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Identification-Robust Factor Pricing: Canadian Evidence

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ABSTRACT

We analyze Arbitrage Pricing Theory (**APT**) based factor models using identification robust inference methods. Such models involve nonlinear reduced rank restrictions whose identification may raise serious non-regularities leading to the failure of standard asymptotics. We next derive confidence set estimates for structural parameters based on inverting, analytically, Hotelling-type type pivotal statistics. Proposed confidence sets have much more informational content than Hotelling-type tests, and extend their relevance beyond reduced forms. Our approach may further be viewed as a multivariate extension of the Fieller problem. We also introduce a formal definition for a redundant factor linking the presence of such factors to unbounded estimation outcomes and document their perverse effects on minimum-root-based model tests. Results are applied to multi-factor asset pricing models with Canadian data, the Fama-French-Carhart benchmarks and monthly returns of 25 portfolios from 1991 to 2010.

Despite evidence of weak identification, several results deserve notice when data is analyzed over 10 year sub-periods. With equally weighted portfolios, the three-factor model is rejected before 2000 and weakly supported thereafter. In contrast, the three factor model is not rejected with value weighted portfolios. Interestingly in this case, the market factor is priced before 2000 along with size, while both Fama-French factors are priced thereafter. The momentum factor severely compromises identification, which calls for caution in interpreting existing work documenting momentum effects for the Canadian market. More generally, our empirical analysis underscore the practical usefulness of our analytical confidence sets.

1 Introduction

Equilibrium based financial econometric models commonly involve reduced rank (**RR**) restrictions. Well known applications include asset pricing works related to the Arbitrage Pricing Theory (**APT**) and to factor models [Black (1972), Ross (1976)].[1]

From a methodological perspective, RR restrictions raise serious statistical challenges. Even with linear multivariate models, discrepancies between standard asymptotic and finite sample distributions can be very severe and are usually attributed to the curse of dimensionality; see *e.g.* Dufour & Khalaf (2002) and the references therein. Nonlinear rank restrictions pose further identification non-regularities that can lead to the failure of standard asymptotics. This provides the motivation for our paper.

From an empirical perspective, we focus on RR-based inference methods relevant to asset pricing factor models. In general multivariate regression based financial models, finite sample motivated testing is important because tests that are only approximate and/or do not account for non-normality can lead to unreliable empirical interpretations of standard financial models; see *e.g.* Shanken (1996), Campbell et al. (1997), Dufour et al. (2003), Dufour et al. (2010) and Beaulieu et al. (2007), Beaulieu et al. (2009), Beaulieu et al. (2010*a*), Beaulieu et al. (2013). In parallel, an emerging literature that builds on Kan & Zhang (1999*a*) and Kan & Zhang (1999*b*) recognizes the adverse effects of a large number of factors; see Kleibergen (2009), Kan et al. (2013), Gospodinov et al. (2014), Kleibergen & Zhan (2013) and Harvey et al. (2015). We consider inference methods immune to both dimensionality and identification difficulties.

[1] See also Gibbons (1982), Barone-Adesi (1985), Shanken (1986), Zhou (1991), Zhou (1995), Bekker et al. (1996), Costa et al. (1997), (Campbell et al. 1997, Chapter 6), Velu & Zhou (1999), and Shanken & Zhou (2007).

The few finite sample rank-based asset pricing methods [see *e.g.* Zhou (1991), Zhou (1995), Shanken (1985), Shanken (1986) and Shanken & Zhou (2007)] have for the most part focused on testing rather than on set inference. Kleibergen (2009) provides numerical identification-robust inference methods backed by a simulation study that reinforces the motivation for our asset-pricing focus.[2] Beaulieu et al. (2013) proposes a confidence set estimate approach, yet the weakly identified parameter is scalar because of a single-factor restriction. In the present paper our results cover parameter vectors, which expands the class of applicable financial models. Overall, the paper has three main contributions.

First, we construct confidence set (**CS**) estimates for the model's parameters of interest, based on inverting minimum-distance pivotal statistics. These include Hotelling's $\mathbf{T}^2$ criterion [Hotelling (1947)]. In multivariate analysis, Hotelling's statistic is mostly used for *test* purposes, and its popularity stems from its least-squares (**LS**) foundations which lead to exact **F**-based null distributions in Gaussian settings. We apply analytical solutions to the test inversion problem. Our CSs provides much more information than Hotelling-type tests, and extend their relevance beyond reduced form specifications; see Beaulieu et al. (2015) for underlying finite sample theory and simulation evidence.

Second, we provide a formal definition of a statistically non-informative factor and linking the presence of such factors to unbounded set estimation outcomes. We also document the perverse effects of adding such factors on J-type minimum-root-based model tests. Although unbounded set estimators are not always uninformative, our results call for caution in relying on model tests alone as measures of fit. This warning has obvious implications for the asset pricing models that motivate our empirical analysis, which we further illustrate empirically.

Third, our results are applied to multi-factor asset pricing models with Canadian data and the benchmark Fama-French-Carhart [Fama & French (1992), Fama &

[2] In contrast, Kan et al. (2013), Gospodinov et al. (2014) and Kleibergen & Zhan (2013) focus on model misspecification.

French (1993), Carhart (1997)] model. We analyze monthly returns of 25 portfolios from 1991 to 2010. The empirical literature on asset pricing models in Canada is scarce. Most studies involving Canadian assets aim to measure North-American financial market integration (see for example Jorion & Schwartz (1986), Mittoo (1992) and Foerster & Karolyi (1993) and more recently Beaulieu et al. (2008)). In some other cases, for example, Griffin (2002) and Fama & French (2012) international multifactor asset pricing models are tested over a large set of countries including Canada to measure the importance of international versus domestic asset pricing factors in different countries.

One of the few articles that relate the importance of a multifactor asset pricing model for Canadian portfolios using exclusively Canadian factors is L'Her et al. (2004); for a larger set of papers, see the references therein. We propose to enlarge this strand of literature to verify on a long time period the importance of domestic Fama and French factors to price Canadian assets adding momentum to the list of factors included in L'Her et al. (2004). The specificities of Canadian assets are put forward and contrasted to American assets for which abundant results are available in the literature. To achieve this goal we use, as described above, improved inference procedures based on a formal definition of a statistically non-informative asset pricing factor.

With reference to the weak-instruments literature, our methodology may be viewed as a generalization of the Dufour & Taamouti (2005) quadrics-based set estimation method beyond the linear limited information simultaneous equations setting. For further quadric based solutions in different contexts, see Bolduc et al. (2010) for inference on multiple ratios, and Khalaf & Urga (2014) for inference on cointegration vectors.

With reference to Kleibergen (2009), our methodology allows to estimate the zero-beta rate [Shanken (1992), (Campbell et al. 1997, Chapter 6) or Lewellen et al. (2010)], which has not been considered by Kleibergen (2009). Theoretically coherent empirical models often impose restrictions on the risk-prices via the zero-beta rate; see *e.g.* (Lewellen et al. 2010, prescription 2). Examples include: (i) whether the zero beta rate is equal to the risk-free rate, and (ii) whether the risk price of a traded portfolio when included as a factor equals the factor's average return in excess of the zero-beta rate [Shanken (1992)]. We also formally control for factors that are tradable portfolios. Furthermore, in contrast to Kleibergen's numerical projections, our analytical solutions can easily ascertain empty or unbounded sets, whereas numerical solutions remain subject to precision and execution time constraints. Avoiding numerical searches is particularly useful for multi-factor models.

With reference to the statistical literature, our test inversion approach can be viewed as an extension of the classical inference procedure proposed by Fieller (1954) [see also Zerbe et al. (1982), Dufour (1997), Beaulieu et al. (2013), Bolduc et al. (2010)] to the multivariate setting. As with standard χ^2-type methods including *e.g.* the *delta* method, our generalized Fieller approach relies on LS. Yet both methods exploit LS theory in fundamentally different ways. In contrast to the former which excludes parameter discontinuity regions out and which by construction yields bounded confidence intervals, our inverted test does not require parameter identification and allows for unbounded solutions.

Empirically, despite overwhelming evidence of weak identification, several interesting results deserve notice particularly when data is analyzed over 10 year subperiods. With equally weighted portfolios, the three-factor model is rejected before 2000. Thereafter, the model is weakly supported with HML confirmed as the only priced factor. With value weighted data, the three factor model is not rejected; interestingly, the market factor is priced before 2000 and unidentified thereafter, while both Fama-French factors are priced post 2000. Although momentum is priced with equally-weighted data before 2000, the resulting sets on other factors are practically uninformative. The momentum factor thus severely compromises identification, which calls for caution in interpreting existing work documenting momentum effects for the Canadian market.

The paper is organized as follows. Section 2 sets the notation and framework. In section 3 we present our test inversion method and associated projections. Our empirical analysis is discussed in section 4. Section 5 concludes followed by a technical appendix.

2 Multifactor pricing model

Let r_i, $i = 1, \ldots, n$, be a vector of T returns on n assets or portfolios for $t = 1, \ldots, T$, and $\mathcal{R} = \begin{bmatrix} \mathcal{R}_1 & \ldots & \mathcal{R}_q \end{bmatrix}$ a $T \times q$ matrix of observations on q risk factors. Building on multivariate regressions of the form

$$r_i = a_i \iota_T + \mathcal{R} b_i + u_i, \qquad i = 1, \ldots, n, \tag{1}$$

our APT based empirical analysis formally accounts for tradable factors; see Shanken (1992), (Campbell et al. 1997, Chapter 6) and Lewellen et al. (2010) and note that the inference method applied by Kleibergen (2009) relaxes tradability restrictions.

Without loss of generality, suppose that $\mathcal{R}_1$ corresponds to a vector of returns on a tradable portfolio, for example, a market benchmark. Partition $\mathcal{R}$ into $\mathcal{R}_1$ and $\mathcal{F}$ so the latter $T \times (q-1)$ submatrix contains the observations on the $(q-1)$ factors $\begin{bmatrix} \mathcal{R}_2 & \ldots & \mathcal{R}_q \end{bmatrix}$ and partition b_i conformably into the scalar b_{i1} and the $(q-1)$ dimensional vector $b_{i\mathcal{F}}$. In this context, the APT implies a zero-intercept in the regression of returns in excess of the zero-beta rate, denoted γ_0, on: (i) $\mathcal{R}_1$ in excess of γ_0, and (ii) on $\mathcal{R}_2, \ldots, \mathcal{R}_q$ in excess of a risk price $(q-1)$ dimensional vector denoted $\gamma_{\mathcal{F}}$, where γ_0 and $\gamma_{\mathcal{F}}$ are unknown parameters:

$$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0)\, b_{i1} + (\mathcal{F} - \iota_T \gamma_{\mathcal{F}}')\, b_{i\mathcal{F}} + u_i, \qquad i = 1, \ldots, n. \tag{2}$$

Rewriting the latter in the (1) form produces a non-linear RR restriction on its intercept that captures the zero-beta rate and risk premia as model parameters:

$$r_i = a_i \iota_T + \mathcal{R}_1 b_{i1} + \mathcal{F} b_{i\mathcal{F}} + u_i, \tag{3}$$

$$a_i + \gamma_0(b_{i1} - 1) + \gamma'_{\mathcal{F}} b_{i\mathcal{F}} = 0. \tag{4}$$

Here we derive confidece set estimators for the q-dimensional vector

$$\theta = (\gamma_0, \gamma'_{\mathcal{F}})' \tag{5}$$

which we interpret via a traditional cross-sectional factor pricing approach [see e.g. (Campbell et al. 1997, Chapter 6) or Shanken & Zhou (2007)]. Indeed, the time series averages of (3) leads (with obvious notation) to

$$\begin{aligned} \bar{r}_i &= \gamma_0(1 - b_{i1}) - \gamma'_{\mathcal{F}} b_{i\mathcal{F}} + \bar{\mathcal{R}}_1 b_{i1} + \bar{\mathcal{F}} b_{i\mathcal{F}} + \bar{u}_i, \qquad i = 1, \ldots, n \\ &= \gamma_0 + \left(\bar{\mathcal{R}}_1 - \gamma_0\right) b_{i1} + \left(\bar{\mathcal{F}} - \gamma'_{\mathcal{F}}\right) b_{i\mathcal{F}} + \bar{u}_i, \qquad i = 1, \ldots, n. \end{aligned} \tag{6}$$

It follows that γ_0 retains its traditional cross-sectional definition, and the coefficients on betas of the non-tradable factors correspond to $\left(\bar{\mathcal{F}} - \gamma'_{\mathcal{F}}\right)$. This implies that $\mathcal{R}_1$ is not priced if $\bar{\mathcal{R}}_1 - \gamma_0 = 0$ and each of the remaining factors will be not be priced in turn, if the corresponding components of $\bar{\mathcal{F}} - \gamma'_{\mathcal{F}} = 0$. Our procedure as described below yields confidence intervals on γ_0 and on the components of $\gamma_{\mathcal{F}}$. Given these intervals, we assess whether the tradable factor is not priced if $\bar{\mathcal{R}}_1$ is covered, and whether each factor besides $\mathcal{R}_1$ is not priced if the average of each factor is, in turn, not covered.

Our intervals are simultaneous in the sense of joint coverage, which implies that decisions on pricing will also be simultaneous. A formal definition of simultaneity is further discussed in the next section which outlines our confidence set estimation method.

3 Identification, estimation and testing

Whether viewed as a RR multivariate or cross sectional regression, identification of θ can be assessed within (2) or (6). If $b_{i1} \simeq 1$ then γ_0 almost drops out of the model.

Furthermore, if any of the factor *betas* is close to zero across i, its risk price effectively drops out. In fact, whenever factor *betas* bunch up across or within test assets, problems akin to cross-sectional colinearity emerge that undermine identification of θ. As may be checked within (6), for θ to be recoverable with no further data and information (in particular in the absence of other instruments), the *betas* per factor need to vary enough across equation. In practice, reliance on portfolios to reduce dimensionality ends up reducing dispersion of *betas*. Identification concerns are thus an empirical reality.

To address this problem, we extend Beaulieu et al. (2013) beyond the single factor and scalar parameter case. In particular, we proceed by inverting a multivariate test of a hypothesis that fixes θ to a given value, say θ_0:

$$H_0(\theta_0) : \theta = \theta_0, \quad \theta_0 \text{ known.} \tag{7}$$

Inverting the test in question at a given level α consists in assembling the θ_0 values that are not rejected at this level. For example, given a right-tailed statistic $\mathcal{T}(\theta_0)$ with α-level critical point $\mathcal{T}_c$, our procedure involves solving, over θ_0, the inequality

$$\mathcal{T}(\theta_0) < \mathcal{T}_c \tag{8}$$

where $\mathcal{T}_c$ controls size for any θ_0. The solution results in a parameter space subset, denoted $\mathrm{CS}(\theta; \alpha)$, that satisfies

$$\mathsf{P}\big[\,\theta \in \mathrm{CS}(\,\theta; \alpha)\,\big] \geq 1 - \alpha \tag{9}$$

whether θ is identified or not.

To derive confidence intervals for the individual components of θ, or more generally, for a given scalar function $g(\theta)$, we proceed by projecting $\mathrm{CS}(\,\theta; \alpha)$, that is, by minimizing and maximizing $g(\theta)$ over the θ values in $\mathrm{CS}(\,\theta; \alpha)$. The resulting intervals so obtained are simultaneous, in the following sense: for any set of m continuous real valued functions of θ, $g_i(\theta) \in R$, $i = 1, ..., m$, let $g_i\big(\mathrm{CS}(\theta; \alpha)\big)$ denote the image of $\mathrm{CS}(\theta;\alpha)$ by the function g_i. Then

$$\mathsf{P}\big[g_i(\theta) \in g_i\big(\,\mathrm{CS}(\theta;\alpha)\,\big), \quad i = 1, \ldots, m\big] \geq 1 - \alpha. \tag{10}$$

If $\mathcal{T}_c$ is defined so that (9) holds without identifying θ then (10) would also hold whether θ is identified or not. A complete description of our methodology requires: (i) defining $\mathcal{T}(\theta_0)$, (ii) deriving $\mathcal{T}_c$ to control size for any θ_0, and (iii) characterizing the solution of (8). As in Beaulieu et al. (2013), we find an analytical solution to all of the above.

We focus on the Hotelling-type statistic

$$\Lambda(\theta) = \frac{(1,\theta')\hat{B}\hat{S}^{-1}\hat{B}'(1,\theta')'}{(1,\theta')(X'X)^{-1}(1,\theta')'} \tag{11}$$

where θ is set by (7)

$$\hat{B} = (X'X)^{-1}X'Y\ , \quad \hat{S} = \hat{U}'\hat{U}, \quad \hat{U} = Y - X\hat{B}, \tag{12}$$

X is a $T \times k$ full-column rank matrix that includes a constant regressor and the observations on all q factors so $k = q + 1$, and Y is the $T \times n$ matrix that stacks the left-hand side returns in deviation from the tradable benchmark. Clearly, $\hat{B}$ and $\hat{U}$ are the OLS estimators for the underlying unrestricted reduced form

$$Y = XB + U \Leftrightarrow Y_t = B'X_t + U_t, \quad t = 1,\ \ldots\ ,\ T \tag{13}$$

where U is the disturbance matrix and the restriction on the intercept (3) is rewritten as $(1,\theta')B = 0$ in which case (7) corresponds to

$$H_0(\theta_0) : (1,\theta_0')B = 0, \quad \theta_0 \text{ known}.$$

We rely on the F-approximation

$$\Lambda(\theta)\frac{\tau_n}{n} \sim F(n,\tau_n)\,, \quad \tau_n = T - k - n + 1. \tag{14}$$

which holds exactly when the regression errors U_t are contemporaneously correlated *i.i.d.* Gaussian assuming we can condition on X for statistical analysis. The latter distributional result does not require any identification restriction.[3] For proofs and further references, see Beaulieu et al. (2015) and references therein. Simulation studies reported in this paper confirm that fat tailed disturbances arising from multivariate-**t** or GARCH do not cause size distortions for empirically relevant designs.

$\Lambda(\theta)$ can be interpreted relative to the financial literature and in particular the well known zero-intercept test of Gibbons et al. (1989), as follows. The classical Hotelling statistics which provide multivariate extensions of the Student-t based significance tests take the form

$$\Lambda_{0j} = \frac{s_k[i]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[i]}{s_k[i]'(X'X)^{-1}s_k[i]}. \tag{15}$$

where $s_k[j]$ denotes a k-dimensional selection vector with all elements equal to zero except for the j-th element which equals 1. The underlying hypotheses assess the joint contribution of each factor in (13), that is

$$H_{0j} : s_k[j]'B = 0, \quad j \in \{1, ..., k\}. \tag{16}$$

So for example $s_k[1]$ provides inference on the unrestricted intercept, and $s_k[2]$ allows one to assess the *betas* on the tradable factor in deviation from one; see Beaulieu et al. (2010a) for recent applications. The same null distribution as in (14) also holds for each of the Λ_{0j} (under the above same conditions). Interestingly, we show that inverting $\Lambda(\theta)$ embeds all of these tests via a sufficient condition for bounded outcomes that avoids pre-testing.

To invert (11)-(14) we rewrite the inequation

$$\Lambda(\theta) \leq f_{n,\tau_n,\alpha} \tag{17}$$

[3] Other than the usual Least Squares assumptions on $X'X$ and $\hat{S}$.

where $f_{n,\tau_n,\alpha}$ denotes the α-level cut off point from the $F(n,\tau_n)$ distribution as

$$(1,\theta')A(1,\theta')' \leq 0 \tag{18}$$

where A is the $k \times k$ data dependent matrix

$$A = \hat{B}\hat{S}^{-1}\hat{B}' - (X'X)^{-1} f_{n,\tau_n,\alpha}(n/\tau_n). \tag{19}$$

Simple algebraic manipulations suffice to show the above. Next, inequality (18) is re-expressed as

$$\theta' A_{22}\theta + 2A_{12}\theta + A_{11} \leq 0 \tag{20}$$

which leads to the set-up of Dufour & Taamouti (2005) so projections based CSs for any linear transformation of θ can be obtained as described in this papers. The solution is reproduced in the Appendix for completion. This requires partitioning A as follows

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{21}$$

where A_{11} is a scalar, A_{22} is $q \times q$, and $A_{12} = A_{21}'$ is $1 \times q$. Using the partitioning

$$\hat{B} = \begin{bmatrix} \hat{a}' \\ \hat{b} \end{bmatrix}, \quad \hat{b} = \begin{bmatrix} \hat{\beta}_2' \\ \vdots \\ \hat{\beta}_k' \end{bmatrix} \tag{22}$$

where $\hat{a}'$ is $1 \times q$ and conformably

$$(X'X)^{-1} = \begin{bmatrix} x^{11} & x^{12} \\ x^{21} & x^{22} \end{bmatrix} \tag{23}$$

where x^{11} is a scalar, $x^{21} = x^{12\prime}$ is $q \times 1$ and x^{22} is $q \times q$, we have $A_{11} = \hat{a}'\hat{S}^{-1}\hat{a} - (f_{n,\tau_n,\alpha}(n/\tau_n))\,x^{11}$, $A_{12} = A_{21}' = \hat{a}'\hat{S}^{-1}\hat{b}' - (f_{n,\tau_n,\alpha}(n/\tau_n))\,x^{12}$ and

$$A_{22} = \hat{b}\hat{S}^{-1}\hat{b}' - (f_{n,\tau_n,\alpha}(n/\tau_n))\,x^{22}. \tag{24}$$

The outcome of resulting projections can be empty, bounded, or the union of two unbounded disjoint sets. Dufour & Taamouti (2005) show that outcomes are unbounded if A_{22} is not positive definite. Applying basic algebra again to (24) reveals that the diagonal terms of A_{22} are

$$F_j = s_k[j]'\hat{B}\hat{S}^{-1}\hat{B}'s_k[j] - s_k[j]'(X'X)^{-1}s_k[j]\frac{nf_{n,\tau_n,\alpha}}{\tau_n}, \quad j = 1, ..., k.$$

Comparing this expression to the definition of Λ_{0j} in (15) implies that

$$\Lambda_{0j}(\tau_n)/n < f_{n,\tau_n,\alpha} \Leftrightarrow F_j < 0, \quad j = 1, ..., k.$$

So if any of the classical Hotelling tests, using the distribution in (14), is not significant at level α, then A_{22} cannot be positive definite and the confidence set will be unbounded. That is, if the tradable beta is not significantly different from one over all portfolios or if any of the factors is jointly redundant, then information on the zero beta rate as well as the risk price *for all factors* is compromised. The above condition is sufficient but not necessary. It follows that although Hotelling tests on each factor are useful, the information they provide is incomplete as for the joint usefulness of the factors in identifying risk price.

An important result from Beaulieu et al. (2013) regarding empty confidence set outcomes also generalizes to our multifactor case. Because the cut-off point underlying the inversion of $\Lambda(\theta)$, which we denoted $f_{n,\tau_n,\alpha}$ above, is the same for all θ values, then

$$\min_{\theta} \Lambda(\theta) \geq f_{n,\tau_n,\alpha} \Leftrightarrow \mathrm{CS}(\theta; \alpha) = \varnothing.$$

Again, basic matrix algebra allows one to show [see e.g. Gouriéroux et al. (1996)] that minimizing $\Lambda(\theta)$ produces the Gaussian-LR statistic to test the non-linear restriction (3) which defines θ. This suggests a bounds J-type test to assess the overall model fit. The outcome of this test will be revealed via the quadric solution we implement for test inversion, so it is built into our general set inference procedure. A potential unbounded confidence set guards the researcher from misreading nonsignificant tests as evidence in favour of models on which data is not informative.

In summary, our confidence sets will summarize the data's informational content (from a least-squares or Gaussian quasi-likelihood perspective) on risk price without compounding type-I errors.

4 Empirical analysis

In our empirical analysis of a multifactor asset pricing model, we use Canadian Fama and French (1992, 1993) factors as well as momentum (Carhart, 1997). We produce results for monthly returns of 25 value weighted portfolios from 1991 to 2010. Portfolios were constructed with all Canadian stocks available on Datastream and Worldscope. The portfolios which are constructed at the end of June are the intersections of five portfolios formed on size (market equity) and five portfolios formed on the ratio of book equity to market equity. The size breakpoints for year s are the Toronto Stock Exchange (TSE) market equity quintiles at the end of June of year s. The ratio of book equity to market equity for June of year s is the book equity for the last fiscal year end in $s-1$ divided by market equity for December of year $s-1$. The ratios of book equity to market equity are TSE quintiles. We use the filter from Karolyi & Wu (2014) to eliminate abnormal observations in our database. This filter is minimally invasive as the filtered database contains an overall sample average of 1700 stocks.

The benchmark factors are: 1) the excess return on the market, defined as the value-weighted return on all TSE stocks (from Datastream and Worldscope database) minus the one-month Treasury bill rate (from the Bank of Canada), 2) SMB (small minus big) defined as the average return on three small portfolios minus the average return on three big portfolios, 3) HML (high minus low) defined as the average return on two value portfolios minus the average return on two growth portfolios, and (4), MOM, the average return on the two high prior return portfolios minus the average return on the two low prior return portfolios. Fama and French benchmark factors, SMB and HML, are constructed from six size/book-to-market benchmark portfolios that do not include hold ranges and do not incur transaction costs. The portfolios for these factors are rebalanced annually using two independent sorts, on size (market equity, ME) and book-to-market (the ratio of book equity to market equity, BE/ME). The size breakpoint (which determines the buy range for the small and big portfolios) is the median TSE market equity. The BE/ME breakpoints (which determine the buy range for the growth, neutral, and value portfolios) are the 30th and 70th TSE percentiles. For the construction of the MOM factor, six value-weighted portfolios formed on size and prior (2–12) returns are used. The portfolios, which are formed monthly, are the intersections of two portfolios formed on size (market equity, ME) and three portfolios formed on prior (2–12) return. The monthly size breakpoint is the median TSE market equity. The monthly prior (2–12) return breakpoints are the 30th and 70th TSE percentiles.

Results over 10 and 5 years subperiods are summarized in Tables 1 and 2, where EW and VW denotes equally weighted and value weighted portofolios. Unless stated otherwise, signficance in what what follows refers to the 5% level. All reported confidence sets are also at the 5% level.

Over 10 year subperiods and with equally-weighted data, the three-factor model is rejected before 2000. Thereafter, while very wide although bounded confidence intervals are observed, HML is confirmed as the only priced factor. Value weighted data supports the three factor model before 2000 albeit weakly as all confidence sets obtained are unbounded. Interestingly, the market factor is priced along with SMB. In sharp contrast, the market risk is unidentified after 2000 and despite evidence of identification difficulties, both Fama-French factors are priced. When the momentum factor is added over and above the Fama-French factor, we find completely uninformative results on all model parameters expect for momentum using equally-weighted data before 2000, in which case we find this factor is priced despite the overwhelming evidence of weak-identification.

Considering 5 year subperiods may help us assess whether the above is driven by instability of *betas*. Results must however be interpreted with caution since sample size considerations can be consequential. Indeed, the bulk of resulting confidence sets are uninformative and the only evidence we can confirm is that the market factor seems to be priced from 1996-2000 while the HML factor is priced post 2006.

Our empirical results collected together reveal that to price Canadian assets, a standard three Fama and French factor model is the best avenue after 2000. Momentum creates important identification problems and it should not be used to price Canadian assets without raising identification questions. This corroborates the empirical results of Beaulieu et al. (2010*b*) and Beaulieu et al. (2015) for American stocks although the case for omitting momentum is stronger in the Canadian context. An important issue further analyzed by Beaulieu et al. (2015) for the U.S. is the portfolio formation method. Using industry portfolios seems to improve identification relative to size-sorting, as the latter compounds factor structure dependences; see also Lewellen & Nagel (2006), Lewellen et al. (2010) and Kleibergen & Zhan (2013).

On balance, given the importance of Fama and French factors for pricing stocks internationally [Fama & French (2012)], and the potential for identification problems presented in this paper, future research should aim at finding ways of choosing the factors that empirically explain the cross-section of returns in a general standard context, as discussed in Harvey et al. (2015).

5 Conclusion

This paper studies the factor asset pricing model for the Canadian market using identification robust inference methods. We derive confidence sets for the zero beta rate and factor price based on inverting minimum-distance Hotelling-type pivotal statistics. We use analytical solutions to the latter problem. Our confidence sets have much more informational content than usual Hotelling tests and have various useful applications in statistics, econometrics and finance. Our approach further provides multivariate extensions of the classical Fieller problem.

Empirical results illustrate, among others, severe problems with redundant factors. These finding concur with the (above cited) emerging literature on redundant factors, on tight factor structures and statistical pitfalls of asset pricing tests, and on the importance of joint (across-portfolios) tests. In practice, our results support a standard three Fama and French factor model for the Canadian market after 2000. In contrast, we find that the momentum factor severely compromises identification which qualifies existing works in this regard.

With regards to the historical debate on the market factor[4], our results suggest an alternative perspective. Perhaps the unconditional market model is neither *dead* nor *alive and well*. Instead, the traditional methods of accounting for additional factors may have confounded underlying inference. Because traditional methods severely understate true uncertainty, identification problems in the literature may have escaped concrete notice so far. More to the point here is that with reference to *e.g.* Harvey et al. (2015) who consider a very large number of factors, we document identification problems with only three to four factors. We thus concur with (Lewellen et al. 2010, prescriptions 5 and 6) that it is by far more useful to report set estimates rather than model tests. However, the proliferation of factors in practice increases the likelihood of redundancies. Their associated costs support the use of our method, and motivate further refinements and improvements as important future research avenues.

[4]See (Campbell et al. 1997, Chapters 5 and 6), Fama & French (2004), Perold (2004), Campbell (2003) and Sentana (2009).

Table 1. Confidence sets for risk premia

Ten years sub-periods

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0)\, b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}})\, b_{i\mathcal{F}} + u_i, \qquad i = 1, \ldots, n$								
$\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$								
EW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	-
91-00	51	$\varnothing$	25	$\varnothing$	28	$\varnothing$	-56	-
00-10	48	[-1165, 302]	149	[-105, 786]	218*	[-1750, 187]	-9	-
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	-
91-00	51*	$]-\infty, -521] \cup [13490, \infty[$	25*	$]-\infty, 15] \cup [4284, \infty[$	28	$]-\infty, 419] \cup [7601, \infty[$	-56	-
00-10	48	$\mathbb{R}$	149*	$]-\infty, -711] \cup [526, \infty[$	218*	$]-\infty, -1611] \cup [3056, \infty[$	-9	-
$\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{HML}})'$								
EW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	δ	$\overline{F}$	δ	$\overline{F}$	δ	$\overline{F}$	δ
91-00	51	$\mathbb{R}$	25	$\mathbb{R}$	28	$\mathbb{R}$	-56*	$]-\infty, -969] \cup [415, \infty[$
00-10	48	$\mathbb{R}$	149	$\mathbb{R}$	218	$\mathbb{R}$	-9	$\mathbb{R}$
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	δ	$\overline{F}$	δ	$\overline{F}$	δ	$\overline{F}$	δ
91-00	51	$\mathbb{R}$	25	$\mathbb{R}$	28	$\mathbb{R}$	-56	$\mathbb{R}$
00-10	48	$\mathbb{R}$	149	$\mathbb{R}$	218	$\mathbb{R}$	-9	$\mathbb{R}$

Note: Sample includes monthly observations from January 1991 to December 2010. Series are constructed with all Canadian observations from Datastream and Worldscope. They include 25 equally weighted (EW) and value weighted (VW) portfolios as well as Canadian factors for market (MKT), size (SMB), book-to-market (HML) and momentum (MOM). Confidence sets are at the 5% level. $\overline{F}$is the factor average over the considered time period; θcaptures factor pricing as defined in (5). * denotes evidence of pricing at the 5% significance level interpreted as follows: given the reported confidence sets, the tradable factor is not priced if $\bar{R}_1$is covered; each other factor is not priced if (its average) is not covered; see section 2.

Table 2. Confidence sets for risk premia

Five years sub-periods

$r_i - \iota_T \gamma_0 = (\mathcal{R}_1 - \iota_T \gamma_0)\, b_{i1} + (\mathcal{F} - \iota_T \gamma'_{\mathcal{F}})\, b_{i\mathcal{F}} + u_i, \quad i = 1, \ldots, n$								
$\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}})'$								
EW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	-
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	-152	-
96-00	80	$\mathbb{R}$	-18	$\mathbb{R}$	25	$\mathbb{R}$	39	-
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	-
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty, \text{-}359\} \cup \{1026, \infty\}$	-89	-
VW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	θ_{MKT}	$\overline{F}$	θ_{SMB}	$\overline{F}$	θ_{HML}	$\overline{F}$	-
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	-152	-
96-00	80*	$\{-\infty, \text{-}384\} \cup \{291, \infty\}$	-18	$\mathbb{R}$	25	$\mathbb{R}$	39	-
00-05	52	$\{-\infty, 307\} \cup \{1687, \infty\}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	-
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty, \text{-}556\} \cup \{874, \infty\}$	-89	-
$\theta = (\gamma_0, \gamma'_{\mathcal{F}})' = (\theta_{\text{MKT}}, \theta_{\text{SMB}}, \theta_{\text{HML}}, \theta_{\text{HML}})'$								
EW	MKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	δ	$\overline{F}$	δ	$\overline{F}$	δ	$\overline{F}$	δ
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	-152	$\mathbb{R}$
96-00	80	$\mathbb{R}$	-18	$\mathbb{R}$	25	$\mathbb{R}$	39	$\mathbb{R}$
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	$\mathbb{R}$
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty, \text{-}172\} \cup \{894, \infty\}$	-89	$\mathbb{R}$
VW	MRKT		SMB		HML		MOM	
$\times 10^{-4}$	$\bar{R}_1$	δ	$\overline{F}$	δ	$\overline{F}$	δ	$\overline{F}$	δ
91-95	21	$\mathbb{R}$	68	$\mathbb{R}$	30	$\mathbb{R}$	-152	$\mathbb{R}$
96-00	80	$\mathbb{R}$	-18	$\mathbb{R}$	25	$\mathbb{R}$	39	$\mathbb{R}$
00-05	52	$\mathbb{R}$	170	$\mathbb{R}$	262	$\mathbb{R}$	72	$\mathbb{R}$
06-10	43	$\mathbb{R}$	128	$\mathbb{R}$	175*	$\{-\infty, \text{-}317\} \cup \{807, \infty\}$	-89	$\mathbb{R}$

Note: See note to Table 1.

Appendix

This appendix summarizes the solution of (20) from Dufour & Taamouti (2005). Projections based confidence sets for any linear transformation of θ of the form $\omega'\theta$ can be obtained as follows. Let $\tilde{A} = -A_{22}^{-1}A_{12}'$, $\quad \tilde{D} = A_{12}A_{22}^{-1}A_{12} - A_{11}$. If all the eigenvalues of A_{22} [as defined in (21)] are positive so A_{22} is positive definite then:

$$\mathrm{CS}_\alpha(\omega'\theta) = \left[\omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}, \omega'\tilde{A} + \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right], \quad if \quad \tilde{D} \geq 0 \tag{A1}$$

$$\mathrm{CS}_\alpha(\omega'\theta) = \varnothing, \quad if \quad \tilde{D} < 0. \tag{A2}$$

If A_{22} is non-singular and has one negative eigenvalue then: (i) if $\omega' A_{22}^{-1}\omega < 0$ and $\tilde{D} < 0$:

$$\mathrm{CS}_\alpha(\omega'\theta) = \left]-\infty, \omega'\tilde{A} - \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}\right] \cup \left[\omega'\tilde{A} + \sqrt{\tilde{D}\left(\omega' A_{22}^{-1}\omega\right)}, +\infty\right[; \tag{A3}$$

(ii) if $\omega' A_{22}^{-1}\omega > 0$ or if $\omega' A_{22}^{-1}\omega \leq 0$ and $\tilde{D} \geq 0$ then:

$$\mathrm{CS}_\alpha(\omega'\theta) = \mathbb{R}; \tag{A4}$$

(iii) if $\omega' A_{22}^{-1}\omega = 0$ and $\tilde{D} < 0$ then:

$$\mathrm{CS}_\alpha(\omega'\theta) = \mathbb{R}\backslash \left\{\omega'\tilde{A}\right\}. \tag{A5}$$

The projection is given by (A4) if A_{22} is non-singular and has at least two negative eigenvalues.

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