Smile Modelling for Exchange-Traded Products on Futures Strategies

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Demand for volatility instruments

- over the last decades, steady increase of volumes for volatility-related products
- since the subprime crisis, demand for volatility instruments accelerated in a context of pronounced market uncertainty
- VIX has become the benchmark instrument for monitoring and trading volatility
- volumes on all instruments related to the VIX have peaked over the last years; eg: in 2015, the VIX became the second most traded underlying in the CBOE options market, right after the S&P 500

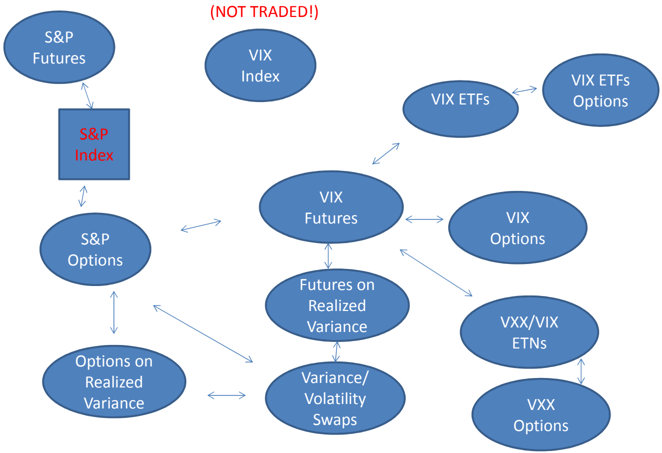
What does it mean trading volatility ?

- using equities and options to build up strategies in order to make profit of the variability of the market

What is market volatility?

- actual (realized) volatility of equities, or
- volatility implied by option prices (implied volatility)

Volatility as an asset class



What can be traded in the market?

- realized volatility
- implied volatility
- implied versus realized
- implied versus implied
- realized versus realized
- correlation between two or more assets
- implied correlation
- dispersion

Historical Volatility

Historical Variance : $\frac{1}{n}\sum_{i=1}^{n}\left(ln\left(\frac{S_i}{S_{i-1}}\right)\right)^2$

- OTC products:
  - volatility swap
  - variance swap
  - corridor variance swap
  - options on volatility variance
- Listed Products
  - Futures on realized variance
  - Options on realized variance

- OTC products:
  - Swap and options
- Listed Products
  - VIX Futures contracts
  - Volax (DAX)

Implied Volatility : $B\&S(S, \sigma^{imp}, r, K, T) = C^{Mkt}(K, T)$

- VIX introduced in 1993 by the CBOE
- Old VIX (since 1992): originally calculated so as to reflect the 30-day at-the-money implied volatility of S&P500 options
- New VIX (since 2000) : calculation changed in 2004 so as to also take into account the skew of S&P500 options
- VIX quickly became a fear gauge, providing a measure for the nervousness of equity markets

- New VIX : proxy for variance swap rate

Fom the Dupire variance swap volatility formula

$$\sigma_T^2 = \frac{2e^{rT}}{T} \int_0^\infty OTM(K, S, T) \frac{dK}{K^2}$$

CBOE created a proxy that uses the first 2 maturities for the expiry of the options (in order to track 30 days)

$$VIX = \sqrt{w_1 \sum_{i=1}^{n} OTM(K_i, S, T_1) \frac{\Delta K}{K_i^2} + w_2 \sum_{i=1}^{n} OTM(K_i, S, T_2) \frac{\Delta K}{K_i^2}}$$

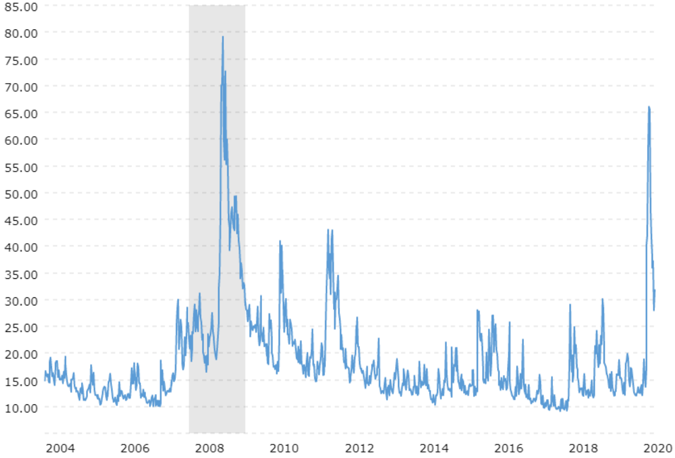


Figure: VIX Volatility Index - Historical Chart from 2004 to 2020.

Trading VIX

- VIX cannot be traded directly
- 2004: CBOE introduced futures on the VIX
- 2006: introduction of options on the VIX
- VIX ETNs : started trading in 2009
- volumes on VIX futures and options skyrocketted since their introduction by CBOE

Trading VIX futures

- VIX futures track implied volatility of the S&P 500 $\rightarrow$ can naturally be used to hedge the volatility risk of a portfolio of options written on the S&P 500
- other natural use for VIX futures: hedge of options written on the VIX itself
- popular use for VIX futures: hedge positions on the US stock market thanks to the negative correlation between the VIX index and the S&P 500.

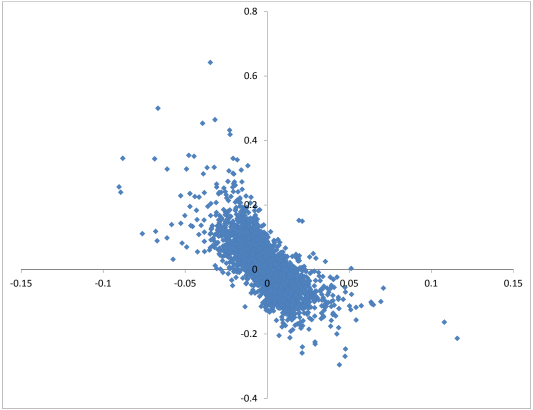


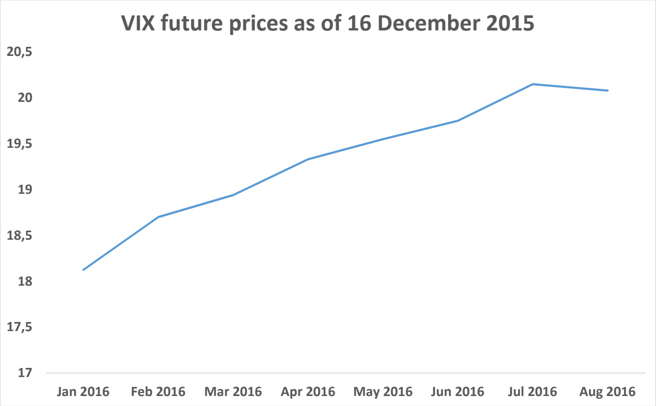
Figure: Scatter plot of VIX returns (Y axis) and S&P 500 returns (X axis)

Jan 1990 to Dec 2010

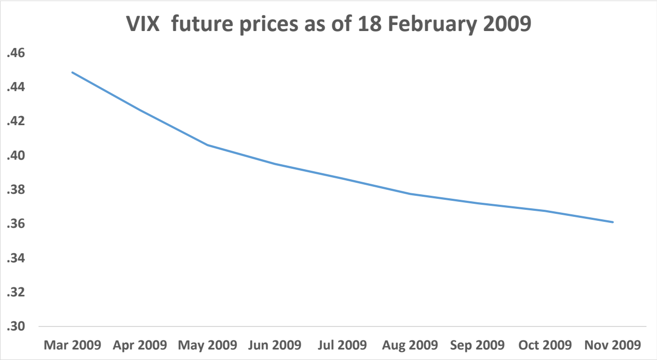
Term structure of VIX futures

- VIX futures have expiries every month
- recently, weekly VIX futures have been introduced
- most often, VIX futures term structure is in contango i.e. market expects (implied) volatility to increase in the future
- VIX futures term structure in backwardation typically following a market shock or in stressed periods

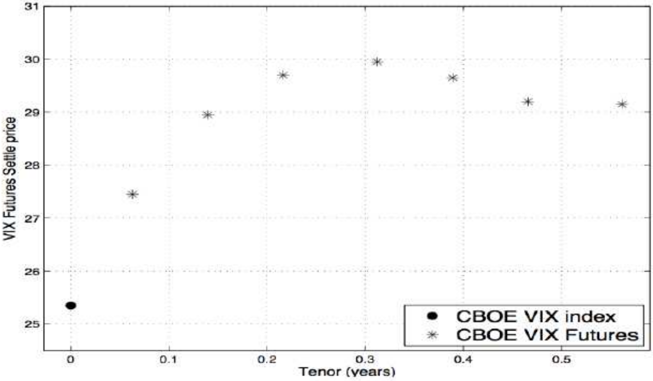
VIX futures term-structure: contango



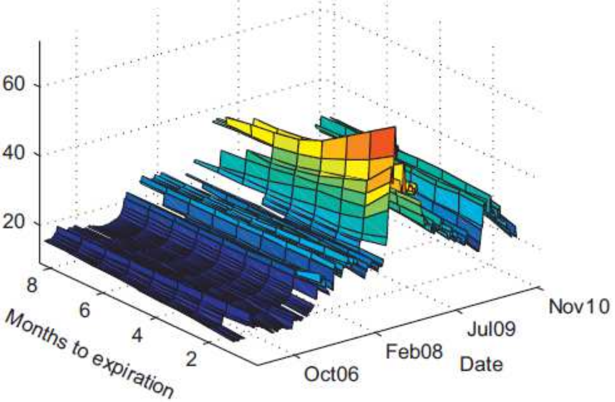
VIX futures term-structure: backwardation



Monday 29, June 2009



Feb 2006 – Dec 2010



Limitations of VIX futures market

- large denominations $\rightarrow$ difficult to trade for non institutional investors
- example: as of August 2016, one VIX futures contract written on a nominal value of around $12,000; minimum block trade size is 200 contracts, i.e. $2,400,000 nominal value
- bid-ask spreads significantly larger than equity markets (in $)
  - $50 per contract
  - $5 per contract if block trade
- rolling risk when futures expire

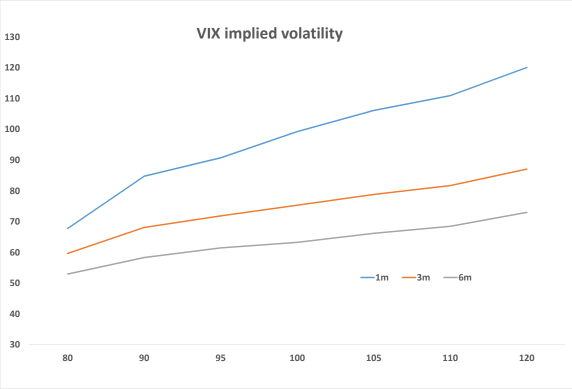


Figure: VIX implied volatility as of 16 December 2015. X axis: percentage strike; Y axis: implied volatility level

Exchange-traded notes on the VIX

- January 2009: Barclays launched the first ETNs on the VIX: VXX and VXZ
- since then, ETNs on the VIX florished (there exists more than thirty of them as of 2016) and their daily trading volumes and market caps amount to $ billions
- VIX ETNs are traded in stock exchanges
- holders of VIX ETN VXX receive a daily return equal to a combination of the daily return of two futures on the VIX, so as to track a synthetic 30-day VIX futures
- note that contrary to ETFs
    - ETNs are not necessarily backed by a basket of assets and the return delivered to the investor is guaranteed by the ETN issuer $\rightarrow$ ETNs are sensitive to the credit risk of the issuer
    - redemption of ETNs are more constrained

- VXX, is the first and most traded ETN on the VIX
- it is designed so as to track the performance of a synthetic 30-day futures contract on the VIX
- VXX provides "daily rolling long position in the first and second month VIX futures contracts" (Barclays prospectus)
- computation of VXX based on two VIX futures with maturities straddling a 30-day maturity $\rightarrow$ similar to the calculation of the VIX itself, calculated from S&P 500 option prices with maturities straddling a fixed 30-day time frame

- VXX, VXZ, XVZ, IVOP, VZZB Barclays
- VIIX, VIIZ, XIV, ZIV, TVIX Credit Suisse
- XVIX, VXAA, VXBB, VXCC, VXDD, VXEE, VXFF, XXV, AAVX,...FFVX UBS
- Options on VXX, VXZ, VIIX are available to trade on CBOE

VXX return

- I_t the value of the VXX at date t and $F(t, T)$ the VIX futures price with expiry T
- return of the VXX defined contractually by:

$$\frac{I_{t+1}-I_t}{I_t} = r + \frac{a_t(F(t+1,T_1)-F(t,T_1))+(1-a_t)(F(t+1,T_2)-F(t,T_2))}{a_t F(t,T_1)+(1-a_t)F(t,T_2)}$$

  where:
  - T_1 is the nearest month expiry and T_2 is the second next-month expiry
  - $a_t = \frac{T_1-t}{T_2-T_1}$ is the rolling factor
  - r is an interest rate indexed on US Treasury bills

VXX properties

- VXX return gives progressively more (resp. less) weight to futures with maturity T_2 (resp. T_1) as the rolling function a is decreasing with respect to time t
- $\rightarrow$ smooths the roll effect
- traded as a stock, with small denominations (price of VXX as of August 2016 around $35), small bid-ask spreads and large volumes
- enables investors to have positive exposure to the VIX index in a cheap and accessible manner

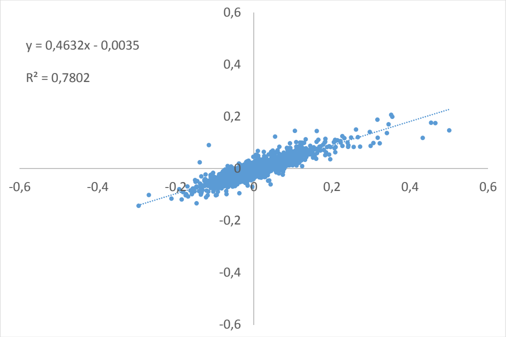


Figure: Linear regression of VIX vs VXX daily returns

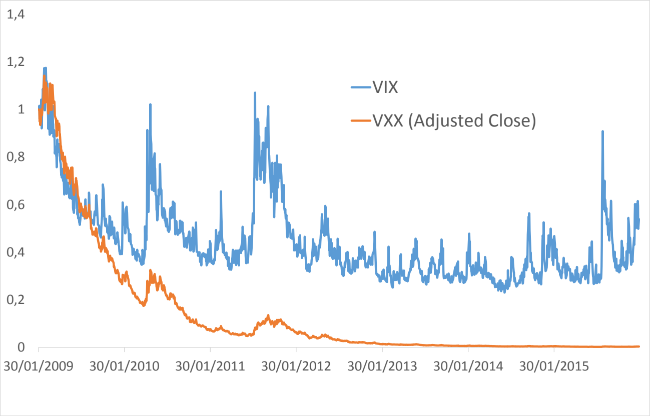
Correlation between VIX and VXX

Figure: One-year realized correlation between VIX and VXX daily returns

VXX underperformance

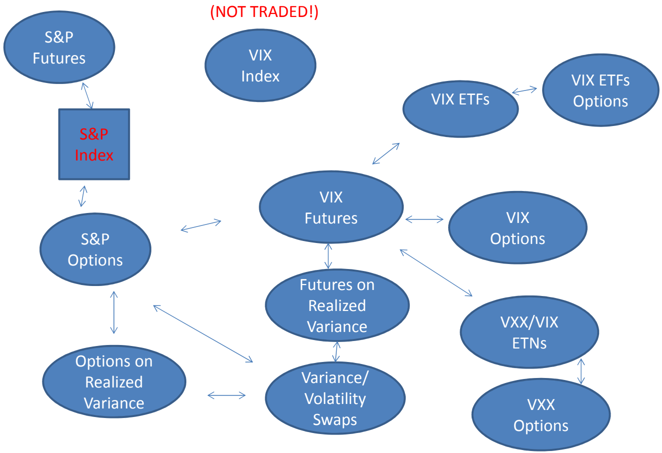
- since its creation in January 2009, VXX has consistently underperformed VIX
- Barclays VXX prospectus: "Your ETN is not Linked to the VIX index and the value of your ETN may be less than it would have been had your ETN been linked to the VIX Index."
- VXX performance has been so poor that it had to undergo 4 reverse stock splits (each time with a 1:4 ratio) in less than 8 years, in order to keep its price away from zero
- underperformance of the VXX when the VIX is in contango as VXX formula implies increasing exposure to the second futures, which is more expensive than the first futures when VIX term structure is in contango

VXX and VIX dynamics

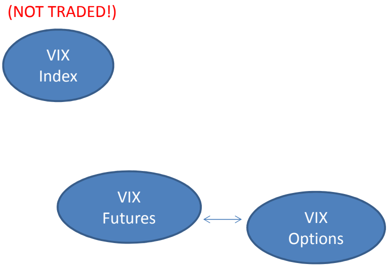


Need for a consistent framework

- On one hand, the introduction of ETNs on VIX enabled new investors to trade market volatility while enjoying the benfits of small nominals and bid-ask spreads
- on the other hand, the performance of those products raised concerns among market players and regulators
- the popularity of VXX calls for a quantitative framework for modeling VIX and its related ETNs in a consistent manner
- This would enable to link the characteristics of the VIX and its derivatives (daily return, implied volatility, skew) to that of its ETNs in a tractable way
- and what is more, better model VIX ETNs, and anticipate their unexpected or undesired behavior


(NOT TRADED!)
S&P Futures
VIX Index
VIX ETFs
VIX ETFs Options
S&P Index
VIX Futures
VIX Options
S&P Options
Futures on Realized Variance
VXX/VIX ETNs
Options on Realized Variance
Variance/ Volatility Swaps
VXX Options

VIX Standalone models



- VIX dynamics are directly specified
- VIX options can be priced in simple formulas!
- Whaley (1993), Schwert (1990), Pagan and Schwert (1990), Schwert (2011), Grunbichler and Longstaff (1996), Detemple and Osakwe (2000), Mencia and Sentana (2013), Psychoyios et al. (2010), Goard and Mazur (2013),...
- mean reverting models (affine but..)
- Alexander and Korovilas (2013) for a comprehensive empirical study of ETNs on the VIX

The GBM model of Whaley (1993)

$$\frac{dVIX_t}{VIX_t} = r_t dt + \sigma dW_t$$

$$\downarrow$$

$$F^{GBM}_{VIX}(t, T) = \mathbb{E}^Q \left[VIX_T | \mathcal{F}_t \right] = VIX_t e^{r(T-t)}$$

- GBM too simple! not consistent with the forward curve
- no mean reversion
- no skew

The SQR model of Grunbichler and Longstaff (1996)

$$dVIX_t = \alpha(\beta - VIX_t)dt + \Lambda\sqrt{VIX_t}dW_t$$

$$\downarrow$$

$$F_{VIX}^{SQR}(t,T) = \mathbb{E}^Q\left[VIX_T|\mathcal{F}_t\right] = \beta + (VIX_t - \beta)e^{-\alpha(T-t)}$$

- VIX mean reverting with known distribution
- simple expression for futures and vanillas
- only negative skew!

The LOU model of Detemple and Osakwe (2000)

$$d \log VIX_t = \alpha(\beta - \log VIX_t)dt + \Lambda dW_t$$

$$\downarrow$$

$$F_{VIX}^{LOU}(t, T) = \mathbb{E}^Q \left[VIX_T|\mathcal{F}_t\right] = VIX_t^{\phi(t,T)} e^{\mu(t,T)+\beta(1-\phi(t,T))}$$

- VIX mean reverting and lognormal
- simple expression for futures and vanillas (Black 1976)
- no skew!

LOU+jumps extension of Psychoyios, Dotsis and Markellos (2009)

$$d \log VIX_t = \alpha(\beta - \log VIX_t)dt + \Lambda dW_t + JdN_t$$

- VIX mean-reverting logarithmic jump process (MRLRJ)
- simple expression for futures and vanillas
- able to fit positive skew!

The SQR and LOU extensions of Mencia and Sentana (2013)

$$dVIX_t = \alpha(\beta_t - VIX_t)dt + \Lambda\sqrt{VIX_t}dW_t^{VIX}$$
$$d\beta_t = \bar{\alpha}(\bar{\beta} - \beta_t)dt + \bar{\Lambda}\sqrt{\beta_t}dW_t^{\beta}$$

- simple expression for futures
  , no positive skew!

$$d\log VIX_t = \alpha(\beta_t - \log VIX_t)dt + \Lambda dW_t^{VIX}$$
$$d\beta_t = \bar{\alpha}(\bar{\beta} - \beta_t)dt + \bar{\Lambda}\sqrt{\beta_t}dW_t^{\beta}$$

- affine! Quasi closed-form formulas for vanillas (FFT)

Calibration issues

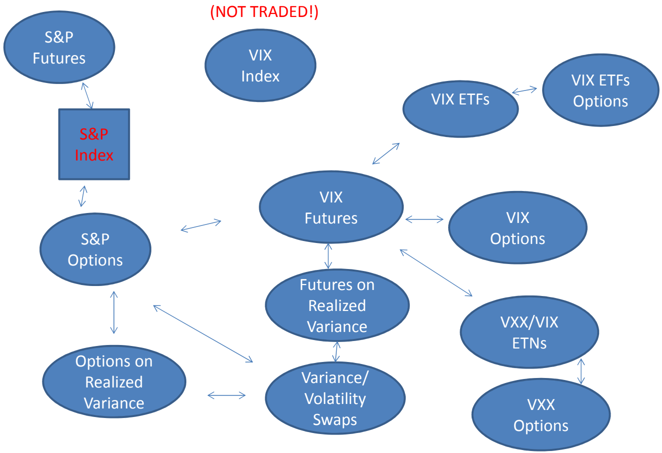
- LOU and SQR can be generalized by adding jumps and/or stochastic vol...
- scarse fit for SQR models (rigid skew..)

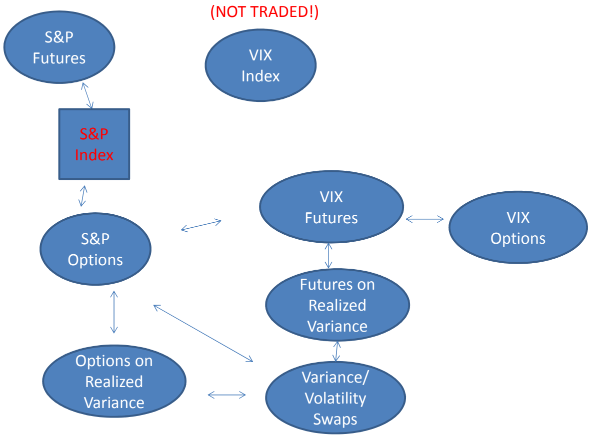
$$\downarrow$$

- MRLR, MRLRJ, MRLRSV and MRLRSVJ
- still possible to compute prices of futures and vanillas (even if.. time dependent Riccati for MRLRSV..)
- almost NO added value when considering jumps AND stochastic vol Kaeck and Alexander (2010), Bao (2013)...
- multifactor specification not easy to handle..

- joint dynamics of S&P500 index (SPX) and its stochastic instantaneous volatility
- VIX is derived by its definition as square root of forward realized variance of the SPX
- Zhang and Zhu (2007), Lin and Chang (2010), Cont and Kokholm (2013),...

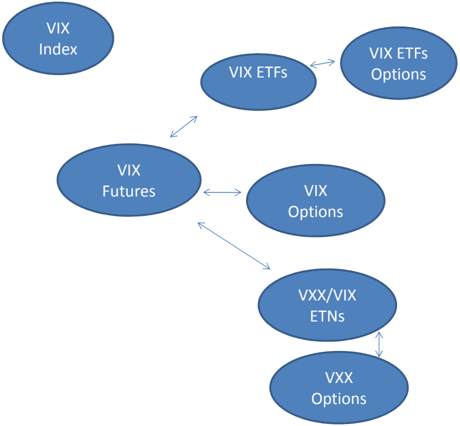
Volatility as an asset class




(NOT TRADED!)
S&P Futures
VIX Index
S&P Index
VIX Futures
VIX Options
S&P Options
Futures on Realized Variance
Options on Realized Variance
Variance/ Volatility Swaps

Our VIX - VXX Consistent Model

(NOT TRADED!)



Desiderata

We want a methodology to model the VIX and VXX in a consistent manner

- that fits the term structure of futures (contango/backwardation)
- and fits the implied volatility surface (positive skew)
- with a mean reversion
- Remark: VIX is not a martingale under the pricing measure (not tradable)

- The stochastic drift of the VIX can be interpreted as a "convenience yield" term, which is commonly used to reproduce the variability of the term structure of futures on commodities (Gibson and Schwartz, 1990; Carmona and Ludkovski, 2004).

$$\frac{dS_t}{S_t} = (r_t - \delta_t)dt + \sigma dW_t^1$$

$$d\delta_t = \kappa(\theta - \delta_t)dt + \gamma dW_t^2$$

- Although the VIX does not pay any dividend, the stochastic drift is mathematically equivalent to a stochastic dividend yield

A stochastic volatility model for VIX

Consider the stochastic factor (note ω redundant!)

$$dX_t = \kappa(\theta - X_t)dt + \omega\sqrt{\alpha X_t + \beta}dW_t, \quad X_0 = x.$$

The dynamics of the VIX:

$$\frac{dS_t}{S_t} = (r - \delta X_t)dt + \sigma\sqrt{\alpha X_t + \beta}dZ_t, \quad S_0 > 0,$$

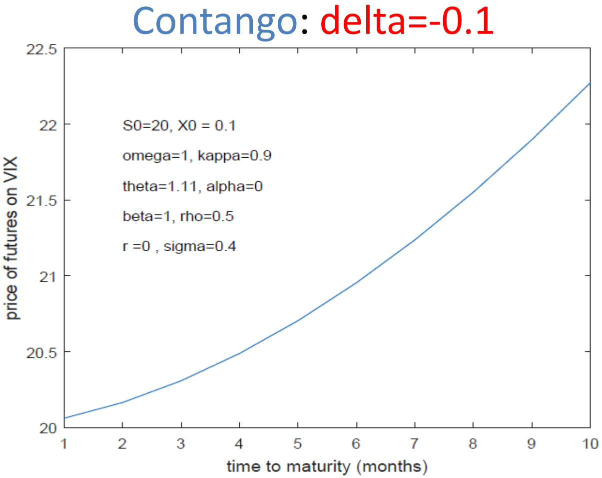
GBM (Whaley, 1993) ($\alpha = \delta = 0$), Heston ($\beta = \delta = 0$), log-normal O-U (Detemple and Osakwe, 2000) ($W = Z$ and $X = \ln(S)$) or Gibson and Schwartz (1990) ($\alpha = 0$)

- Futures prices are explicit!

$$F(t,T) = S_t e^{\Phi(t,T)+X_t\Psi(t,T)},$$

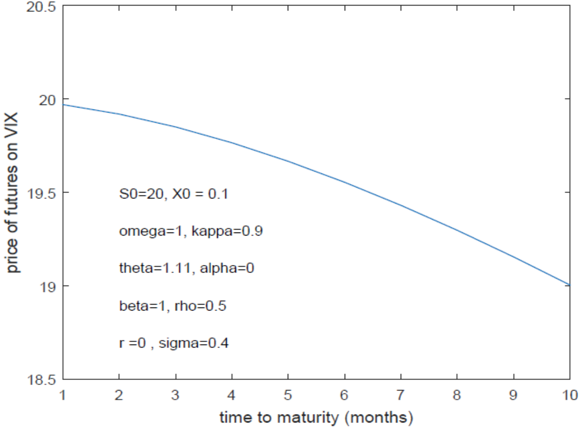
- Fourier transform of the log price $\mathbb{E}[e^{i\epsilon \ln(S_T)}]$ can be computed explicitly!
- prices of vanillas can be efficiently computed through the FFT method
- the model can be calibrated on the vanillas and futures on VIX!
- easy to calibrate also in the multifactor specification

δ controls the Contango/Backwardation



δ controls the Contango/Backwardation

Backwardation: delta=0.1



Easy in different ways:

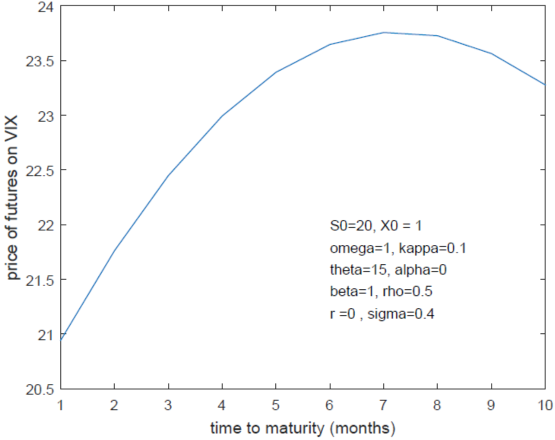
- adding another factor with δ_1, δ_2
- changing parametrization $Y_t = \alpha X_t + \beta$

$$\frac{dS_t}{S_t} = (r + \bar{\delta} - \tilde{\delta} Y_t)dt + \tilde{\sigma}\sqrt{Y_t}dZ_t$$

$$dY_t = \tilde{\kappa}(\tilde{\theta} - Y_t)dt + \tilde{\omega}\sqrt{Y_t}dW_t$$

- more stable in the calibrations (β negative!)
- less problems in the MC simulations

Humped: deltabar=1, delta=0.5



From the pricing supplement of the VXX Prospectus of Barclays

$$\frac{dI_t}{I_t} = rdt + \frac{a(t)dF(t,T_1) + (1-a(t))dF(t,T_2)}{a(t)F(t,T_1) + (1-a(t))F(t,T_2)},$$

- $F(t, T_i), i = 1, 2$ are the first and second expiring futures
- $a(t) = \frac{T_2 - (t+\tau)}{T_2 - T_1}$

Then

$$\begin{aligned} \frac{dI_t}{I_t} &= \frac{dS_t}{S_t} + \delta X_t dt \\ &+ \omega \frac{a(t)F(t,T_1)\Psi(t,T_1) + (1-a(t))F(t,T_2)\Psi(t,T_2)}{a(t)F(t,T_1) + (1-a(t))F(t,T_2)} \sqrt{\alpha X_t + \beta} dW_t \end{aligned}$$

we can identify the term $\delta X_t dt$ as the systematic loss due to contango, in the case $\delta < 0$.

Order zero in ω:

- X becomes deterministic
- VIX is lognormal

$$\frac{d\bar{S}_t}{\bar{S}_t} = (r - \delta\bar{X}_t)dt + \sigma\sqrt{\alpha\bar{X}_t + \beta}dZ_t$$

- vol(VXX) $=$ vol(VIX) ! and vol(VIX futures)

$$\frac{dI_t}{I_t} = rdt + \sigma\sqrt{\alpha\bar{X}_t + \beta}dZ_t$$

- no smile, no skew !

$$\frac{dI_t}{I_t} = rdt + \sigma\sqrt{\alpha\bar{X}_t + \beta}\left(1 + \omega\frac{\alpha Y_t}{2(\alpha\bar{X}_t + \beta)} + o(\omega)\right)dZ_t + \omega c(t)\sqrt{\alpha\bar{X}_t + \beta}dW_t + o(\omega).$$

- Y_t stochastic : $Y_t = e^{-\kappa t}\int_0^t e^{\kappa s}\sqrt{\alpha\bar{X}_s + \beta}dW_s$
- $c(t)$ deterministic : $c(t) = \frac{a(t)\bar{F}(t,T_1)\bar{\Psi}(t,T_1)+(1-a(t))\bar{F}(t,T_2)\bar{\Psi}(t,T_2)}{a(t)\bar{F}(t,T_1)+(1-a(t))\bar{F}(t,T_2)}$
- Due to the stochastic term Y_t in the volatility, we expect an impact on the skew in VIX and VXX at the first order in ω

$$
\begin{aligned}
\langle \frac{dI_t}{I_t}, dVar_I \rangle \approx \\
\color{red}{Skew_{VXX}} =& \frac{\rho}{\left(1+\omega\left(\frac{\alpha Y_t}{\alpha \bar{X}_t+\beta}+\frac{2\rho c(t)}{\sigma\sqrt{\alpha \bar{X}_t+\beta}}\right)\right)^{1/2}} \\
=& \rho\left(1-\frac{\omega}{2}\left(\frac{\alpha Y_t}{\alpha \bar{X}_t+\beta}+\frac{2\rho c(t)}{\sigma\sqrt{\alpha \bar{X}_t+\beta}}\right)\right)+o(\omega).
\end{aligned}
$$

- At the order 0 in ω the skew is equal to the constant ρ
- At the order 1, we have a term structure of the implied volatility and a stochastic skew

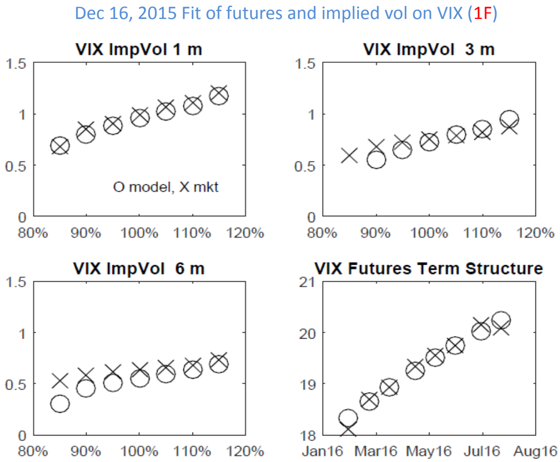
Link between the price of a Call on the VXX and that of a Call on the VIX:

$$\frac{1}{I_0} C_I(T,k) = \frac{1}{S_0} e^{\overline{\delta} T} C_S(T, ke^{-\overline{\delta} T})$$

- The price of the Call option on the VXX with strike k can hence be deduced from the (scaled) price of a Call option on the VIX, but with equivalent strike $ke^{-\overline{\delta} T}$
- the link between the skew of the VXX $\frac{\partial v_I}{\partial k}$ and that of the VIX $\frac{\partial v_S}{\partial k}$ can be found by deriving the price with respect to k:

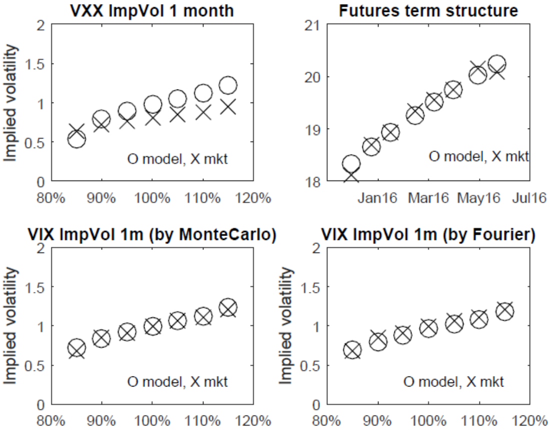
$$\frac{\partial C_{B\&S}}{\partial k}(\overline{\delta} = 0) + \frac{\partial C_{B\&S}}{\partial v}\frac{\partial v_I}{\partial k} = \frac{\partial C_{B\&S}}{\partial k}(\overline{\delta}) + e^{-\overline{\delta} T}\frac{\partial C_{B\&S}}{\partial v}\frac{\partial v_S}{\partial k}$$

- we take 2 trading days
- 16 dec 2015 contango
- 19 january 2016 backwardation
- fit the futures term structure and the imp vol (1, 3, 6 months)
- we compare market prices of imp vol on VXX with our MC imp vol !


Dec 16, 2015 Fit of futures and implied vol on VIX (1F)
VIX ImpVol 1 m
O model, X mkt
VIX ImpVol 3 m
VIX ImpVol 6 m
VIX Futures Term Structure
1.5
1
0.5
0
80%
90%
100%
110%
120%
21
20
19
18
Jan16
Mar16
May16
Jul16
Aug16

Seat belt fastened: here we go!

Dec 16, 2015 Fit of futures and 1 month implied vol on VIX and VXX



Dec 16, 2015 Fit of futures and 3 months implied vol on VIX and VXX

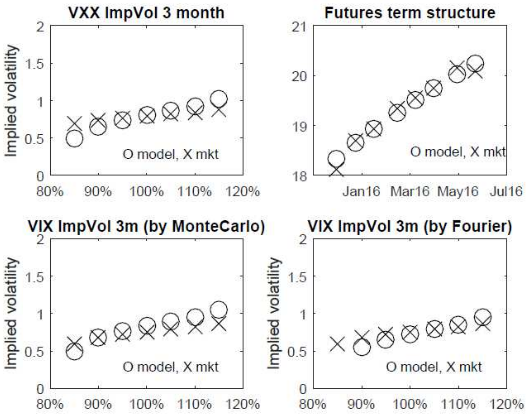


Figure: Dec 16, 2015 Fit of futures and 3 month implied vol on VIX and VXX

6 months fit

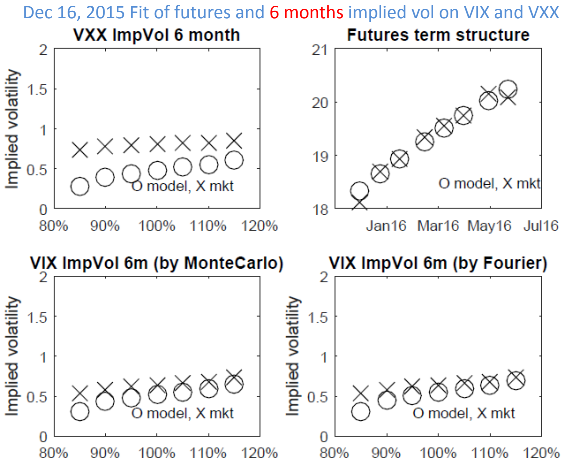
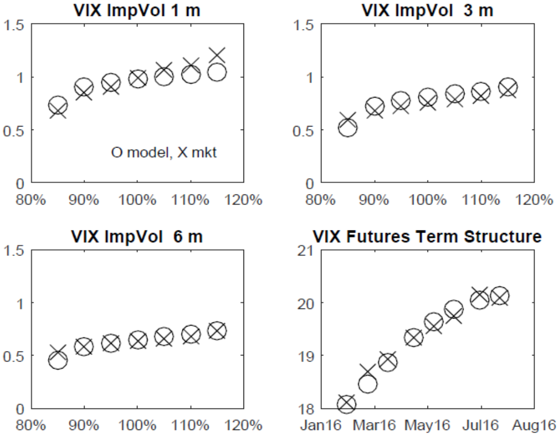


Figure: Dec 16, 2015 Fit of futures and 6 month implied vol on VIX and VXX

Not satisfied? Add another factor!

Dec 16, 2015 Fit of futures and implied vol on VIX (2F)



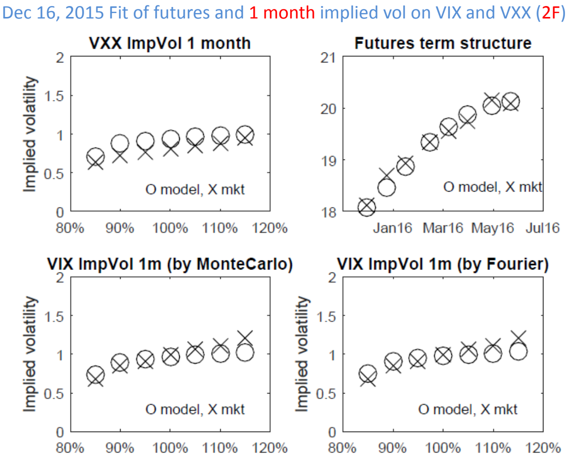


Figure: Dec 16, 2015 Fit of futures and 1 month implied vol on VIX and VXX with 2 factors specification

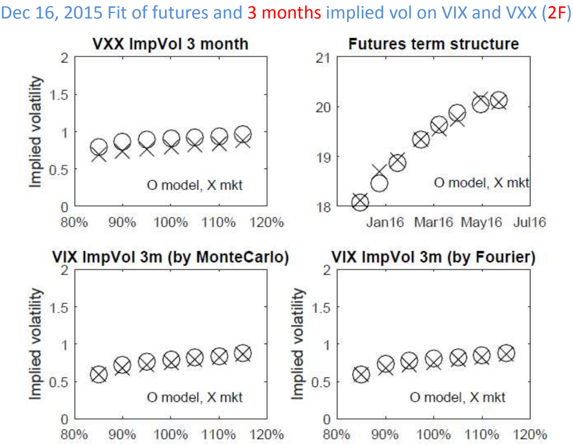


Figure: Dec 16, 2015 Fit of futures and 3 month implied vol on VIX and VXX with 2 factors specification

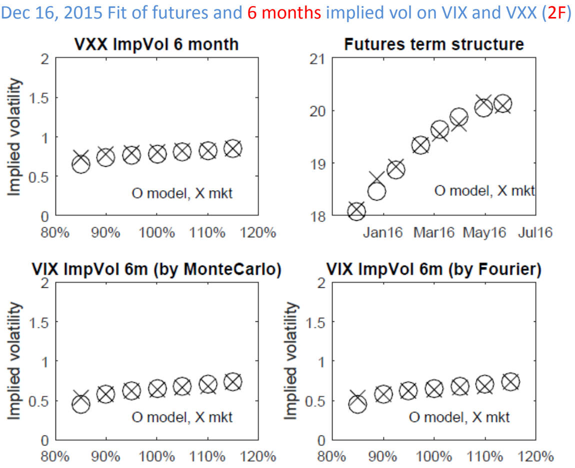


Figure: Dec 16, 2015 Fit of futures and 6 month implied vol on VIX and VXX with 2 factors specification

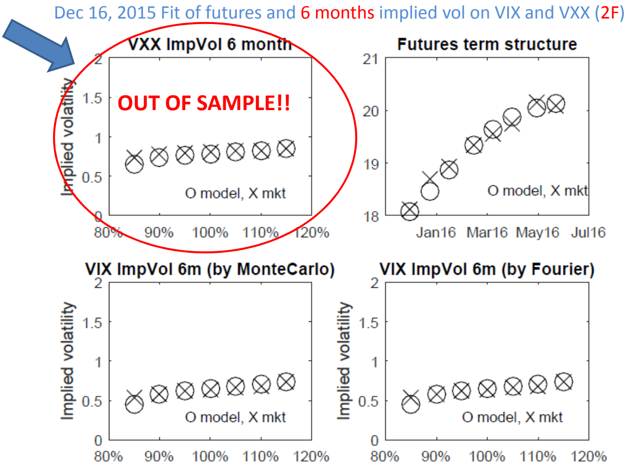
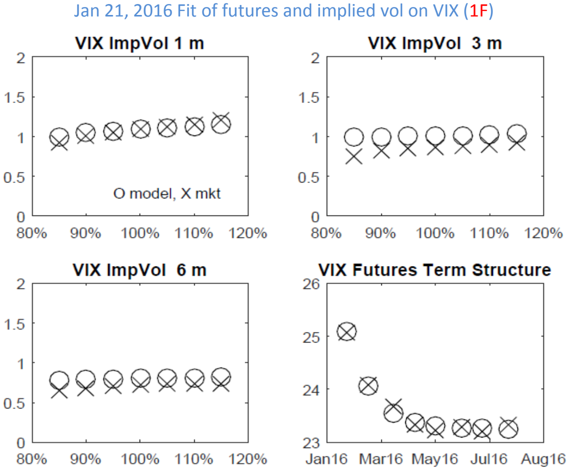
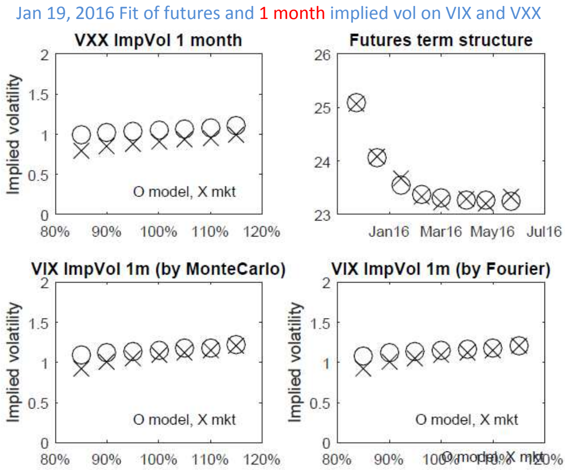


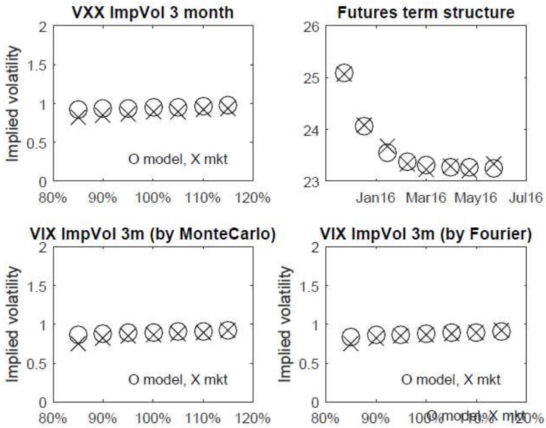
Figure: Dec 16, 2015 Fit of futures and 6 month implied vol on VIX and VXX with 2 factors specification

Fit of the VIX market on 19 January 2016: backwardation

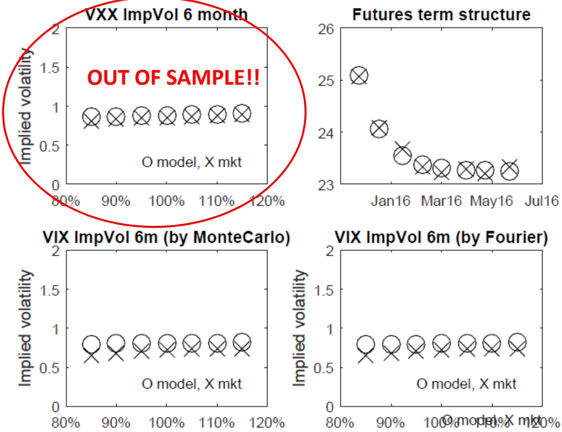



Jan 19, 2016 Fit of futures and 1 month implied vol on VIX and VXX
VXX ImpVol 1 month
Implied volatility
O model, X mkt
80% 90% 100% 110% 120%
Futures term structure
Jan16 Mar16 May16 Jul16
VIX ImpVol 1m (by MonteCarlo)
Implied volatility
O model, X mkt
VIX ImpVol 1m (by Fourier)
Implied volatility
O model, X mkt

Jan 19, 2016 Fit of futures and 3 months implied vol on VIX and VXX



Jan 19, 2016 Fit of futures and 6 months implied vol on VIX and VXX



Conclusion in the stochastic volatility framework

- first VIX-VXX consistent approach
- affine model for VIX able to reproduce futures term structure and implied volatility
- efficient pricing $\rightarrow$ calibration on market data on VIX options
- MC simulation of implied volatility surface for VXX
- quite good global fit for the imp vol on real data on VXX!
- ready to price other products on VIX (VXZ, UVXY, ...)

Main problem

The main problem to solve is the fact that the price process of the ETP strategy is non-Markov, since it depends on the futures contract prices.

In the case of an ETP based on N futures contracts $F_t(T_i)$, $i = 1, \dots, N$, we have the following dynamics

ETP dynamics

$$\frac{dV_t}{V_t} = (r_t - \phi_t)dt + \sum_{i=1}^{N} \omega_t^i \frac{dF_t(T_i)}{F_t(T_i)} \tag{1}$$

where ω_t^i represent the investment percentage (or weights) in the futures contracts

We start by introducing a generic stochastic volatility dynamics for futures prices under the risk-neutral measure:

Futures dynamics

$$dF_t(T_i) = \nu_t(T_i) \cdot dW_t \tag{2}$$

where

- $F_t(T_i)$ is the vector of futures prices (each entry refers to a different quoted maturity T_i);
- $\nu_t(T_i)$ are vector processes to be defined (and possibly depending on the future price itself);
- W_t is a vector of standard Brownian motions under the risk-neutral measure.

We shall use the following version of the Markovian projection

Gyongy Lemma (1986)

Let $X(t)$ be given by

$$dX(t) = \alpha(t)dt + \beta(t)dW(t), \tag{3}$$

where $\alpha(\cdot)$, $\beta(\cdot)$ are adapted bounded stochastic processes such that (3) admits a unique solution. If we define $a(t,x)$, $b(t,x)$ by

$$\begin{aligned} a(t,x) &= \mathbb{E}(\alpha(t) \,|\, X(t) = x), \\ b^2(t,x) &= \mathbb{E}(\beta^2(t) \,|\, X(t) = x), \end{aligned} \tag{4}$$

then the SDE

$$\begin{aligned} dY(t) &= a(t, Y(t))dt + b(t, Y(t))dW(t), \\ Y(0) &= X(0), \end{aligned} \tag{5}$$

admits a weak solution $Y(t)$ that has the same one-dimensional distributions as $X(t)$.

Markovian projection

The process $Y(\cdot)$ is called the **Markovian projection** of the process $X(\cdot)$.

Heston Model

Starting with

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sqrt{\nu(t)}S(t)dW(t), \\ d\nu(t) &= \kappa(\theta - \nu(t))dt + \xi\sqrt{\nu(t)}dZ(t), \end{aligned} \tag{6}$$

in order to have an equivalent process, we need to compute the following quantities

$$\begin{aligned} a(t,x) &= \mathbb{E}(\mu S(t) \,|\, S(t) = x), \\ b^2(t,x) &= \mathbb{E}(\nu(t)S(t)^2 \,|\, S(t) = x), \end{aligned} \tag{7}$$

and simulate the process

$$dY(t) = a(t, Y(t))dt + b(t, Y(t))dW(t). \tag{8}$$

Markovian projection

$$d\bar{F}_t(T_i) = \eta_F\left(t, T_i, \bar{F}_t(T_i)\right) d\bar{W}_t^i, \quad \eta_F(t, T_i, K) := \sqrt{\mathbb{E}\left[\|\nu_t(T_i)\|^2 \mid F_t(T_i) = K\right]}, \tag{9}$$

and

$$\frac{d\bar{V}_t}{\bar{V}_t} = (r_t - \phi_t)\, dt + \eta_V(t, \bar{V}_t) d\bar{W}_t^0, \quad \eta_V(t, K) := \sqrt{\mathbb{E}\left[\sum_{i,j=1}^N \omega_t^i \nu_t(T_i) \cdot \nu_t(T_j)\, \omega_t^j \,\middle|\, V_t = K\right]}, \tag{10}$$

where

- $\eta_F(t, T_i, K)$ is the futures local volatility function,
- $\eta_V(t, K)$ is the VXX local volatility function.

Calibration of local volatility functions

We calibrate $\eta_F(t, T_i, K)$ to options on futures and $\eta_V(t, K)$ to options on the VXX.

In order to do so, we model futures prices in terms of a common process s_t which we can identify as the price of a rolling futures contract: $F_t(T_i) = F_0(T_i)s_t$, where s_t has the following dynamics

Driving risk process

$$ds_t = a(t)\,(1 - s_t)\,dt + \eta\,(t, s_t)\,s_t dW_t^s, \quad s_0 = 1, \tag{11}$$

where

- a is the mean reversion speed;
- η is a function of time and price to be determined by the calibration of options of futures;
- W^s is a standard Brownian motion under the risk-neutral measure.

We can finally get a closed formula for the futures price:

Futures dynamics

$$F_t\,(T_i) = F_0\,(T_i)\left(1 - (1 - s_t)\,e^{-\int_t^{T_i} a(u)du}\right), \tag{12}$$

where $F_0\,(T_i)$ is the futures with expiry T_i observed today in the market.

The next step is to ensure the two constraints given by the Markov projection by properly choosing the vector processes $\nu_t(T_i)$. We use the following representation

Vector process

$$\nu_t(T_i) \doteq \eta_F(t, T_i, F_t(T_i)) \frac{v_t}{\sqrt{\mathbb{E}\left[v_t^2 | F_t(T_i)\right]}} R(t, T_i, V_t) \tag{13}$$

where

- v_t is a scalar process independent of the futures and VXX dynamics;
- R has the role of local correlation and it is a vector function of the VXX price such that $\|R\|^2 = 1$.

Remark

Equation (13) satisfies by construction equation (9), so that the model reprices plain-vanilla options on futures correctly.

Then, we analyze the constraint on the VXX dynamics by substituting equation (13) into equation (10). We get

Main equation

$$\eta_V^2(t,K) = \sum_{i,j=1}^{N} R\left(t,T_i,K\right) \cdot R\left(t,T_j,K\right) \mathbb{E}\left[v_i^2 \omega_t^i \omega_t^j \frac{\eta_F\left(t,T_i,F_t\left(T_i\right)\right)}{\sqrt{\mathbb{E}\left[v_t^2 \mid F_t\left(T_i\right)\right]}} \frac{\eta_F\left(t,T_j,F_t\left(T_j\right)\right)}{\sqrt{\mathbb{E}\left[v_t^2 \mid F_t\left(T_j\right)\right]}} \,\middle|\, V_t = K \right] \tag{14}$$

Hence, we have to solve Equation (14) for the unknown local function $R(t, T_i, K)$ to complete the calibration to VXX plain vanilla prices.

Then, we choose to implement our modelling framework by using as local correlation vectors the following parameterization:

Local correlation parametrization

$$R\,(t, T_1, K) := [1, 0], \quad R\,(t, T_2, K) := [\rho(t,K), \sqrt{1-\rho(t,K)^2}] \tag{15}$$

where the local function $\rho(t,K)$ has to be derived solving Equation (14). For simplicity we initially assume $v_t \doteq 1$, which gives

$$\rho(t,K) = \frac{\eta_V^2(t,K) - A_1(t,K) - A_2(t,K)}{2A_{12}(t,K)}, \tag{16}$$

where

$$\begin{aligned} A_1(t,K) &= \mathbb{E}\left[\tfrac{\alpha_1^2(t)F_t^2(T_1)\eta_F^2(t,T_1,F_t(T_1))}{(\alpha_1(t)F_t(T_1)+\alpha_2(t)F_t(T_2))^2} \mid V_t = K\right] \\ A_2(t,K) &= \mathbb{E}\left[\tfrac{\alpha_2^2(t)F_t^2(T_2)\eta_F^2(t,T_2,F_t(T_2))}{(\alpha_1(t)F_t(T_1)+\alpha_2(t)F_t(T_2))^2} \mid V_t = K\right] \\ A_{12}(t,K) &= \mathbb{E}\left[\tfrac{\alpha_1(t)\alpha_2(t)F_t(T_1)F_t(T_2)\eta_F(t,T_1,F_t(T_1))\eta_F(t,T_2,F_t(T_2))}{(\alpha_1(t)F_t(T_1)+\alpha_2(t)F_t(T_2))^2} \mid V_t = K\right] \end{aligned} \tag{17}$$

In order to succeed in our aim, we need to be able to compute the conditional expectation appearing in the previous formula.
We refer to the procedure developed in *Guyon and Henry-Labordre* (2012):

Conditional Expectation computation

We are using the following approximation

$$\mathbb{E}\left[X_t \mid Y_t = K\right] \approx \frac{\mathbb{E}\left[X_t \delta^{\epsilon}\left(Y_t - K\right)\right]}{\mathbb{E}\left[\delta^{\epsilon}\left(Y_t - K\right)\right]}, \tag{18}$$

where δ^{ϵ} is a suitably defined mollifier of the Dirac delta depending on a smoothing coefficient ϵ. Standard choice may include Gaussian kernel or quartic kernel.

The option market as on November 7, 2019

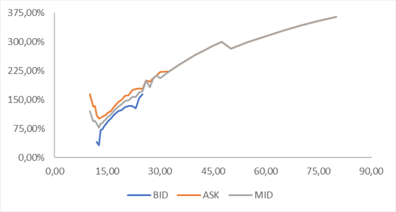
VIX futures call options		
Maturities	Number of options	Strikes range (min − max)
T_1: November 20, 2019	40	10 − 80
T_2: December 18, 2019	40	10 − 80
T_3: January 22, 2020	35	10 − 80
T_4: February 19, 2020	35	10 − 80

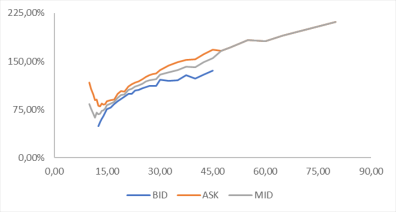
Table: VIX futures call options data set as on T_0: November 7, 2019.

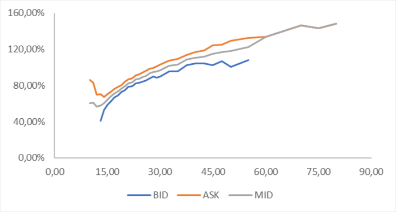
VXX call options		
Maturities	Number of options	Strikes range (min − max)
T_1: November 20, 2019	65	8 − 60
T_2: December 18, 2019	59	2 − 60
T_3: January 22, 2020	53	10 − 90

Table: VXX call options data set as on T_0: November 7, 2019.

The VIX option market as on November 7, 2019







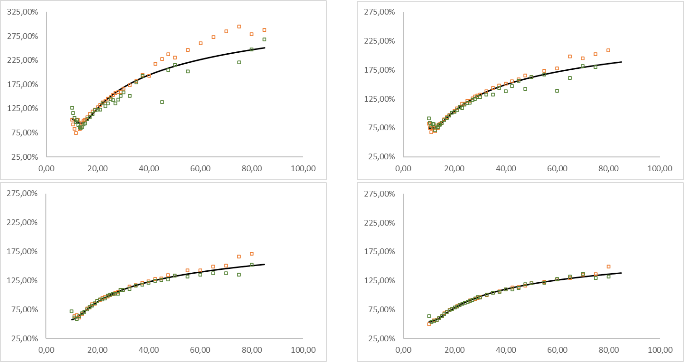
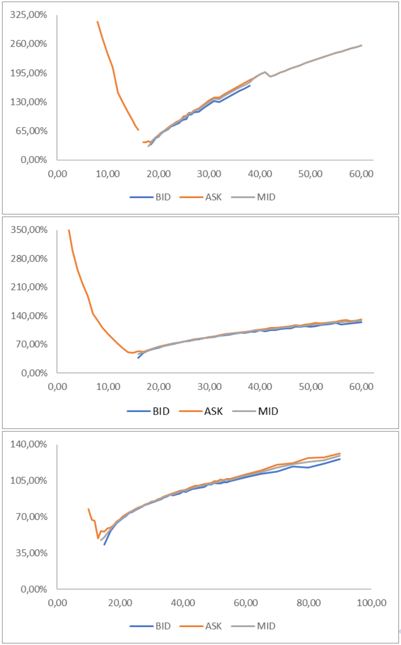


Figure: Application of SVI methodology to VIX options: the continuous black line represents the volatility smile using the SVI formulae. Orange Call market quotes, green Put market quotes

The VXX option market as on Nov. 7, 2019



SVI for VXX

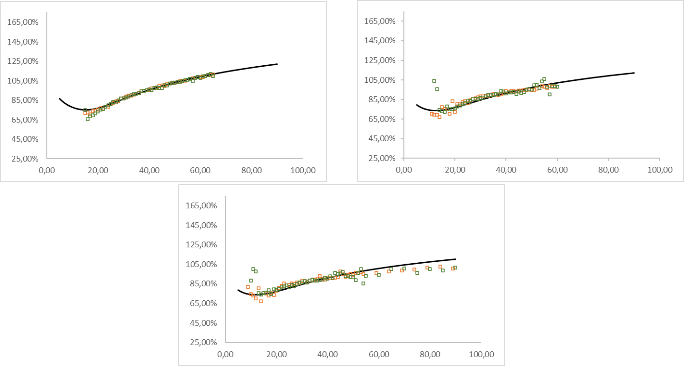


Figure: Application of SVI methodology to VXX options: the continuous black line represents the volatility smile using the SVI formulae. Orange squares represent Call market quotes, while the green squares represent Put market quotes. From the top left corner, as on May 24, 2019, pictures refer to maturities September 20, 2019, December 20, 2019 and January 17, 2020, respectively.

Thanks to the previous formulae, plain-vanilla options on futures can be calculated as plain-vanilla options on the process s_t:

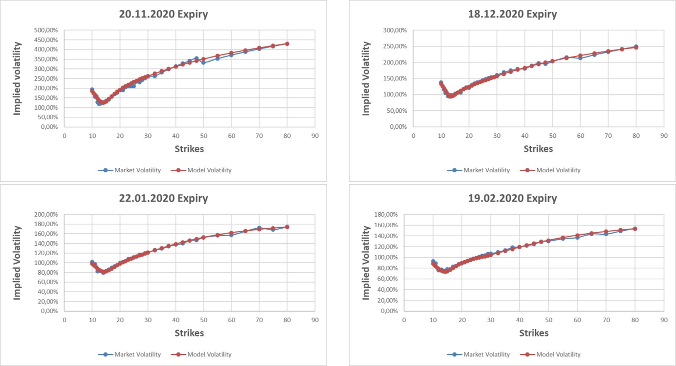


Figure: Market plain vanilla of VIX and model plain vanilla, with a mean reversion speed $a = 8$.

Similar to the previous slide, we have the analogous for VXX:

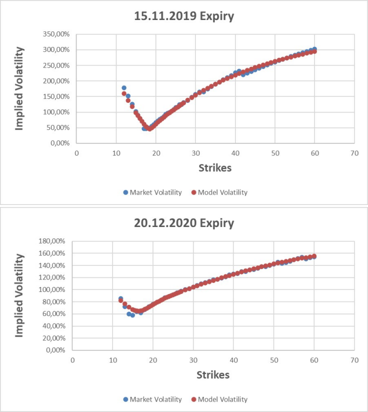


Figure: Market plain vanilla of VXX and model plain vanilla.

Conclusions

- Purely stochastic vol not satisfactory
- Purely local vol not flexible enough
- Stochastic/local vol framework necessary to fit both VIX-VXX markets
- more extensive empirical work has to be done..

THANKS FOR YOUR ATTENTION!