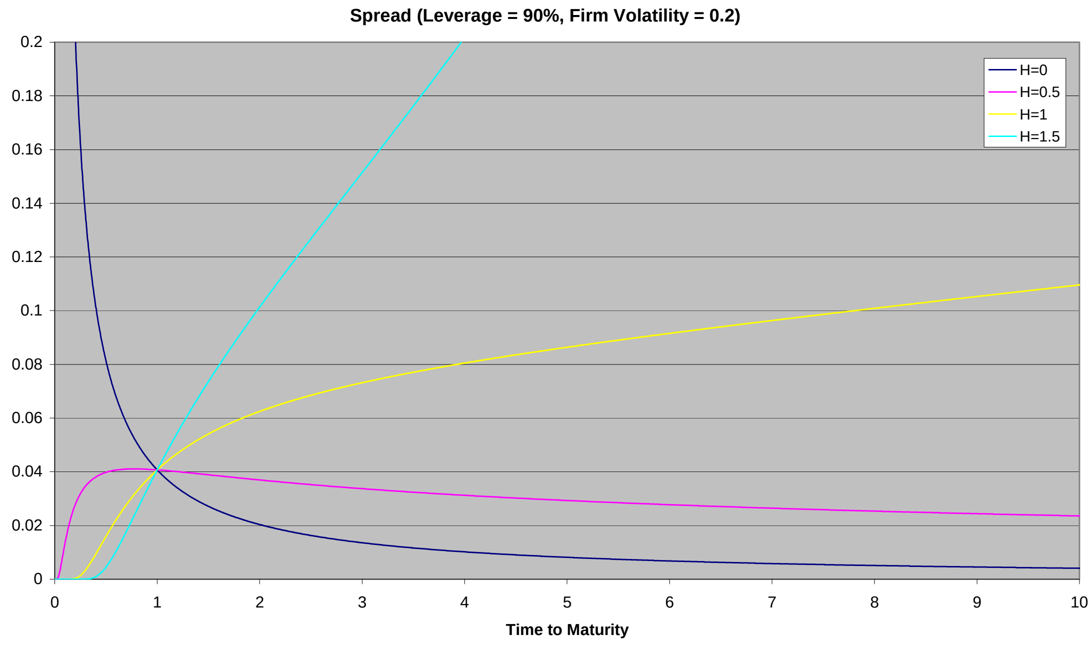

Figure 15 – Spread Sensitivity to Time to Maturity (L=100%)

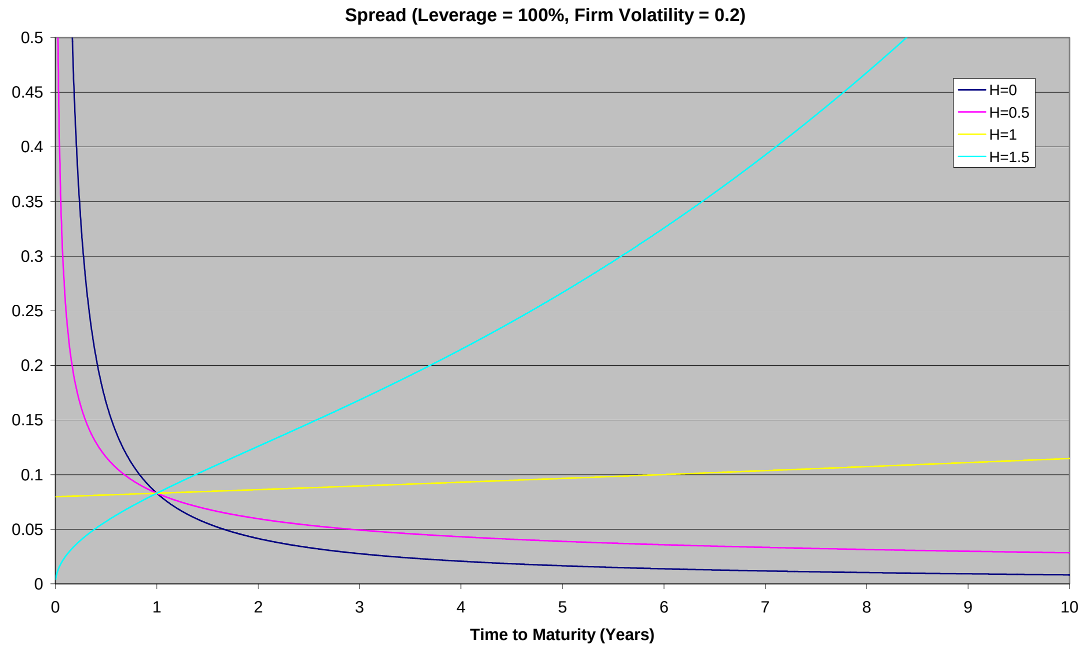

Figure 16 – Spread Sensitivity to Time to Maturity (L=110%)

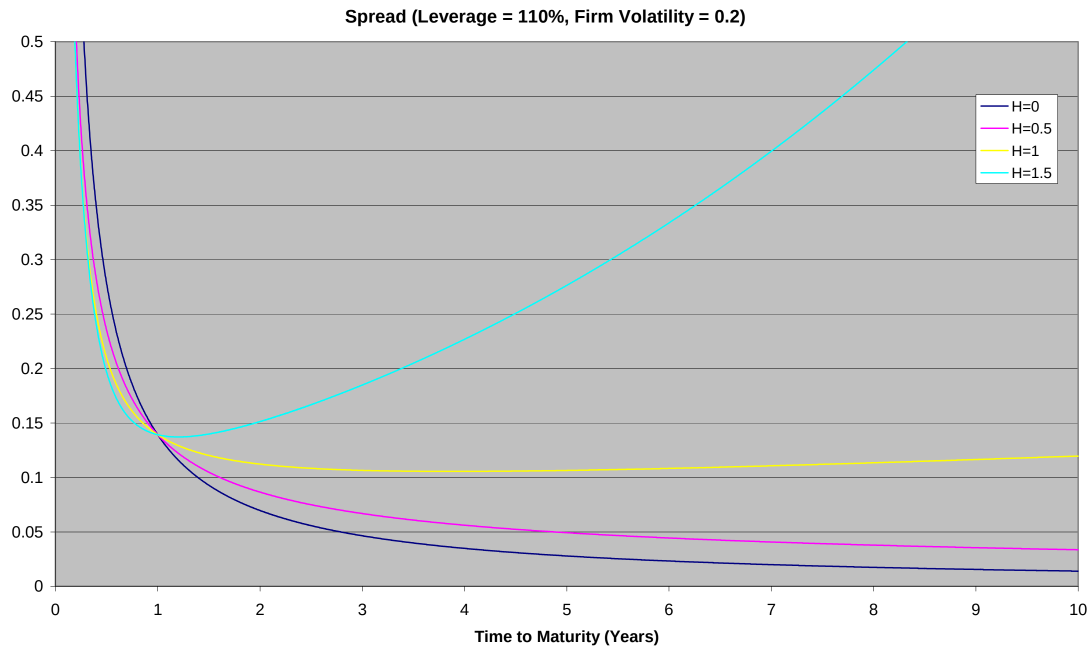

Figure 17 – Spread Sensitivity to Time to Maturity (L=120%)

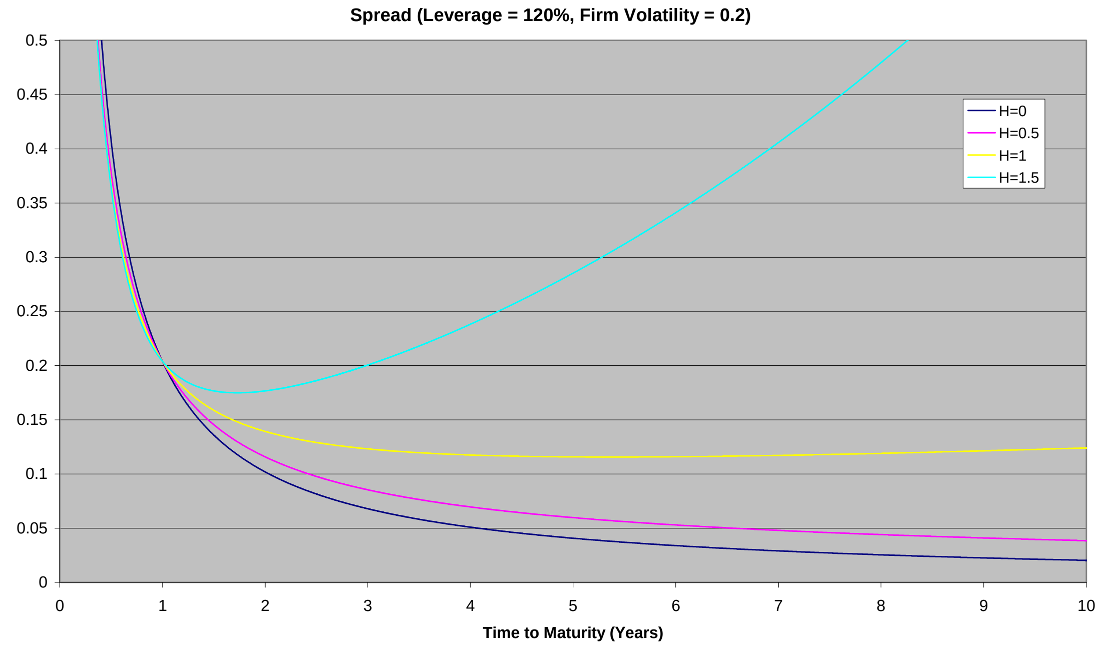

Figure 18 – Firm Equity Sensitivity to Long Memory (T>1)

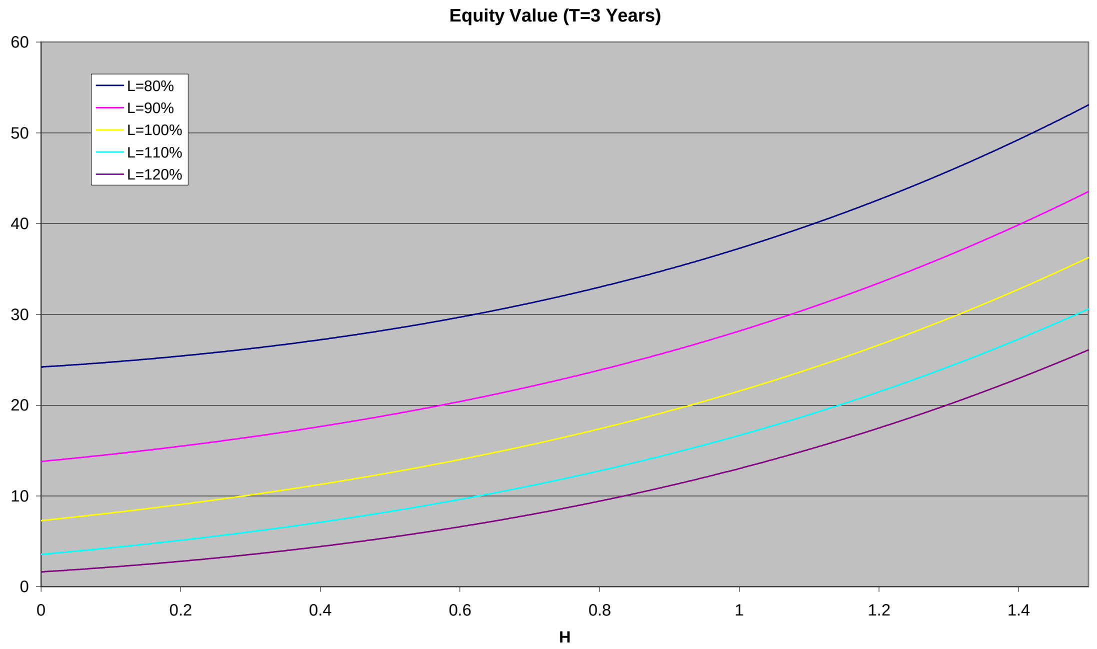

Figure 19 – Firm Equity Sensitivity to Long Memory (T<1)

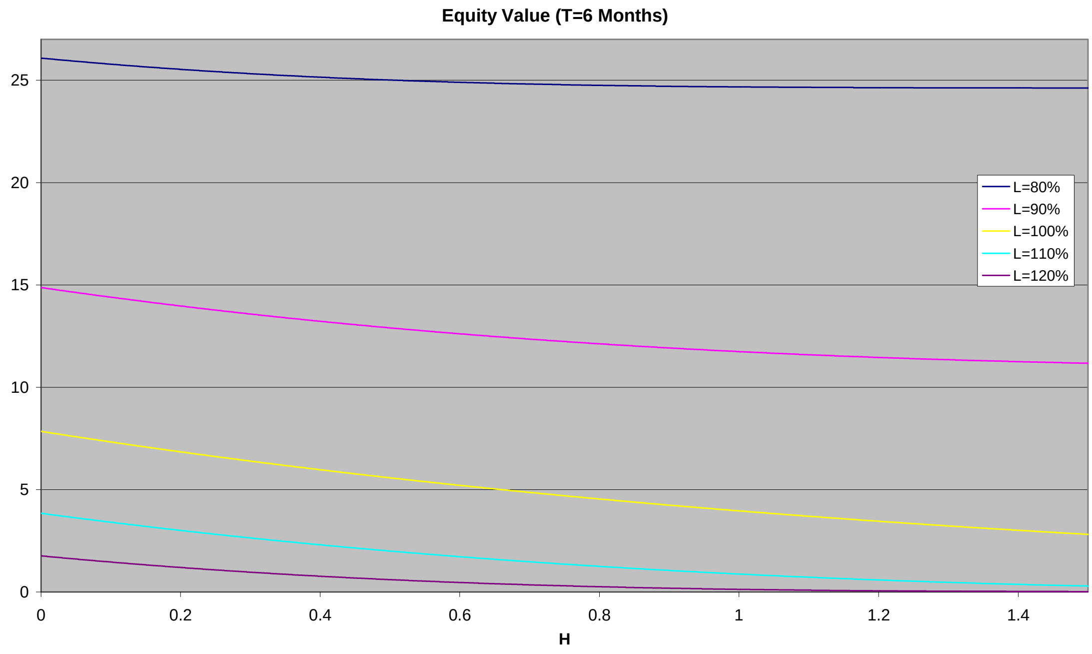

Figure 20 – Firm Debt Sensitivity to Long Memory (T>1)

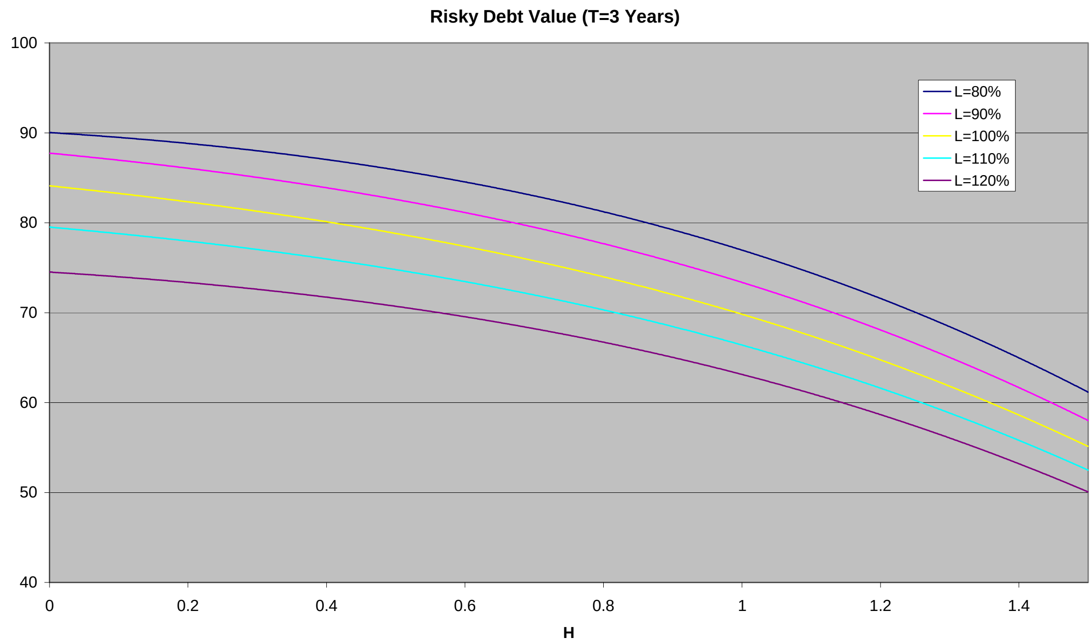

Figure 21 – Firm Debt Sensitivity to Long Memory (T<1)

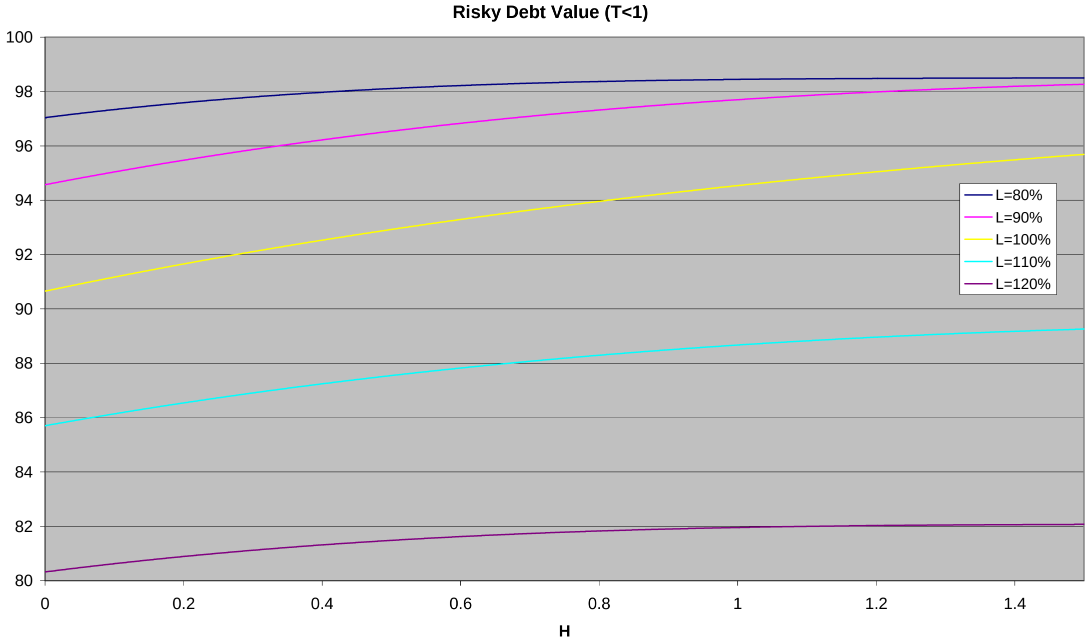

Figure 22 – Default Probability Sensitivity to Long Memory (T>1)

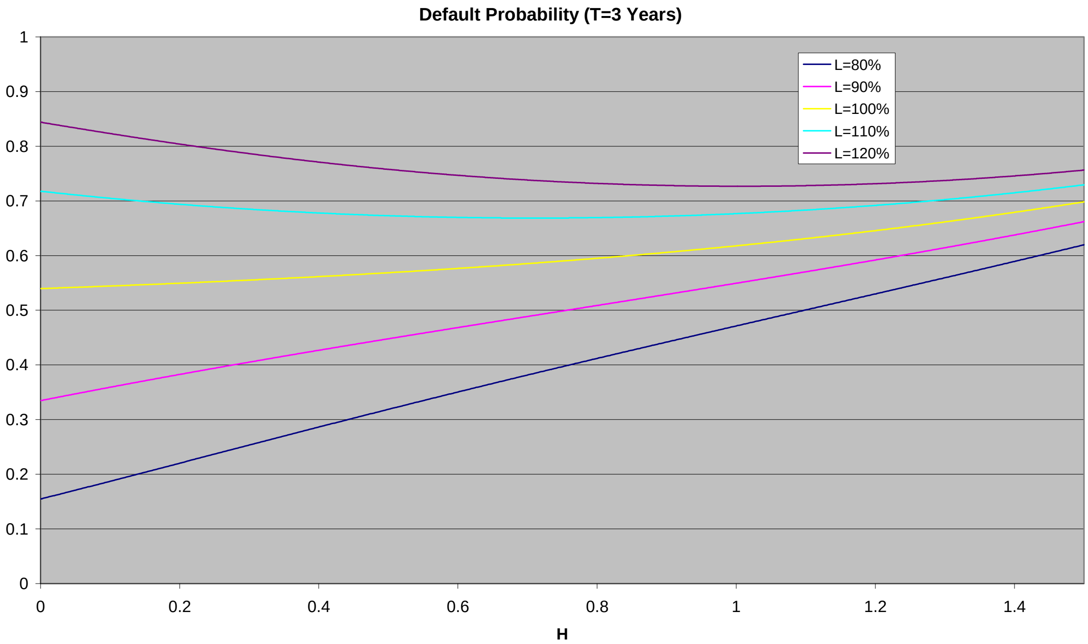

Figure 23 – Default Probability Sensitivity to Long Memory (T<1)

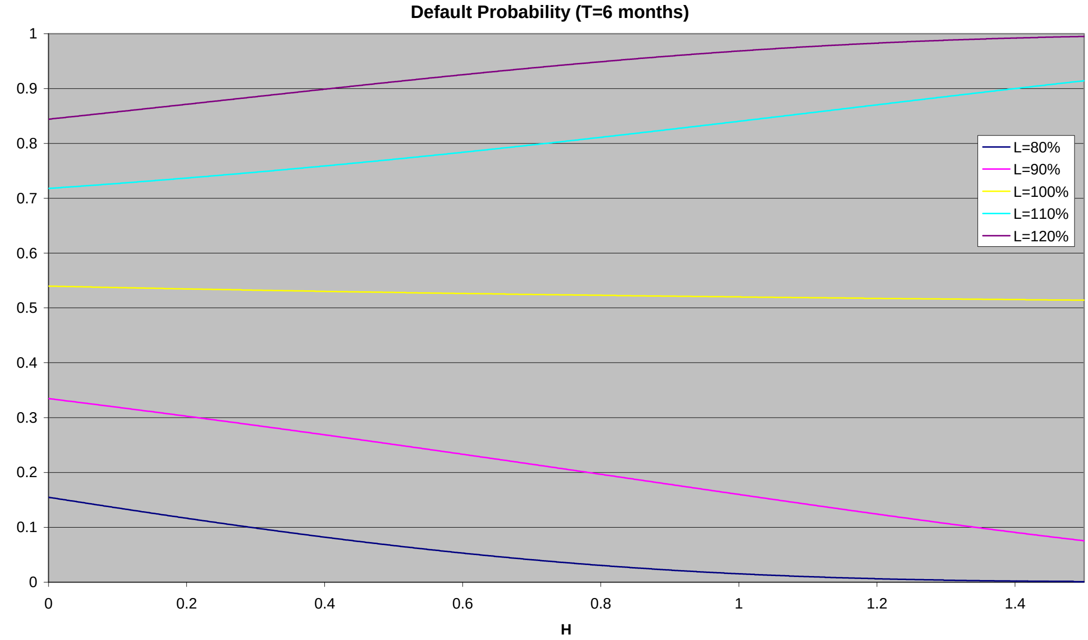

Figure 24 – Sensitivity of the Option to Default to Long Memory (T>1)

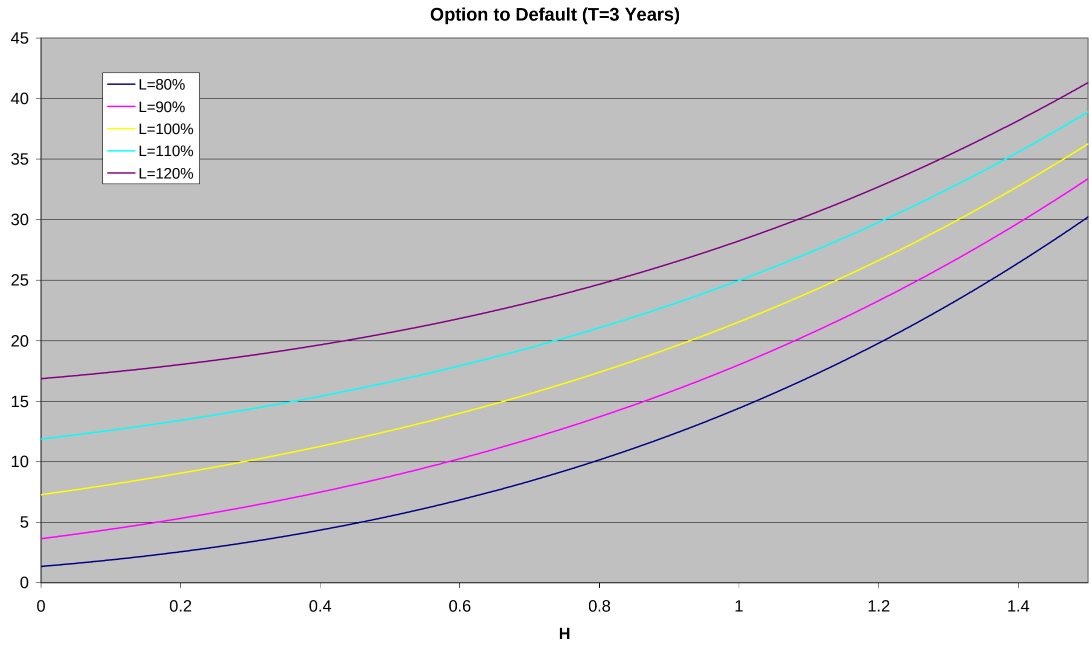

Figure 25 – Sensitivity of the Option to Default to Long Memory (T<1)

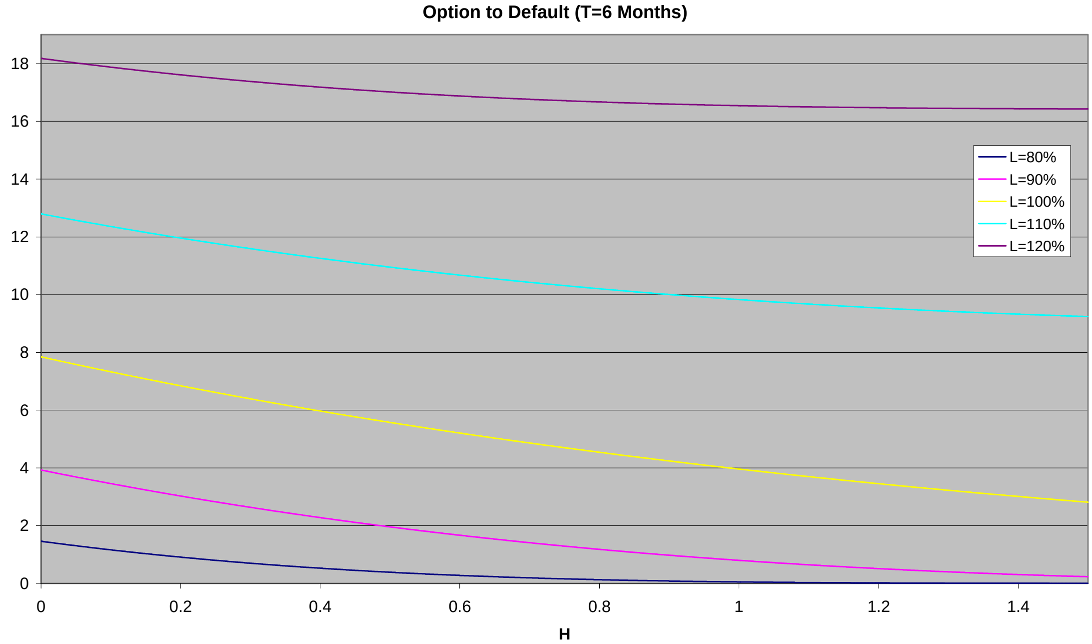

Figure 26 – Empirical Densities of the Implied H^* Values