Term structure modelling with overnight rates beyond stochastic continuity

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The LIBOR reform

- London Interbank Offered Rate (LIBOR), computed as the trimmed average of rates reported by a panel of banks, for five currencies (CHF, EUR, GBP, JPY, USD) and seven tenors (1D, 1W, 1M, 2M, 3M, 6M, 1Y).
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- July 2017: *The future of LIBOR* speech by Andrew Bailey (FCA): announcement of LIBOR discontinuation after 2021.
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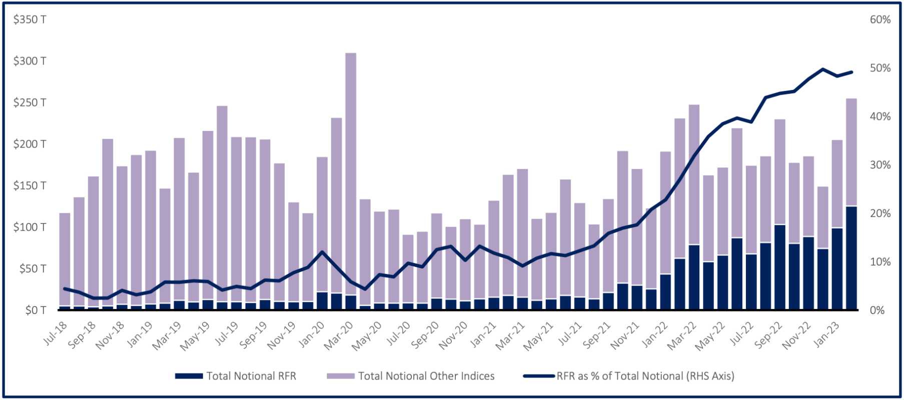
- Transition towards transaction-based overnight rates as benchmark rates. ARRC, June 2017: Secured Overnight Funding Rate (SOFR) in the US.
- FCA, March 2021: cessation of LIBOR on 31/12/2021. Complete discontinuation after June 2023.
- May 2021: *Life after LIBOR* speech by Andrew Bailey:

  > *"transition to the most robust overnight rates, underpinned by deep underlying markets, will support a stronger more transparent financial system and ultimately benefit all market participants".*

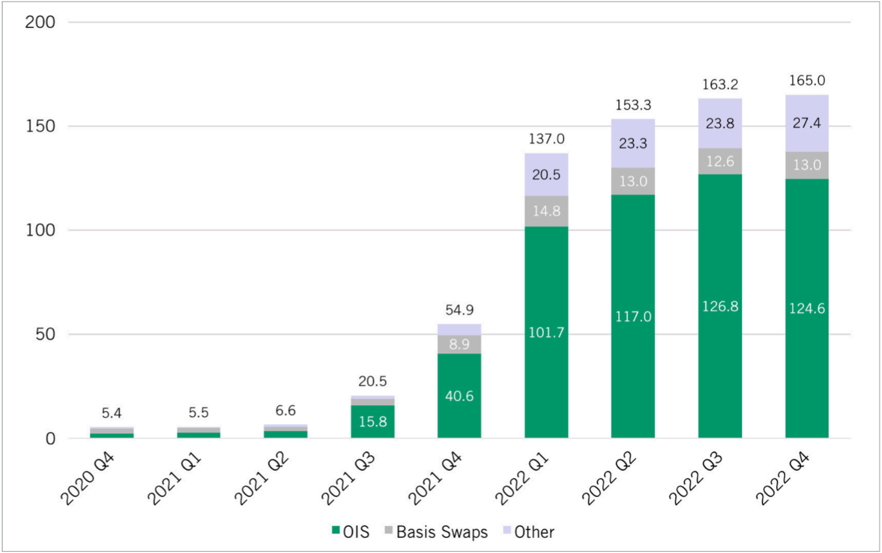
Adoption of overnight rates



<u>Source</u>: ISDA-Clarus RFR adoption indicator, February 2023.

Trades in SOFR

Chart 5: SOFR Trade Count by Product (thousands)



Source: DTCC SDR

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- RFRs are prone to upward/downward spikes at regulatory reporting dates: SOFR is on average 20.25 bps higher at quarter-ends compared to other dates (source: Klingler and Syrstad (2021), period: 08/2014 - 12/2019).

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These facts bring evidence of the presence of **stochastic discontinuities**: new information arriving at pre-determined dates that affects the level of the rates.

SOFR behavior: spikes and hikes

SOFR time series from 01/01/2018 until 12/12/2022 (source: Refinitiv).

SOFR behavior: spikes and hikes

- Let us consider the spike observed on 17/09/2019.
  According to Anbil et al. (2020):

  > *Strains in money markets in September seem to have originated from routine market events, including a corporate tax payment date and Treasury coupon settlement. The outsized and unexpected moves in money market rates were amplified by a number of factors.*

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- Presence of stochastic discontinuities in the RFR dynamics.
  This phenomenon is playing an important role in recent works:
  - Andersen and Bang (2020): spikes in the SOFR dynamics, both at totally inaccessible times and at anticipated times.
  - Gellert and Schlögl (2021): a diffusive HJM model for instantaneous forward rates, with jumps/spikes at fixed times in the short rate, inspired by SOFR.
  - Brace et al. (2022): diffusive HJM model with stochastic volatility.
  - Backwell and Hayes (2022): a short-rate model for the SONIA rate, based on a pure jump process with expected and unexpected jumps times.
  - Fontana et al. (2020): multi-curve framework with stochastic discontinuities.

A quick overview of the literature on RFR modelling

- General aspects of the Libor reform: Henrard (2019), Piterbarg (2020), Klingler and Syrstad (2021), Baig and Winters (2022).
- Mercurio (2018): short rate model for SOFR, adding a deterministic spread to the OIS rate.
- Lyashenko and Mercurio (2019): one of the first and most influential contributions, extending the classical Libor market model.
- Macrina and Skovmand (2020): rational model driven by an affine process.
- Willems (2020): extended SABR model applied to caplet pricing.
- Extensions of the Hull-White model: Hofman (2020), Turfus (2020), Hasegawa (2021), Xu (2022).
- Fontana (2023): general affine models for RFRs and pricing formulae.
- Skov and Skovmand (2021), Skov and Skovmand (2022): multi-factor Gaussian models for SOFR futures.
- Rutkowski and Bickersteth (2021): Vasiček model for SOFR, discussing pricing and hedging in the presence of funding costs and collateralization.

Outline

1. Numéraire, backward-looking and forward-looking rates;
2. an extended HJM framework;
3. the affine semimartingale setup;
4. an extended Hull-White model;
5. hedging problems.

The RFR numéraire

- We consider a continuous-time RFR process $\rho = (\rho_t)_{t \geq 0}$. In line with empirical evidence, ρ is allowed to have expected and unexpected jumps.
- The numéraire S^0 asset:

$$S_t^0 = \exp\left(\int_{(0,t]} \rho_u \, \eta(du) \right),$$

  where $\eta(du) = du + \sum_{n \in \mathbb{N}} \delta_{\{t_n\}}(du)$.
- The set $\mathcal{T} := \{t_n; n \in \mathbb{N}\}$ of roll-over dates, at which S^0 is expected to jump.

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- The set $\mathcal{T} := \{t_n; n \in \mathbb{N}\}$ of roll-over dates, at which S^0 is expected to jump.
- Depending on the specification of ρ and η, this setup includes:
  - classical short-rate approach (corresponding to $\mathcal{T} = \emptyset$);
  - discretely updated bank account at overnight frequency:

$$S^0_t = \prod_{t_{n+1}\leq t} \left(1 + r_{t_n}(t_{n+1} - t_n)\right),$$

where r_{t_n} is the overnight rate for the time interval $[t_n, t_{n+1}]$.

- Denote by $P(t, T)$ the zero-coupon bond (ZCB) price at t for maturity T.

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$$R(S,T) := \frac{1}{T-S} \prod_{n \in N(S,T)} \left(\frac{1}{P(t_n, t_{n+1})} - 1 \right),$$

  where $N(S,T) := \{n \in \mathbb{N} : S \leq t_n \text{ and } t_{n+1} \leq T\}$.
- According to the ISDA protocol, $R(S,T)$ is chosen as the LIBOR fallback, up to an additive spread determined from historical data.
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- CME Term SOFR and Refinitiv Term SONIA are forward-looking rates. 12/29/2021: ARRC endorsed CME term SOFR as forward-looking rate.
- The use of term SOFR for derivatives is currently restricted by ARRC, but there is increasing market pressure for derivatives referencing term SOFR.

Forward term rates

As in Lyashenko and Mercurio (2019), we can consider two types of forward rates:

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Consequence of the above definitions:

$$F(t, S, T) = R(t, S, T), \qquad \text{for all } t \in [0, S].$$

The forward-looking forward rate $F(t, S, T)$ stops evolving at time S, while the backward-looking forward rate $R(t, S, T)$ continues to evolve until time T, with

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$\Rightarrow$Forward-looking and backward-looking forward rates can be consolidated into a single process $R(\cdot, S, T)$. We call this process the forward term rate.

Forward term rates

Payoff $1 + (T - S)R(S, T)$ at maturity T can be statically replicated as follows:

- buy-and-hold strategy in one ZCB with maturity S;
- at time S, invest 1 in a roll-over strategy remunerated at the overnight rate.

This implies the following (classical) representation of forward term rates:

$$R(t, S, T) = \frac{1}{T - S}\left(\frac{P(t, S)}{P(t, T)} - 1\right),$$

where we extend ZCB bond prices beyond maturity by setting

$$P(t, S) = \frac{P(t, t_{n(t)})}{P(t_{n(t)-1}, t_{n(t)})} \prod_{n \in N(S,t)} \frac{1}{P(t_n, t_{n+1})}, \qquad \text{for } t > S,$$

with $n(t) := \inf\{n \in \mathbb{N} : t_n \geq t\}$.

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Similarly to classical (single-curve) interest rate models, the family of ZCB prices $\{P(\cdot, T); T > 0\}$ constitutes the fundamental basis of a term structure model.

An extended HJM framework

We start by specifying ZCB prices as follows:

$$P(t,T) = \exp\Big(-\int_{(t,T]} f(t,u)\eta(du)\Big),$$

where $\eta(dt) = dt + \sum_{n\in\mathbb{N}} \delta_{\{t_n\}}(dt)$ and we assume that

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \varphi(s,T)dW_s + V(t,T),$$

with W a d-dim. Brownian motion and $V(\cdot,T)$ a pure jump process such that

$$\{\Delta V(\cdot,T) \neq 0\} \subseteq \Omega \times \mathcal{S}, \qquad \text{where } \mathcal{S} = \{s_1,\ldots,s_M\}.$$

The set $\mathcal{S}$ contains expected jump dates, i.e., dates at which the RFR ρ and forward term rates are expected to exhibit jumps.

Remarks:

- Lévy-type jumps can be included;
- we do not exclude the case $\mathcal{S} \cap \mathcal{T} \neq \emptyset$;
- $\mathcal{S}$ can be generalized to a countable family of predictable times.

Martingale representation property

The representation of instantaneous forward rates implicitly uses the following.

Assumption

There exists a family $(\xi_1, \ldots, \xi_M)$ of random variables such that ξ_i is $\mathcal{F}_{s_i}$-measurable, for all $i = 1, \ldots, M$, and every local martingale $N = (N_t)_{t \geq 0}$ can be represented as

$$N = N_0 + \int_0^{\cdot} \theta_t dW_t + \sum_{i=1}^{M} f_i(\xi_i) \mathbf{1}_{[\![s_i, +\infty [\![},$$

where $f_i(\cdot) : \Omega \times X \to \mathbb{R}$ is a $(\mathcal{F}_{s_i-} \otimes \mathcal{B}_X)$-measurable function such that

$$E[f_i(\xi_i) | \mathcal{F}_{s_i-}] = 0 \quad \text{a.s.}$$

We denote by $\mathcal{H}$ the space of all such functions $f = (f_1, \ldots, f_M)$.

Technical assumptions

The following conditions hold a.s.:

(i) the *initial forward curve* $T \to f(0, T)$ is $(\mathcal{F}_0 \otimes \mathcal{B}_{\mathbb{R}_+})$-measurable, real-valued and satisfies $\int_0^T |f(0, u)| du < +\infty$, for all $T > 0$;

(ii) the *drift process* $\alpha : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is progressively measurable, satisfies $\alpha(t, T) = 0$ for $T < t$, and

$$\int_0^T \int_0^u |\alpha(s, u)| ds \, \eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iii) the *volatility process* $\varphi : \Omega \times \mathbb{R}_+^2 \to \mathbb{R}^d$ is progressively measurable and satisfies $\varphi(t, T) = 0$ for $T < t$, and

$$\sum_{i=1}^d \int_0^T \left(\int_0^u |\varphi^i(s, u)|^2 ds \right)^{1/2} \eta(du) < +\infty, \qquad \text{for all } T > 0;$$

(iv) the *stochastic discontinuity process* $V(\cdot, T)$ satisfies $\int_0^T |\Delta V(s, u)| du < +\infty$ for all $s \in \mathcal{S}$ and $\Delta V(t, T) = 0$ for $T < t$.

An extended HJM framework

Objective: characterize when Q is a risk-neutral measure, i.e., S^0-denominated ZCB prices are local martingales under Q. This ensures absence of arbitrage in the sense of *no asymptotic free lunch with vanishing risk* (NAFLVR, see Cuchiero et al. (2016)), with respect to the numéraire S^0.

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As a preliminary to the next theorem, we define

$$\bar{\alpha}(t,T) := \int_{[t,T]} \alpha(t,u)\eta(du),$$

$$\bar{\varphi}(t,T) := \int_{[t,T]} \varphi(t,u)\eta(du),$$

$$\bar{V}(t,T) := \int_{[t,T]} \Delta V(t,u)\eta(du).$$

HJM-type conditions

Theorem

Q is a risk-neutral measure if and only if (some integrability properties hold) and the following four conditions are satisfied:

(i)
$$f(t,t) = \rho_t,$$

(ii)
$$\bar{\alpha}(t,T) = \frac{1}{2}\|\bar{\varphi}(t,T)\|^2$$

(iii) for every $j = 1, \ldots, N$ it holds that
$$f(t_j-, t_j) = \rho_{t_j-} - \log\left(E[e^{-\Delta\rho_{t_j}} | \mathcal{F}_{t_j-}]\right),$$

(iv) for every $i = 1, \ldots, M$ it holds that
$$E\left[e^{-\Delta\rho_{s_i}\delta_{\mathcal{T}}(s_i)}\left(e^{-\int_{(s_i,T]}\Delta V(s_i,u)\eta(du)} - 1\right)\Big|\mathcal{F}_{s_i-}\right] = 0.$$

Remark: if $\mathcal{S} \cap \mathcal{T} = \emptyset$, then conditions (i) and (iii) can be jointly written as
$$f(t,t) = \rho_t, \qquad \eta(dt) \otimes dQ\text{-a.e.}$$

Example: a Cheyette-type model

An extension of the Cheyette model with stochastic discontinuities:

- for simplicity, no roll-over dates ($\mathcal{T} = \emptyset$), so that $S^0 = \exp(\int_0^{\cdot} r_u du)$;
- forward rates are specified as follows:

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \varphi(s,T)dW_s + \sum_{s_i \le t} \bigl(\alpha_i(T) + \xi_i g_i(T)\bigr),$$

  with independent $\xi_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, for $i = 1, \ldots, M$;
- separable volatility structure (one factor, for illustration):

$$\varphi(t,T) = \frac{a(T)}{a(t)} b(t) \qquad \text{and} \qquad g_i(T) = a(T)B_i.$$

- Under this volatility structure, it holds that

$$f(t,T) = f(0,T) + \frac{a(T)}{a(t)} X_t + U(t,T),$$

  where X is a mean-reverting Gaussian Markov process with mean-reversion speed $\partial_t \log(a(t))$, diffusion coefficient b and jumps at dates $\{s_1, \ldots, s_M\}$, and $U(t,T)$ is a deterministic function.

The affine framework

The presence of expected jump times requires an extension of affine processes: affine semimartingales generalize affine processes by allowing for jumps at fixed times with possibly state-dependent jump sizes (see Keller-Ressel et al. (2019)).

An affine semimartingale $X = (X_t)_{t \geq 0}$ taking values in $\mathbb{R}_+^m \times \mathbb{R}^n$ satisfies

$$E\left[e^{\langle u, X_T \rangle} | \mathcal{F}_t\right] = \exp\left(\phi_t(T, u) + \langle \psi_t(T, u), X_t \rangle\right),$$

for all $u \in \mathcal{U} = \mathbb{C}_-^m \times \mathrm{i}\mathbb{R}^n$, where the functions $\phi_t(T, u)$ and $\psi_t(T, u)$ satisfy generalized Riccati equations.

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Short-rate approach: let the RFR be given by

$$\rho_t = \ell(t) + \langle \Lambda, X_t \rangle, \qquad \text{for all } t \geq 0,$$

where the function ℓ fits the initially observed term structure.

Proposition

The joint process $(X, \int_0^\cdot \rho_u \, \eta(du))$ is an affine semimartingale.

- Similar to the enlargement of the state-space approach of Duffie et al. (2003).
- Fourier-based methods for pricing a variety of interest rate derivatives.

An example: an extended Hull-White model

Assume that $\rho = (\rho_t)_{t \geq 0}$ satisfies

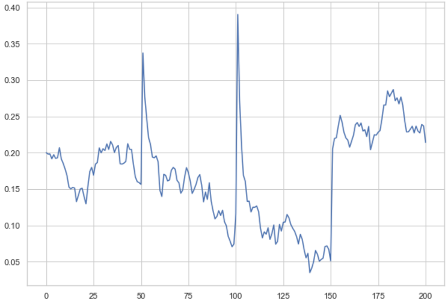
$$d\rho_t = \big(\alpha(t) + \beta\rho_t\big)dt + \sigma dW_t + dJ_t,$$

where J is a pure jump process independent of W:

$$J = \sum_{i=1}^{M} \xi^i \mathbf{1}_{[\![s_i,+\infty[\![},$$

In the Gaussian case (i.e., $(\xi_i)_{i=1,\ldots,M}$ independent and Gaussian):

- explicit formula for ZCB prices;
- Black-type formula for post-Libor caplets/floorlets.



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- Suppose that the market contains a single risky asset with price process

$$X = X_0 + A + M,$$

  where
  - A is predictable process of finite-variation,
  - $M = \int_0^{\cdot} \eta_t dW_t + \sum_{s_i \leq \cdot} w_i(\xi_i)$ is a square-integrable martingale.
- For instance, X can represent the price process of a SOFR future contract (currently the most liquid SOFR product).

Hedging with stochastic discontinuities

Let $H \in L^2$ be an $\mathcal{F}_T$-measurable payoff. We denote by Θ the set of predictable processes ζ such that $E[\int_0^T \zeta_u^2 d\langle M\rangle_u + (\int_0^T \zeta_u dA_u)^2] < +\infty$.

Definition

- We call *H-admissible strategy* a pair $\varphi = (\zeta, V)$, where $\zeta = (\zeta_t)_{t\in[0,T]} \in \Theta$ and $V = (V_t)_{t\in[0,T]}$ is an adapted process such that $V_T = H$ a.s.
- We say that an H-admissible strategy $\varphi = (\zeta, V)$ is *locally risk-minimizing* if the associated cost process

$$C_t(\varphi) := V_t - \int_0^t \zeta_u dX_u, \qquad \text{for } t \in [0, T],$$

is a square-integrable martingale strongly orthogonal to M.

Remarks:

- ζ_t and V_t represent respectively the positions held in the traded security and the portfolio value at time t, for all $t \in [0, T]$;
- if X satisfies the so-called structure condition, the above definition is equivalent to the original definition of Schweizer (1991).

Hedging with stochastic discontinuities

- By absence of arbitrage, there exists a predictable process $\boldsymbol{\lambda}$ such that $A = \int_0^{\cdot} \lambda_u \, d\langle M \rangle_u$. In particular, this implies that

$$\Delta A_{s_i} = \lambda_{s_i} E[(\Delta M_{s_i})^2 | \mathcal{F}_{s_i -}] \text{ a.s., for all } i = 1, \dots, M.$$

- Assume that $\widehat{\boldsymbol{Z}} := \mathcal{E}(-\int_0^{\cdot} \lambda_u dM_u)$ is a strictly positive square-integrable martingale and define the minimal martingale measure by $d\widehat{Q} = \widehat{Z}_T dQ$.

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- We can then define the $\widehat{Q}$-martingale $\widehat{\boldsymbol{H}} = (\widehat{H}_t)_{t \in [0,T]}$ by

$$\widehat{H}_t := \widehat{E}[H | \mathcal{F}_t], \qquad \text{for all } t \in [0, T],$$

where we denote by $\widehat{E}$ the expectation with respect to $\widehat{Q}$.

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  where we denote by $\widehat{E}$ the expectation with respect to $\widehat{Q}$.
- By Bayes' formula, $\widehat{H} = N/\widehat{Z}$, with $N_t := E[\widehat{Z}_T H|\mathcal{F}_t]$, for all $t \in [0, T]$.
- As a consequence of the martingale representation assumption, we have that

$$N = N_0 + \int_0^\cdot \boldsymbol{\theta_u} dW_u + \sum_{s_i \leq \cdot} \Delta N_{s_i}.$$

Hedging with stochastic discontinuities

Proposition

Let H be an $\mathcal{F}_T$-measurable random variable such that $\sup_{t\in[0,T]} \widehat{H}_t \in L^2$. Define the predictable process

$$\zeta_t^H := \left(\widehat{Z}_{t-}^{-1}\eta_t^{-1}\theta_t + \widehat{H}_{t-}\lambda_t\right)\delta_{\mathcal{S}^c}(t) + \frac{E[\Delta\widehat{H}_t\Delta M_t|\mathcal{F}_{t-}]}{E[(\Delta M_t)^2|\mathcal{F}_{t-}]}\delta_{\mathcal{S}}(t).$$

If $\zeta^H \in \Theta$, then the strategy $\varphi^H = (\zeta^H, V^H)$ is locally risk-minimizing, where $V_t^H = \widehat{H}_t$, for all $t \in [0,T]$.

Remarks:

- perfect replication at all times $t \in [0,T] \setminus \mathcal{S}$, when the only active source of randomness is the Brownian motion W;
- at the discontinuity dates $\mathcal{S} = \{s_1, \ldots, s_M\}$, the strategy $\zeta_{s_i}^H$ is determined by a linear regression of $\Delta\widehat{H}_{s_i}$ onto ΔX_{s_i}, conditionally on $\mathcal{F}_{s_i-}$:

$$\zeta_{s_i}^H = \frac{\mathrm{Cov}(\Delta\widehat{H}_{s_i}, \Delta X_{s_i}|\mathcal{F}_{s_i-})}{\mathrm{Var}(\Delta X_{s_i}|\mathcal{F}_{s_i-})},$$

- In the paper, explicit formula for the locally risk-minimizing strategy of a SOFR term caplet with respect to a SOFR future.

Thank you for your attention

For more information:

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