Emergence of shocks in large pool credit contagion models and resulting optimal bailout strategies

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CHRISTOPH REISINGER

with Christa Cuchiero and Stefan Rigger

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Outline

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- Background structural credit models with mutual obligations.
- Large pool limit and the supercooled Stefan problem.
- A central agent controlling the number of defaults:
  - well-posedness and ‘propagation of chaos’;
  - numerical solution of mean-field control problem;
  - illustration of strategies, losses, and cost to central agent.

‘Structural’ credit risk model

à la Merton, Cox,...

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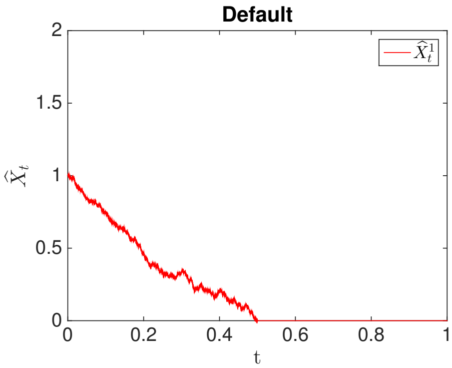
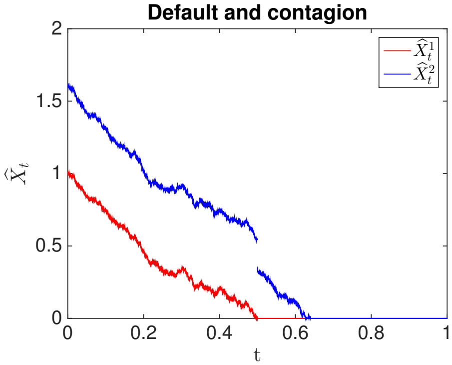
- The assets of a bank minus its liabilities are modelled by a process X_t.
- The bank is considered defaulted if X_t hits zero.



$\widehat{X}_t^1$ the stopped process of bank 1.

Mutual liabilities

- On default, bank 1 fails to meet some of its liabilities to bank 2.
- Bank 2's equity process jumps downwards.



$\widehat{X}^1_t$, $\widehat{X}^2_t$ the stopped processes of banks 1 and 2.

Credit contagion

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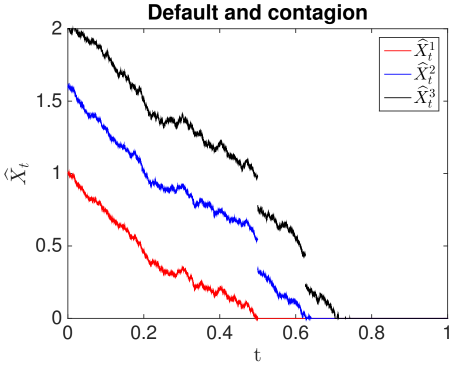
- On default of bank 2, it fails to meet some of its liabilities to bank 3.
- Bank 3's equity process jumps downwards.



$\widehat{X}_t^1$, $\widehat{X}_t^2$, $\widehat{X}_t^3$ the stopped processes of banks 1, 2, and 3.

N-bank model

We assume the $X^{i,N}$ satisfy, for $i = 1, \ldots, N$,

$$X_t^{i,N} = X_{0-}^{i,N} + B_t^{i,N} - \alpha \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau_{i,N} \le t\}},$$

where

$$\tau_{i,N} = \inf\{t : X_t^{i,N} \le 0\},$$

- $X_{0-}^{i,N}$ are non-negative i.i.d. random variables,
- $(B^{i,N})_{1\le i\le N}$ is an N-dimensional standard Brownian motion, independent of $X_{0-} = (X_{0-}^{i,N})_{1\le i\le N}$,
- and $\alpha \ge 0$ is a contagion parameter measuring the amount of inter-firm lending.

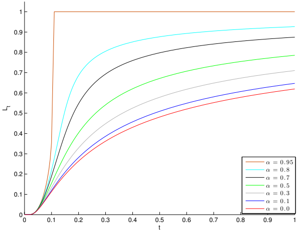
Mean-field limit

e.g., Hambly, Ledger, & Sojmark (2019)

For $N \to \infty$, a representative bank follows

$$X_t = X_{0-} + B_t - \alpha\,\mathbb{P}\Big(\inf_{0 \le s \le t} X_s \le 0 \Big), \quad t \ge 0,$$

where B is a standard Brownian motion independent of X_{0-}.



$L_t = \mathbb{P}\Big(\inf_{0 \le s \le t} X_s \le 0 \Big)$, $X_{0-} = 0.5$

- Cartoon model for credit risk with default contagion.
- Realistic α depends on mutual liabilities, recovery rates, asset vol – anything 0.5 to 5, or even higher (see Lipton, Kaushansky, & R, 2019)
- Similar models in neuroscience (see Delarue, Inglis, Rubenthaler, &Tanré, 2015)

(Probabilistic) problem statement

McKean–Vlasov structure

Let X_{0-} be a random variable with probability density f.

Consider the problem of finding a non-decreasing Λ such that the stochastic process

$$X_t = X_{0-} + B_t - \Lambda_t, \quad t \geq 0$$

satisfies the constraint

$$\Lambda_t = \alpha\,\mathbb{P}\Big(\inf_{0\leq s\leq t} X_s \leq 0\Big), \quad t \geq 0,$$

where B is a standard Brownian motion independent of X_{0-}.

Let $\tau := \inf\{t \geq 0 :\ X_t \leq 0\}$.

‘Blow up’ scenario

Hambly, Ledger, & Sojmark (2019); see also Delarue, Inglis, Rubenthaler, &Tanré (2015)

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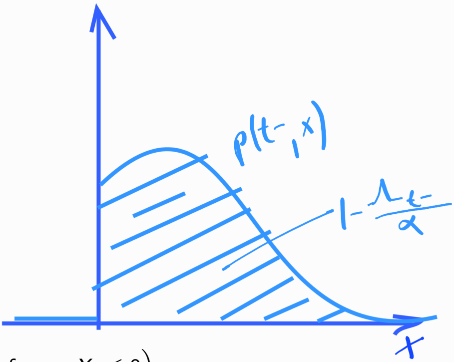
HLS19: If $\alpha > 2\mathbb{E}[X_0]$, $t \to \Lambda_t$ cannot be continuous for all t.*

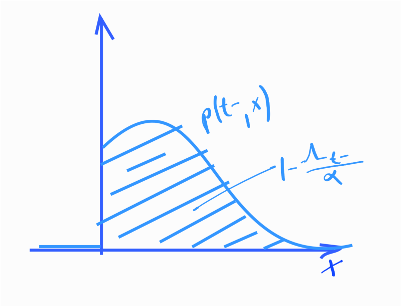


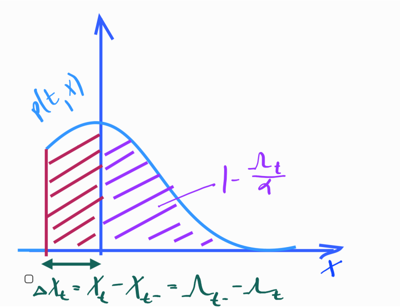
- $\Lambda_t/\alpha = \mathbb{P}\Big(\inf_{0\leq s\leq t} X_s \leq 0\Big)$
- $p(\cdot, t)$ the density of $\widehat{X}_t = X_t \mathbb{1}_{\{\tau>t\}}$.

*No jumps for α sufficiently small (Bayraktar, Guo, Tang, Zhang, 2020).

‘Blow up’ scenario

Hambly, Ledger, & Sojmark (2019); see also Delarue, Inglis, Rubenthaler, &Tanré (2015)

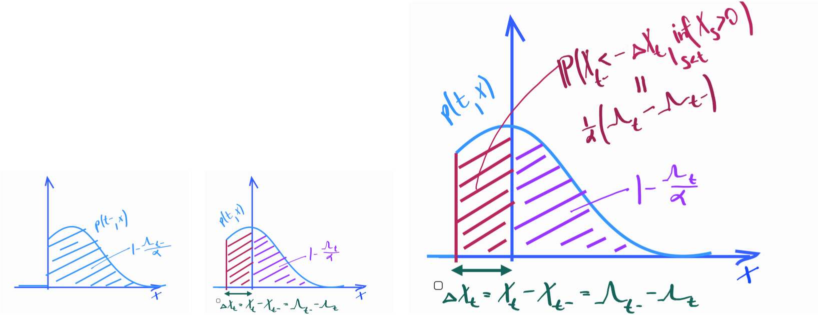




- Recall $X_t = X_{0-} + B_t - \Lambda_t$.

‘Blow up’ scenario

Hambly, Ledger, & Sojmark (2019); see also Delarue, Inglis, Rubenthaler, & Tanré (2015)



Physical and minimal solutions

Hambly, Ledger, & Sojmark (2019); see also Delarue, Inglis, Rubenthaler, &Tanré (2015)

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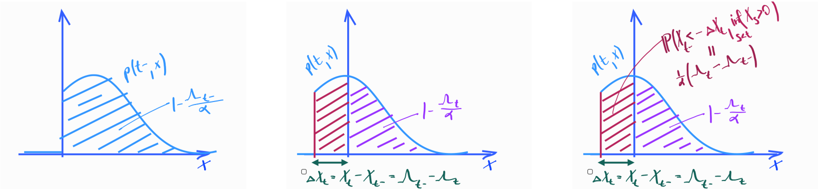

Definition (Physical and minimal solutions)

Call a solution (X, Λ) *physical* if for all t

$$\Lambda_t - \lim_{s \to t^-} \Lambda_s = \left\{ \inf_x \left(\mathbb{P}(X_{t-} < x, \inf_{s<t} X_s > 0) < \frac{1}{\alpha} x \right) \right\}.$$

Call a solution $\underline{\Lambda}_t$ *minimal* if for any other solution Λ we have

$$\underline{\Lambda}_t \leq \Lambda_t, \quad t \geq 0.$$

Well-posedness and regularity

Delarue, Nadtochiy, Shkolnikov (2019)

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a. [Existence, Cuchiero, Rigger, & Svaluto-Ferro (2020); see also Ledger and Sojmark (2019)]: If $\mathbb{E}[X_{0-}] < \infty$, there is a unique minimal solution, which is also a physical solution.

b. [Uniqueness]: Let X_{0-} possess a density f on $[0, \infty)$ that is bounded and changes monotonicity finitely often on compacts. Then physical solutions are unique.

- For any $t > 0$, $\Lambda \in C^1(t, t+\epsilon)$.
- The densities $p(s, \cdot)$, $s \in (t, t+\epsilon)$ are classical solutions of

$$\partial_t p = \frac{1}{2}\partial_{xx} p + \dot{\Lambda}_t \partial_x p, \quad x \geq 0, \quad t \in [0, T],$$
$$\dot{\Lambda}_t = \frac{\alpha}{2}\partial_x p(t, 0), \quad t \in [0, T] \quad \text{and} \quad \Lambda_0 = 0,$$
$$p(0, x) = f(x), \quad x \geq 0 \quad \text{and} \quad p(t, 0) = 0, \quad t \in [0, T].$$

Supercooled Stefan problem

Sherman (1970), Fasano, Primicerio, Howison, & Ockendon (80's), and Stefan (1889) for standard case

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The transformation $u(t,x) := p(t, x - \Lambda_t)$, $x \geq \Lambda_t$ leads to

$$\partial_t u = \frac{1}{2}\partial_{xx} u, \quad x \geq \Lambda_t, \quad t \geq 0,$$
$$\dot{\Lambda}_t = \frac{\alpha}{2}\partial_x u(t, \Lambda_t), \quad t \geq 0 \quad \text{and} \quad \Lambda_0 = 0,$$
$$u(0,x) = f(x), \quad x \geq 0 \quad \text{and} \quad u(t,\Lambda_t) = 0, \quad t \geq 0.$$

This is the classical one-dimensional supercooled Stefan problem:

- $-f$ is the initial temperature in a liquid relative to its freezing point;
- Λ_t is the location of the liquid-solid boundary at time t;
- $-u(t,\cdot)$ is the temperature in the liquid relative to its freezing point at time t;
- and $\alpha > 0$ is a physical parameter.

We consider the *supercooled* regime, i.e., when $f \geq 0$.

▸ Link

Introducing a central agent

A central agent injects cash at a rate $\beta_t^{i,N}$ into bank i,

$$X_t^{i,N} = X_{0-}^{i,N} + \int_0^t \beta_s^{i,N}\, ds + B_t^{i,N} - \alpha \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_{\{\tau_{i,N} \le t\}},$$

where $\tau_{i,N} := \inf\{t \geq 0 :\ X_t^{i,N} \leq 0\}$.

In analogy to before, we will consider the mean-field limit (TBC)

$$X_t = X_{0-} + \int_0^t \beta_s\, ds + B_t - \Lambda_t,$$

$$\Lambda_t = \alpha \mathbb{P}\Big(\inf_{0 \le s \le t} X_s \le 0 \Big).$$

For given β, the central agent has a total cost

$$C_T(\beta) = \mathsf{E}\Big[\int_0^T \beta_t\, dt\Big],$$

and observes losses prior to T,

$$L_{T-}(\beta) = \mathbb{P}\Big(\inf_{0 \le s < T} X_s(\beta) \le 0\Big) = \Lambda_{T-}(\beta)/\alpha.$$

We will study the constrained optimisation problem

$$C_T(\beta) \longrightarrow \min_{\beta} \qquad \text{subject to} \qquad L_{T-}(\beta) \le \delta.$$

Lagrangian multiplier

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We minimise the following objective function

$$J(\beta) = \mathsf{E}\Big[\int_0^T \beta_t\,dt\Big] + \gamma\,\mathbb{P}\Big(\inf_{0\le s<T} \underline{X}_s(\beta) \le 0\Big)$$
$$= \mathsf{E}\Big[\int_0^T \beta_t\,dt + \gamma\,\mathbb{1}_{\{\widehat{\underline{X}}_{T-}=0\}}\Big],$$

where γ traces out optimal pairs $(C_T^\star(\gamma), L_{T-}^\star(\gamma))$.

As function of δ, γ is a shadow price of preventing defaults,

$$\partial_\delta C_T^\star(\delta) = -\gamma(\delta).$$

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In all examples, we choose a gamma initial density,

$$f(x) = 1/\Gamma(k)\,\theta^{-k} x^{k-1} \mathrm{e}^{-x/\theta}, \qquad x \geq 0.$$

- Default parameters $k = 2$, $\theta = 1/3$.
- Smooth solutions for small t, but blow-up is guaranteed for $\alpha > k\theta = 2/3$.
- We will consider various values of α around 1.
- The terminal time is typically $T = 0.02$.
- We fix $b_{\max} = 30$ at first, but investigate changes later on.

Optimal strategies

Feedback controls $\beta_t = \beta(t, X_t)$

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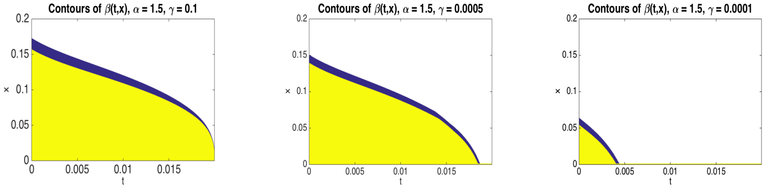

$\gamma = 0.1$ $\gamma = 0.0005$ $\gamma = 0.0001$

Contour plots of $(t, x) \to \beta^\star(t, x)$ for $\alpha = 1.5$ and different γ. The white region is $\{\beta^\star \leq 0.05\ b_{\max}\}$, the (yellow) shaded region $\{\beta^\star \geq 0.95\ b_{\max}\}$, the dark (blue) zone the transition.

Changing γ and $b_{\max}$

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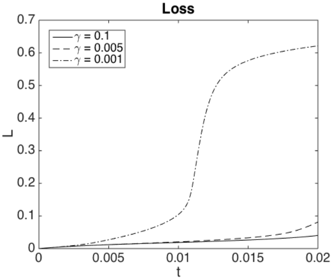
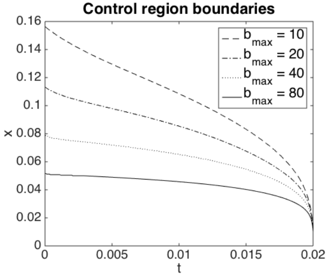


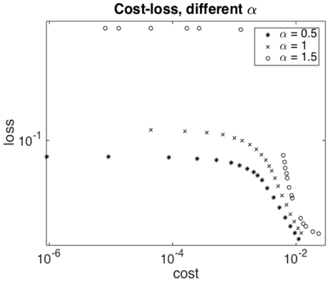
Loss for $\gamma \in \{0.1, 0.005, 0.001\}$



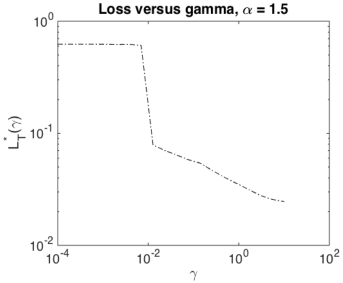
Control regions for different $b_{\max}$

Optimal cost-loss pairs

Small cash withdrawal can cause systemic events.



Pairs $(C^\star_T, L^\star_T)$ for different α



$L^\star_T$ as function of γ, $\alpha = 1.5$

Cost $C^\star_T$ and loss $L^\star_T$ in the optimal regime for logarithmically spaced $\gamma \in [0.0001, 0.1]$.

Comparison with heuristic strategies

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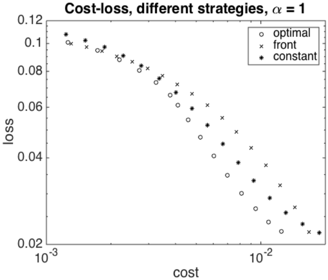

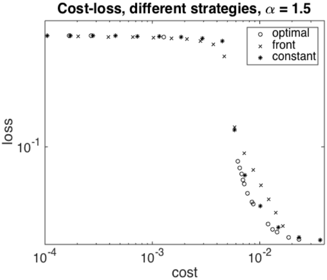

$\alpha = 1$, no jumps

$\alpha = 1.5$, jump possible

Cost-loss pairs $(C_T^\star, L_T^\star)$ under optimal strategy compared to those for a

- constant strategy, $(C_T^{\mathrm{u}}(c), L_T^{\mathrm{u}}(c))$, where cash is injected for $0 < X_t \leq c$,
- front-up strategy, $(C_T^{\mathrm{f}}(d), L_T^{\mathrm{f}}(d))$, where X_0 is lifted to $d > 0$.

Existence of optimiser

$0 \leq \beta_t \leq b_{\max}$

$$\begin{aligned} S_T &:= \{ f \in L^2[0,T] \mid 0 \leq f \leq b_{\max} \text{ a.e.} \} \\ \mathcal{B}_T &:= \{ \beta \text{ progressively measurable} \mid \mathbb{P}(\beta \in S_T) = 1 \}. \end{aligned}$$

Theorem (Existence of solutions for general drift)

For any $\beta \in \mathcal{B}_T$, there is a unique minimal solution to

$$X_t = X_{0-} + \int_0^t \beta_s \, ds + B_t - \alpha \mathbb{P}\Big(\inf_{0 \leq s \leq t} X_s \leq 0 \Big).$$

Theorem (Existence of optimiser)

There is $\beta^\star \in \mathcal{B}_T$ such that

$$V_\infty = \inf_{\beta \in \mathcal{B}_T} J(\beta) = J(\beta^\star).$$

Propagation of chaos

Consider the N-bank setting

$$X_t^{i,N} := X_{0-}^{i,N} + \int_0^t \beta_s^{i,N}\,\mathrm{d}s + B_t^{i,N} - \alpha L_t^N,$$

$$L_t^N := \frac{1}{N}\sum_{i=1}^N \mathbb{1}_{\{\tau^{i,N}\le t\}}, \quad \text{where} \quad \tau_{i,N} := \inf\{t \ge 0 : X_t^{i,N} \le 0\},$$

$$V_N := \inf_{\beta^N \in \mathcal{B}} \mathbb{E}\left[\int_0^T \frac{1}{N}\sum_{i=1}^N \beta_s^{i,N}\,\mathrm{d}s + \gamma \underline{\widetilde{L}}_T^N(\beta^N)\right],$$

where $\underline{\widetilde{L}}^N(\beta^N)$ is the minimal solution with drift β^N and initial condition $X_{0-}^{i,N} = X_{0-}^N + N^{-\kappa}$ for $\kappa \in (0, 1/2)$ and all $i = 1, \ldots, N$.

Theorem

Then it holds that $\lim_{N\to\infty} V_N = V_\infty$.

A non-standard MFC problem

Recall $\underline{X}(\beta)$ corresponding to the minimal solution $\underline{\Lambda}(\beta)$, and

$$J(\beta) = \mathrm{E}\Big[\int_0^T (\beta_t + \gamma \dot{L}_t)\, dt\Big], \qquad L_t = \mathbb{P}\Big(\inf_{0 \le s < T} \underline{X}_s(\beta) \le 0\Big).$$

For regular solutions, $\widehat{\underline{X}} = \underline{X}_t \mathbb{1}_{\{\tau > t\}}$ has a sub-probability density p supported on $(0, \infty)$ and an atomic mass at 0:

$$\partial_t p + \partial_x(\beta p) = \frac{1}{2}\partial_{xx} p + \dot{\Lambda}_t \partial_x p, \;\; x \ge 0, \qquad \Lambda_t = \alpha\Big(1 - \int_0^\infty p(t,x)\, dx\Big).$$

Assuming again regularity, $\dot{L}_t = \frac{1}{2}\partial_x p(t,0)$ and

$$dX_t = (\beta_t - \alpha \dot{L}_t)\, dt + dB_t.$$

Regularisation

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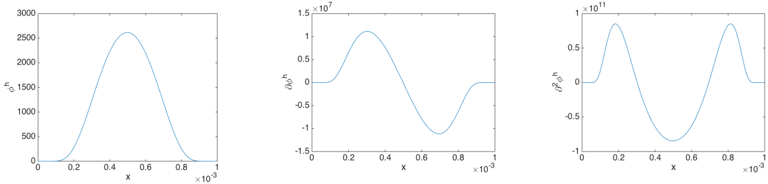
For (small) $h > 0$, approximate $p_x(t, 0)$ by way of the measure ν_t,

$$\dot{L}_t^h = \frac{1}{2}\int_{-\infty}^{\infty}(-\partial\phi^h)(x)\,\nu_t(dx) = \frac{1}{2}\langle -\partial\phi^h, \nu_t\rangle.$$

- Dynamics and objective can be rewritten in these terms.
- We smoothly transition SDE coefficients to 0 over $[-h, 0]$.



Smoothed Dirac delta and its derivatives, for $h = 10^{-3}$.

Densities

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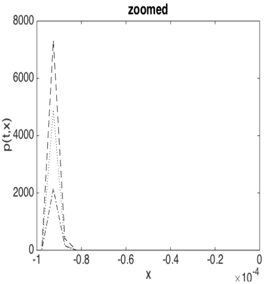
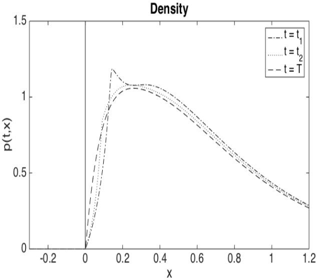


Density $p(t, \cdot)$ for small negative x.



Density $p(t, \cdot)$ in macroscopic range.

Parameters $\alpha = 1.5$, $\gamma = 0.1$; $\Gamma(2, 1/3)$ initial density

(Proximal) policy gradient method

We consider the objective

$$\inf_{\beta \in \mathcal{H}^2(\mathbb{R})} \mathrm{E}\left[\int_0^T \left(\beta_t + \frac{\gamma}{2} \langle -\partial \phi^h, \nu \rangle + g(\beta_t) \right) \mathrm{d}t \right],$$

where $g(b) = 0$ for $b \in [0, b_{\max}]$ and ∞ otherwise.

- We apply the method from R., Stockinger, Zhang, *A fast iterative PDE-based algorithm for feedback controls of nonsmooth mean-field control problems*, arXiv, 2021.
- We compute the measure by a forward PDE, and the gradient by a backward PDE for a 'decoupling field'.

Finite difference convergence

Finite termination

			$10^3 h$	1	2^{-1}	2^{-2}	2^{-3}	2^{-4}	2^{-5}	CPU
$\frac{N}{10^2}$	$\frac{N_x}{10^3}$	$10^3\theta_N$	ρ_N							(s)
1	3.75	1.956	-2.42	0.5643	0.6430	0.7137	0.832	4.775	0	0.44
2	7.5	-0.806	1.31	0.5663	0.6260	0.6693	0.726	0.838	5.121	1.2
4	15	-0.614	1.82	0.5655	0.6164	0.6481	0.677	0.822	0.033	4.2
8	30	-0.337	1.94	0.5649	0.6118	0.6376	0.658	0.688	0.609	17
16	60	-0.173	—	0.5645	0.6096	0.6327	0.648	0.664	0.689	83
32	120	—	—	0.5643	0.6085	0.6304	0.644	0.654	0.668	427
			$10^2\vartheta_h$	4.414	2.192	1.354	1.084	1.32	—	
			ϱ_h	2.01	1.61	1.24	0.81	—	—	

Mesh convergence, losses, $\alpha = 1.5$ and $\gamma = 0.1$.

Convergence of iterations

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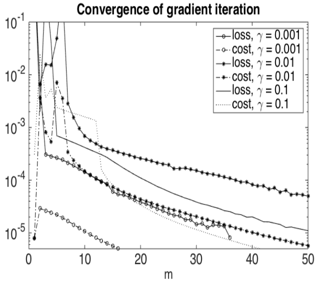
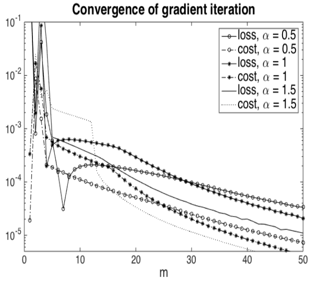


$\alpha = 0.5$, varying γ, $N = 800$



varying α, $\gamma = 1$, $N = 800$

Convergence of C and L in the PGM for varying γ and α. Shown are $|L^{(m+1)} - L^{(m)}|$ and $|C^{(m+1)} - C^{(m)}|$.

Delayed contagion

Hambly, Petronilia, R, Rigger & Sømark, 2023

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$$dX_t^i = b(t, X_t^i, \nu_t^N)\, dt + \sigma(t, X_t^i)\sqrt{1 - \rho(t, \nu_t^N)^2}\, dW_t^i + \sigma(t, X_t^i)\rho(t, \nu_t^N)\, dW_t^0 - \alpha(t)\, d\mathcal{L}_t^{N,\epsilon},$$

$$\nu_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}, \qquad \tau^i = \inf\{t \geq 0 : X_t^i \leq 0\},$$

$$L_t^N = 1 - \nu_t^N(\mathbb{R}^+), \qquad \mathcal{L}_t^{N,\epsilon} = \int_0^t \epsilon^{-1}\kappa(\epsilon^{-1}(t-s))L_s^N\, ds,$$

where all W^i, W^0 B.M., all independent.

- W^0 is a common noise;
- smoothed default contagion through Λ_t.

Control and killing at state-dependent intensity

Hambly & Jettkant, 2023

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$$
\begin{aligned}
dX_t^i &= b(t, X_t^i, \nu_t^N, \gamma_t^i)\, dt + \sigma(t, X_t^i, \nu_t^N)\, dW_t^i + \sigma_0(t, X_t^i, \nu_t^N)\, dW_t^0 \\
&\quad -\alpha(t, X_{t-}^i, \nu_{t-}^N)\, dL_t^N, \\
d\Lambda_t^i &= \lambda(t, X_t^i, \nu_t^N)\, dt, \\
\nu_t^N &= \frac{1}{N}\sum_{i=1}^{N} 1_{\theta_i > \Lambda_t^i}\delta_{X_t^i}, \qquad L_t^N \;=\; 1 - \nu_t^N(\mathbb{R}),
\end{aligned}
$$

where all W^i, W^0 B.M. and θ_i i.i.d. exponential, all independent.

- W^0 is a common noise;
- γ_t^i are controls;
- default contagion through Λ_t.

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