Exotic options and Fourier transforms

The story is far from over

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Outline

Outline

What did Heston ever do for us?

Heston's work was ground-breaking at the time.

- One of the very first stochastic volatility models for a risky asset S_t
- Introduced characteristic function techniques
- Calculates characteristic function explicitly
- Provides explicit semi-analytic formula for vanilla European options

Heston assumes

$$\begin{aligned} dS_t &= r_t S_t dt + \sigma\sqrt{v_t} S_t dW_t, \\ dv_t &= k\left(\mu - v_t\right) dt + \sigma_v \sqrt{v_t} dW_t^v \end{aligned} \tag{1}$$

with

$$d[W, W^v]_t = \rho dt$$

Heston's breakthrough

Theorem (Heston 1993)

Under the previous assumptions the price of a European call option is given by

$$SP_1 - Ke^{-rT}P_2,$$

where

$$P_j = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-i\xi\log K} f_j(\xi, T, x, v)}{i\xi}\right) d\xi \quad j = 1, 2$$

with $f_1 \equiv \Phi(\xi - i)$, $f_2 = \Phi(\xi)$ *and* $\Phi = \mathbb{E}_{\mathbb{Q}}[e^{i\xi \log S_T}]$.

- Pros:
  1. Provides usable alternative to GBM, more realistic dynamics.
  2. Better calibration to market prices.
- Cons:
  1. Semi-analytic formula not easily numerically integrated.

Enter Peter Carr and Dilip Madan

Carr and Madan realise that issues arising from the numerical integration of complex logarithm (still being discussed by Kahl-Jäckel as late as 2005) are not due to Heston's CF, but by the Gil-Pelaez formula. Their main ideas are

- Use the Fourier transform with respect to log-strike
- Introduce damping to achieve integrability

Theorem (Carr-Madan, 1999)

Let $\epsilon > 0$. Then the price of a call option is given by

$$C_T(k) = \frac{e^{-\epsilon k}}{2\pi} \int_{\mathbb{R}} e^{-i\xi k} \frac{e^{-rT}\Phi_T(\xi - (\epsilon+1)i)}{\epsilon^2 + \epsilon - \xi^2 + i(2\epsilon+1)\xi} d\xi,$$

with $k = \log K$ *and* $\Phi_T = \mathbb{E}_{\mathbb{Q}}[e^{i\xi \log S_T}]$.

Carr-Madan declassified

- The formula is valid for any asset dynamics, as long as the CF is known, Gil-Pelaez can enjoy retirement.
- The formula holds for *any* $\epsilon > 0$.
- Don't be scared by the choice of ϵ; all we are saying is $e^{\alpha} \times e^{-\alpha} = 1 \quad \forall \alpha \in \mathbb{R}$.
- Involved numerical integration procedures can be replaced by an efficient FFT routine, since there are no singularities.
- We obtain option prices for an array of log-strikes.
- Does not address exotic option pricing.

Fang and Oosterlee - all options allowed

- Fang and Oosterlee make a very simple (200 year old) observation: *any function $f \in L^2([a,b])$ can be written as a Fourier series.*
- Thus, a Fourier series can potentially be used in place of the Fourier transform for option pricing purposes.
- Any real valued function with finite support can be expressed as a cosine series. This is the birth of the COS method.

We may write

$$f(x) = \sum_{k=0}^{\infty} A_k \cos\left(\frac{x-a}{b-a}\pi\right), \quad A_k = \frac{2}{b-a}\int_a^b f(x)\cos\left(\frac{x-a}{b-a}\pi\right)dx$$

with

$$A_k = \frac{2}{b-a}\Re\left\{\phi_1\left(\frac{k\pi}{b-a}\right) \times \exp\left(-i\frac{ka\pi}{b-a}\right)\right\}.$$

$$\phi_1(\xi) = \int_a^b e^{i\xi y} f(y)dy \approx \int_{-\infty}^{\infty} e^{i\xi y} f(y)dy \equiv \phi(\xi).$$

In the case of option pricing we have the following.

Theorem (Fang-Oosterlee, 2008)

The price V of an option with payoff $c \equiv c(x,k)$ can be approximated by

$$V(x,k) = e^{-rT} \int_{-\infty}^{\infty} c(x,k)\varphi(x)dx \approx e^{-rT} \int_a^b c(x,k)\varphi(x)dx$$

$$\approx e^{-rT} \sum_{k=0}^{N-1} \Re \left\{ \phi\left(\frac{k\pi}{b-a}\right) \times \exp\left(-i\frac{ka\pi}{b-a}\right) \right\} V_k,$$

where

$$V_k = \frac{2}{b-a} \int_a^b c(k,x) \cos\left(\frac{x-a}{b-a}\pi\right) dx$$

Outline

Fourier transform techniques in higher dimension

- Hurd and Zhou want to address the pricing of rainbow options, in particular basket options; existing techniques not sufficient.
- They observe that yet another classical result may be used. Plancherel's theorem states

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{g}(\xi)}d\xi$$

- If g represents the pdf associated to a random vector X with respect to a probability measure $\mathbb{Q}$ then

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \widehat{f}(\xi)\mathbb{E}_{\mathbb{Q}}[e^{i\xi\cdot X}]d\xi$$

- To ensure convergence, choose $\epsilon \in \mathbb{R}^d$ so that $e^{\epsilon\cdot x}f(x), e^{-\epsilon\cdot x}g(x) \in L^2(\mathbb{R}^d)$.

Semi-analytic formula for basket options

- Now

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}dx = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{f}(\xi + i\epsilon)\mathbb{E}_{\mathbb{Q}}[e^{i(\xi+i\epsilon)\cdot X}]d\xi \qquad (2)$$

- Just pick f to be the option payoff and assume $\mathbb{E}_{\mathbb{Q}}[e^{i\xi\cdot X}] = e^{i\xi\cdot X_0}\Phi(\xi, T) \implies$ apply FFT with respect to X_0.

Theorem (Hurd-Zhou, 2011)

The price $V(x, T)$ of a put option on a basket of assets $(S_1, \dots, S_d)$ is given by equation (2) with

$$\widehat{f}(\xi) = \frac{\prod_{j=1}^{d}\Gamma(-i\xi_j)}{\Gamma(-i\sum_{j=1}^{d}\xi_j + 2)},$$

where $f(x) = (1 - e^{x_1} - \dots e^{x_d})_+$.

Semi-analytic formula for basket options

Pros:

1. Semi-analytic formula is exact - a similar formula for spread options is also available.
2. May be applied to any asset dynamics as long as joint characteristic function of log-prices is known.
3. May use the FFT algorithm in higher dimension after simple discretisation.
4. Provides prices for a grid of initial asset values X_0.

Cons:

1. Curse of dimensionality - calculations in dimension greater than 3 are probably still computationally too burdensome.

COS method to the rescue

Ruijters and Oosterlee provide a natural extension of the COS method to higher dimension - the theory of Fourier series is valid in any dimension after all. Implementation is computationally feasible up to $d = 4$.

Theorem (Ruijters-Oosterlee, 2012)

The price V of an option with payoff $c \equiv c(\mathbf{x}, k), \mathbf{x} = (x_1, x_2)$ can be approximated by

$$V(\mathbf{x}, T) \approx e^{-rT} \sum_{k_1=0}^{N_1-1} \sum_{k_2=0}^{N_2-1} \frac{1}{2} \Re \left(\phi \left(\frac{k_1 \pi}{b_1 - a_1}, \frac{k_2 \pi}{b_2 - a_2} \right) \right.$$
$$\times \exp \left(i k_1 \pi \frac{x_1 - a_1}{b_1 - a_1} + i k_2 \pi \frac{x_2 - a_2}{b_2 - a_2} \right) \Bigg)$$
$$+ \frac{1}{2} \Re \left(\phi \left(\frac{k_1 \pi}{b_1 - a_1}, - \frac{k_2 \pi}{b_2 - a_2} \right) \right.$$
$$\left. \times \exp \left(i k_1 \pi \frac{x_1 - a_1}{b_1 - a_1} - i k_2 \pi \frac{x_2 - a_2}{b_2 - a_2} \right) \right) V_{k_1, k_2}$$

Recent developments

A number of new techniques have been introduced in recent years. Among others:

- Kirkby, Nguyen et. al. use Markov Chain Monte Carlo methods for rainbow option pricing under the assumption of GBM dynamics.
- Colldeforms, Oosterlee et. al. introduce wavelet decomposition techniques to address the same problem for general asset dynamics whose joint characteristic function is known. They observe similar performance to the COS method.
- Zhao and Li introduce frame duality for the 2-dimensional problem.

Personal remark. The increasing level of mathematical sophistication, although impressive, might make numerical implementation arduous.

Outline

Calls on maximum and minimum of multiple assets

We are interested in pricing options with payoff (in log-coordinates)

$$P(X) = \left(\max\left(e^{X_1}, \dots, e^{X_d}\right) - 1\right)_+ \quad Q(X) = \left(1 - \min\left(e^{X_1}, \dots, e^{X_d}\right)\right)_+$$

To do so, we observe that

$$\mathbb{R}^d \cong \dot{\bigcup}_{k \in \{1,\dots,d\}} \left\{x \in \mathbb{R}^d \mid x_k > x_j,\ \forall j \neq k\right\},$$

modulo a set of measure zero, implying

$$P(x) = \sum_{k=1}^{d} P(x) \prod_{j \neq k}^{d} \mathbb{1}_{x_k > x_j} = \sum_{k=1}^{d} P^k(x) \quad a.e.$$

and by the same argument

$$Q(x) = \sum_{k=1}^{d} Q(x) \prod_{j \neq k}^{d} \mathbb{1}_{x_k < x_j} = \sum_{k=1}^{d} Q^k(x) \quad a.e.$$

Calls on maximum and minimum of multiple assets

Now

$$\mathbb{E}_{\mathbb{Q}}\left[\left(\max\left(e^{X_T^1},\ldots,e^{X_T^d}\right)-1\right)_+\right]=\int_{\mathbb{R}^d}P(x)\Phi(x)dx$$

$$=\sum_{k=1}^{d}\int_{\mathbb{R}^d}P^k(x)\Phi(x)dx=\sum_{k=1}^{d}\int_{\mathbb{R}^d}e^{\epsilon^k\cdot x}P^k(x)e^{-\epsilon^k\cdot x}\Phi(x)dx$$

$$=\frac{1}{(2\pi)^d}\sum_{k=1}^{d}\int_{\mathbb{R}^d}\widehat{P^k}(\xi+i\epsilon^k)\mathbb{E}_{\mathbb{Q}}\left[e^{i(\xi+i\epsilon^k)\cdot X}\right]d\xi$$

for an appropriate choice of ϵ^k - the same applies to the put payoff Q.

Calls on maximum and minimum of multiple assets

Theorem

Let $\epsilon^k \in \mathbb{R}^d$ *with* $\sum_{j=1}^d \epsilon_j^k < -1$ *and* $\epsilon_j^k > 0$ *for* $j \neq k$. *Then*

$$\mathcal{F}\left[x \mapsto e^{\epsilon^k \cdot x} P^k(x)\right](\xi)$$

$$= \frac{1}{\left(1 + \sum_{j=1}^d (\epsilon_j^k - i\xi_j)\right)\left(\sum_{j=1}^d (\epsilon_j^k - i\xi_j)\right)} \prod_{j \neq k}^d \frac{1}{\left(\epsilon_j^k - i\xi_j\right)}.$$

Now let $\epsilon^k \in \mathbb{R}^d$ *with* $\sum_{j=1}^d \epsilon_j^k > 0$ *and* $\epsilon_j^k < 0$ *for* $j \neq k$, *one has*

$$\mathcal{F}\left[x \mapsto e^{\epsilon^k \cdot x} Q^k(x)\right](\xi)$$

$$= \frac{(-1)^{d-1}}{\left(1 + \sum_{j=1}^d (\epsilon_j^k - i\xi_j)\right)\left(\sum_{j=1}^d (\epsilon_j^k - i\xi_j)\right)} \prod_{j \neq k}^d \frac{1}{\left(\epsilon_j^k - i\xi_j\right)}.$$

Outline

The models

- GBM:

$$dS_t^j = rS_t^j dt + \sigma_j S_t^j dW_t^j, \quad j = 1, \dots, d$$

- Heston:

$$dS_t^j = r_t S_t^j dt + \sigma_j \sqrt{v_t} S_t^j dW_t^j, \quad j = 1, \dots, d$$
$$dv_t = k(\mu - v_t)\, dt + \sigma_v \sqrt{v_t} dW_t^v$$

- VG:

$$S_t^j = S_0^j e^{(r+\omega)t + Y_t^j + Y_t^0}, \quad j = 1, 2$$

where Y_t^j are independent Lévy processes associated to $\nu(x) = \lambda_j \left[e^{-a_+ x}\mathbb{1}_{x>0} + e^{-a_- x}\mathbb{1}_{x<0}\right]/|x|$, with $\lambda, a_+, a_- > 0$, and $\lambda_1, \lambda_2 = (1-\alpha)\lambda$, $\lambda_0 = \alpha\lambda$, $0 \le \alpha \le 1$.

- JD:

$$dS_t^j = (r - \lambda\kappa_j)\, S_t^j dt + \sigma_j S_t^j dW_t^j + \left(e^{J_j} - 1\right) S_t^j dN_t, \quad j = 1, 2$$

N_t Poisson process with arrival rate λ, and $\mathbf{J} = (J_1, J_2)$ normally distributed jumps with $\mu^{\mathbf{J}} = (\mu_1^J, \mu_2^J)$ and covariance $\Sigma_{ij}^J = \sigma_i^J \sigma_j^J \rho_{ij}^J$.

Discretisation

Truncate integration region to $[-\bar{u}, \bar{u}] \subset \mathbb{R}^d$, and consider the lattice

$$\Gamma = \left\{ u(n) = (u_1(n_1), \dots, u_d(n_d)) | n = (n_1, \dots, n_d) \in \{0, \dots, N-1\}^d \right\},$$

where $u_i(n_i) = -\bar{u} + n_i \eta$ with $\bar{u} := N\eta/2$, and N chosen by the user. If initial log-prices $X_0 = x(\ell)$, $\ell = (\ell_1, \dots, \ell_d)$ lie on the dual lattice Γ^*, then

$$C_{max}(S_0) \equiv C_{max}(e^{X_0}) \approx \sum_{k=1}^{d} (-1)^{|\ell|} \left(\frac{\eta N}{2\pi} \right)^d e^{-\epsilon^k \cdot x(\ell)} \left[\text{ifftd}(H_k) \right](\ell),$$

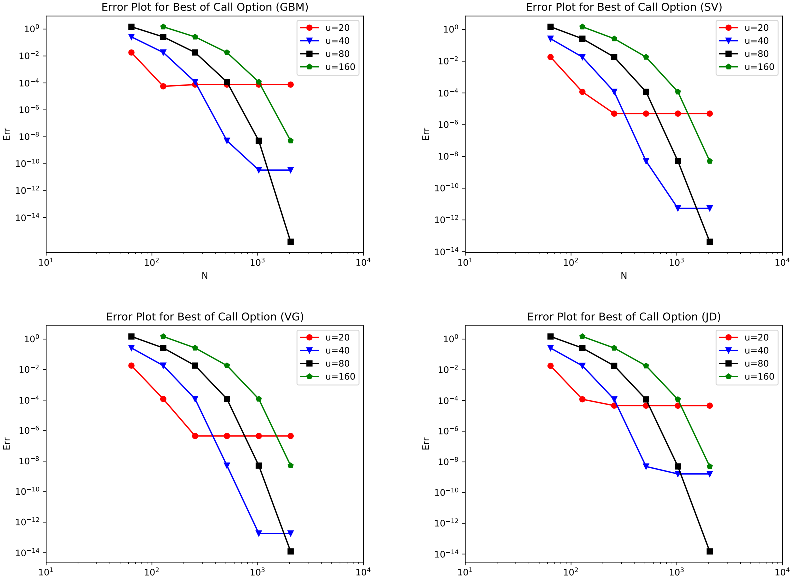
where

$$H_k(n) = (-1)^{|n|} \Phi(u(n) + i\epsilon^k) \widehat{P^k}(u(n) + i\epsilon^k),$$

ifftd is the inverse FFT in d dimensions. We choose $d = 2$ and select initial log-asset prices $\log S_i^1 = -\frac{\pi}{5} + \frac{i\pi}{10}$, $\log S_j^2 = -\frac{\pi}{5} + \frac{j\pi}{10}$, $i, j = 0, \dots, 5$; we then set an error function

$$\text{Err} = \frac{1}{36} \sum_{i,j=0}^{5} |M^{ij} - B^{ij}|, \quad M^{ij} = \text{model } B^{ij} = \text{benchmark}$$

Accuracy



Error Plot for Best of Call Option (GBM)
Error Plot for Best of Call Option (SV)
Error Plot for Best of Call Option (VG)
Error Plot for Best of Call Option (JD)
u=20
u=40
u=80
u=160
Err
N

Accuracy

- Benchmark prices are achieved by using Stulz's formula for GBM and the COS method (whose accuracy has been established) for the other models
- Best empirical choice of discretisation parameters is given by

$$\frac{u}{N} = \frac{10}{2^8} \iff N = \frac{2^8 \times u}{10}, \quad u \in \{20, 40, 80\}.$$

Table: Value of the Err function for the two-dimensional cosine method.

N	GBM	SV	VG	JD
32	8.67E-06	4.22E-03	2.01E-04	3.34E-03
64	6.22E-13	1.81E-06	1.39E-08	4.74E-06
128	1.86E-15	1.30E-15	5.60E-16	4.25E-14

Table: Computing time (in seconds) for an FFT panel of prices.

Discretisation	GBM	SV	VG	JD
64	0.014001	0.010999	0.007	0.018999
128	0.032001	0.039999	0.027998	0.093003
256	0.127999	0.171868	0.114001	0.286
512	0.521000	0.682000	0.428999	0.925999
1024	2.042001	2.666260	1.352998	3.361972
2048	8.125552	10.754999	5.147999	13.079627

Table: Computing time (in seconds) for a single price using the cosine method.

N	GBM	SV	VG	JD
32	0.092998505	0.1179986	0.117999077	0.114000
64	0.378000736	0.427000523	0.439803362	0.432999
128	1.501997948	1.721000195	1.755997658	1.733844

Outline

Margrabe and dimensionality reduced by 1

Margrabe showed that by choosing one of the assets as numéraire the dimensionality of the problem may be reduced by one,

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{\max\left(S_T^1, S_T^2\right)}{B_T}\right] = \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T^2}{B_T}\right] + \mathbb{E}_{\mathbb{Q}}\left[\frac{\left(S_T^1 - S_T^2\right)_+}{B_T}\right] =$$

$$S_0^2 + \mathbb{E}_{\mathbb{Q}}\left[\frac{S_T^2}{B_T}\left(\frac{S_T^1}{S_T^2} - 1\right)_+\right] = S_0^2 + \mathbb{E}_{\mathbb{S}^2}\left[\frac{d\mathbb{Q}}{d\mathbb{S}^2}\frac{S_T^2}{B_T}\left(\frac{S_T^1}{S_T^2} - 1\right)_+\right] =$$

$$S_0^2 + \mathbb{E}_{\mathbb{S}^2}\left[\frac{B_T S_0^2}{S_T^2 B_0}\frac{S_T^2}{B_T}\left(\frac{S_T^1}{S_T^2} - 1\right)_+\right] = S_0^2 + S_0^2\mathbb{E}_{\mathbb{S}^2}\left[\left(\frac{S_T^1}{S_T^2} - 1\right)_+\right],$$

this is true in any dimension.

- Need explicit dynamics of the asset ratio under new numéraire.
- Simple for GBM, extremely difficult for certain stochastic processes.

Theorem

Let $P \equiv P(x) = \max\left(e^{x_1}, \ldots, e^{x_d}\right)$, $x \in \mathbb{R}^d$,*and define* $P^k \equiv P^k(x)$ *as above. Then* $\forall \epsilon^k \in \mathbb{R}^d$ *with* $\epsilon^k_k = 0$ *and* $\epsilon^k_j > 0$*, for* $j \neq k$*, one has*

$$\mathcal{F}\left[x \mapsto e^{\epsilon^k \cdot x} P^k(x)\right](\xi) = 2\pi\delta\left(\sum_{j=1}^{d} \xi_j + i \sum_{j \neq k} \epsilon^k_j + i\right) \prod_{j \neq k} \frac{1}{\left(\epsilon^k_j - i\xi_j\right)}$$

in the sense of distributions; here δ *represents the Dirac delta. Further,*

$$\mathbb{E}_{\mathbb{Q}}\left[P\left(X_T\right)\right] = \frac{1}{(2\pi)^{d-1}} \sum_{k=1}^{d} \int_{\mathbb{R}^{d-1}} \left(\prod_{j \neq k} \frac{1}{(\epsilon^k_j - i\xi_j)}\right) \mathbb{E}_{\mathbb{Q}}\left[e^{i\eta \cdot X}\right] d\xi^{(-k)},$$

where $\xi^{(-k)} = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_d)$*, whilst* $\eta_j = \xi_j + \epsilon^k_j$*, for* $j \neq k$ *and* $\eta_k = -\sum_{j \neq k}(\xi_j + i\epsilon^k_j) - i$.

Best of asset option pricing

- Best of asset option may be evaluated by calculating d integrals over $d-1$ dimensions.
- Confirms Margrabe's insight about dimensionality reduced by 1 *independently of asset dynamics*.

Corollary

For $d=2$ the best of asset option price is given by

$$\mathbb{E}_{\mathbb{Q}}\left[\max\left(e^{X_T^1}, e^{X_T^2}\right)\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(\xi - i\epsilon_2 - i)x_0}}{\epsilon_2 + i\xi}\Phi(\xi - i\epsilon_2 - i, -\xi + i\epsilon_2)d\xi + \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(\xi + i\epsilon_1)x_0}}{\epsilon_1 - i\xi}\Phi(\xi + i\epsilon_1, -\xi - i\epsilon_1 - i)d\xi.$$

where we can always assume $S_0 \equiv \left(S_0^1, S_0^2\right) = \left(S_0^1, 1\right) \Longrightarrow X_0 = (x_0, 0)$

Geometric basket options are 1-dimensional

These options have payoffs

$$\left(\prod_{j=1}^{d}(S_T^j)^{\frac{1}{d}} - K\right)_+, \quad \left(K - \prod_{j=1}^{d}(S_T^j)^{\frac{1}{d}}\right)_+,$$

or

$$\left(e^{\frac{1}{d}\sum_{j=1}^d X_T^j} - K\right)_+, \quad \left(K - e^{\frac{1}{d}\sum_{j=1}^d X_T^j}\right)_+$$

in log-space; we may assume $K = 1$ by scaling. If S^j are assumed to be GBMs, then one may use classical Black-Scholes, with initial spot price $S_0 = \prod_{j=1}^d (S_0^j)^{\frac{1}{d}} = e^{\frac{1}{d}\sum_{j=1}^d X_0^j}$, dividend $\widehat{d}$ and volatility $\widehat{\sigma}$, where

$$\widehat{d} = \frac{1}{d}\sum_{j=1}^{d}\frac{1}{2}\sigma_j^2 - \frac{1}{2}\widehat{\sigma}^2, \quad \widehat{\sigma} = \frac{1}{d}\sqrt{\langle C\mathbf{e}, \mathbf{e}\rangle},$$

and $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^d$; C is the covariance matrix.

Geometric basket options are 1-dimensional

Theorem

Let $P \equiv P(x) = \left(\sqrt[d]{e^{x_1+\ldots+x_d}} - 1\right)_+ = \left(e^{\frac{1}{d}\sum_{j=1}^d x_j} - 1\right)_+$, $x \in \mathbb{R}^d$, *and let* $\bar{\epsilon} = (\epsilon, \ldots, \epsilon)$. *Then* $\forall \bar{\epsilon} \in \mathbb{R}^d$ *with* $\epsilon < -1/d$ *one has*

$$\mathcal{F}\left[x \mapsto e^{\bar{\epsilon}\cdot x} P(x)\right](\xi) = \frac{(2\pi)^{d-1}}{(i\xi_d - \epsilon)\left(d(i\xi_d - \epsilon) - 1\right)} \prod_{j=1}^{d-1} \delta\left(\xi_d - \xi_j\right)$$

Further,

$$\mathbb{E}_{\mathbb{Q}}\left[\left(e^{\frac{1}{d}\sum_{j=1}^d X_T^j} - 1\right)_+\right] = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{e^{i(\xi+i\epsilon)x_0}\Phi(\xi+i\epsilon, \ldots, \xi+i\epsilon)}{(i\xi-\epsilon)\left(d\times(i\xi-\epsilon)-1\right)} d\xi,$$

where $x_0 = \sum_{j=1}^d X_0^j$. *The same formula provides the put price* $\forall \bar{\epsilon} \in \mathbb{R}^d$ *with* $\epsilon > 0$.

Outline

Lower dimensional discretisation

For best of asset option the lattice Γ has dimension $d-1$, and $S_0 = e^{X_0} = e^{(x_0,0)}$. Set $\delta^k = (\epsilon'(k), -\sum_{j\neq k}\epsilon_j^k - 1, \epsilon''(k))$ for $k \leq d-1$, discretisation is given by

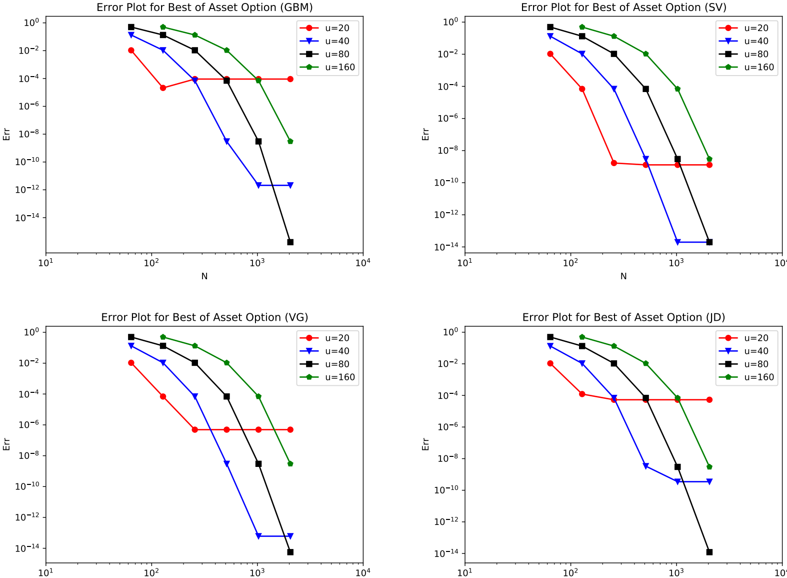
$$B_{max}(e^{(x_0,0)}) \approx \sum_{k=1}^{d-1}(-1)^{|\ell|}\left(\frac{\eta N}{2\pi}\right)^{d-1} e^{-\delta^k\cdot x(\ell)}\left[\mathrm{ifftdm1}(H_k)\right](\ell)$$
$$+(-1)^{|\ell|}\left(\frac{\eta N}{2\pi}\right)^{d-1} e^{-\tilde{\epsilon}(d)\cdot x(\ell)}\left[\mathrm{ifftdm1}(H_d)\right](\ell).$$

Choose $d=2$ and define the error function

$$\mathrm{Err} = \frac{1}{6}\sum_{i=0}^{5}|\log M^i - \log B^i|,$$

for initial log-asset prices $\log S_i^1 = -\frac{\pi}{5} + \frac{i\pi}{10}, i = 0,\ldots,5; \log S^2 \equiv 1$.

Accuracy



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Err
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Performance

- Benchmark prices are achieved by using Margrabe's formula for GBM, and COS method for other models
- COS method performance very similar to the call and put on maximum and minimum:

  Table: Computing time (in seconds) for a single best of asset option using the cosine method.

N	GBM	SV	VG	JD
32	0.080999374	0.101999521	0.097000599	0.095001
64	0.319999218	0.373001099	0.382999182	0.372999
128	1.292999744	1.512083292	1.542000055	1.516999

- Our methodology calculations are nearly instantaneous due to dimensionality reduction

Basket option discretisation

Lattice Γ and its dual Γ^* are strictly one-dimensional. Set $S_0 = e^{\sum_{j=1}^{d} X_0^j} = e^{x_0}$. We have the discretised equation

$$C(S_0) \equiv C(e^{x_0}) \approx (-1)^{\ell} \left(\frac{\eta N}{2\pi}\right) e^{-\epsilon x(\ell)} \left[\text{ifft1}(H)\right](\ell),$$

with

$$H(n) = \frac{\Phi\left(u(n) + i\epsilon, \ldots, u(n) + i\epsilon\right)}{\left(iu(n) - \epsilon\right)\left(d \times \left(iu(n) - \epsilon\right) - 1\right)}$$

Select initial log asset prices $x_0^i = -\frac{\pi}{5} + \frac{i\pi}{10}, i = 0, \ldots, 5$; we also define a new objective function

$$\text{Err} = \frac{1}{6} \sum_{i=0}^{5} |M^i - B^i|,$$

Accuracy

- We choose $d = 4$ to make a direct accuracy estimation in the GBM case
- Calculations instantaneous for any choice of model

Table: Value of the Err function for the FFT method.

N	64	128	256	512	1024	2048
u	Basket call					
20	4.74E-04	2.60E-07	7.38E-14	1.02E-16	1.48E-16	1.39E-16
40	3.91E-01	4.74E-04	2.60E-07	7.38E-14	1.02E-16	1.30E-16
80	1.65E+00	3.91E-01	4.74E-04	2.60E-07	7.38E-14	1.02E-16
u	Basket put					
20	1.54E-05	7.89E-10	4.60E-17	4.55E-17	4.54E-17	4.54E-17
40	1.14E-03	1.54E-05	7.89E-10	4.60E-17	4.55E-17	4.54E-17
80	3.87E-03	1.14E-03	1.54E-05	7.89E-10	4.59E-17	4.55E-17

Outline

Conclusion

- We introduced some rainbow option pricing formulas building on the work of Hurd and Zhou
- The formulas demonstrate the "real" dimensionality of certain pricing problems, confirming intuition
- We do not solve the issue of the curse of dimensionality
- More research and new methodologies needed to make calculation of higher dimensional options computationally competitive vs Monte Carlo.

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