A comparison of two approaches to identify postponement and compression

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Quick summary

Try to reconcile the finding of Zhuo et al (2018, PNAS) that mortality postponement is a period phenomenon with the findings of McCarthy and Wang (2023, PLoS One) that the dominant pattern of mortality improvement has been compression, with episodes of postponement. We:

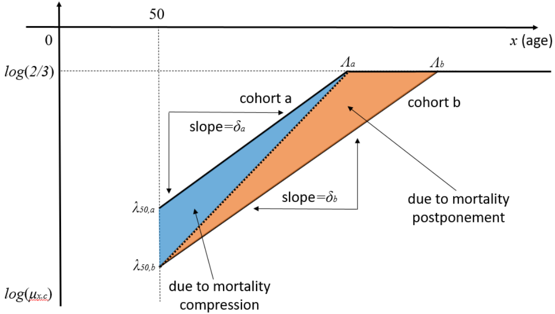
1. Use the Gompertz law to divide mortality changes between compression and postponement and show that widening percentile bands (even at higher percentiles) are not unambiguous evidence of mortality postponement
2. Show that the Gompertz law is an increasingly poor model for senescent mortality in both periods and cohorts, because some deaths appear to being shifted in a non-Gompertzian way
3. Divide historical period and cohort mortality changes between compression and postponement, and show that both appear to be occurring

Gompertz Law

$$log(m_x) = \lambda + \delta(x - x_0) + \epsilon_x \tag{1}$$

1. Captures broad features of shape of mortality curve above age (say) x_0 (we use 65)
2. Fails at very high ages due to differential frailty within each cohort and resulting survivor bias
3. Define the Gompertzian Maximum Age (GMA, denoted Λ) as that age where the Gompertz Law would predict that mortality intensity first reaches 2/3 (so annual probability of death ~ 0.5); (contested) evidence that mortality plateaus at this rate.

Using the Gompertz Law to divide changes in mortality between compression and postponement



Effect of compression and postponement on percentile bands

Parameterized with Γ and λ, the x'th percentile of age-at-death is

$$P(x) = log(1 - \frac{log(2/3) - \lambda}{\Gamma - x_0} e^{-\lambda} log(1 - x)) \frac{\Gamma - x_0}{log(2/3) - \lambda} + x_0. \qquad (2)$$

Writing the width of a percentile band with lower percentile l and upper percentile u as $W(u, l) = P(u) - P(l)$, we show that:

- $\frac{\delta W}{\delta \Gamma} > 0$
- $\frac{\delta W}{\delta \lambda} > 0$ if $l > x^* = 1 - exp(-(\Gamma - x_0)e^{\lambda})$
- $\frac{\delta W}{\delta \lambda} < 0$ if $u < x^* = 1 - exp(-(\Gamma - x_0)e^{\lambda})$

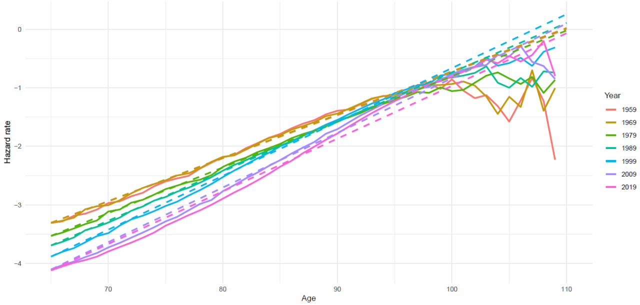
(in other cases, sign of $\frac{\delta W}{\delta \lambda}$ is indeterminate) More info

Interpretation

- Postponement is an increase in Γ, so will always lead to widening percentile bands
- Compression is a decrease in λ, and will only lead to unambiguously narrowing percentile bands if the lower percentile of the band is greater than some number $x^*(\Gamma, \lambda)$
- If the upper percentile of the band is lower than x^*, compression will cause the width of the percentile band to increase (think of the first percentile less x_0!)
- Where neither condition holds, the position is ambiguous
- **Widening percentile bands are *only* unambiguous evidence in favor of postponement if the lower percentile is greater than x^*, otherwise it could just be compression!**

Example: US males, 1959-2019, period population mortality (similar to Zhuo at el, 2018)

Fitted Gompertz model to pop mortality 65+ from HMD in each year, fixing $\lambda = log(m_{65})$ and weighting each data point by the number of deaths (Fay and Herriot, 1979)



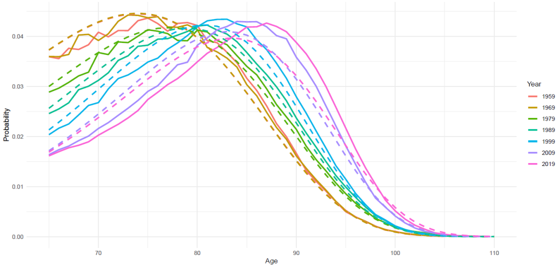
Parameter estimates, US males, 1959-2019, period population mortality

USA Male	1959	1969	1979	1989	1999	2009	
λ (intercept)	-3.31	-3.31	-3.53	-3.69	-3.88	-4.10	-
δ (slope)	0.07367	0.07396	0.07798	0.08450	0.09203	0.09340	0.0
Γ (GMA)	104.40	104.23	105.08	103.91	102.80	104.61	10
x^* (critical percentile)	0.763	0.761	0.755	0.622	0.542	0.481	0
K-S statistic	0.03	0.02	0.02	0.02	0.03	0.05	

- For US males, dominant pattern until 1999 is compression, after then postponement and compression
- But mortality compression would only result in unambiguous narrowing of all percentile ranges above the median after 2009; in 1959-1979 even percentile ranges starting at the 75th percentile aren't safe!
- Increasingly poor fit of the Gompertz law

US Males: Theoretical distribution of age-at-death

- Gompertz law fits OK-ish at younger ages until 1989, good at older ages throughout
- After 1989, evidence of another type of compression: some deaths are being shifted from the seventh and eight decades of life to the ninth and tenth



Changes in percentiles of theoretical distribution of age-at-death

USA Male	1959	1969	1979	1989	1999	2009	2019	Total Δ
Expected age at death	77.58	77.55	78.92	79.61	80.39	81.94	82.39	4.81
Δ due to: postponement	-0.02	0.13	-0.21	-0.23	0.43	0.38		0.48
compression	-0.01	1.23	0.91	1.02	1.12	0.07		4.33
model error	-0.09	-0.04	0.03	0.12	0.25	0.18		0.46
Median age at death	76.56	76.29	77.82	78.81	80.06	82.18	82.98	6.43
Δ due to: postponement	-0.02	0.11	-0.18	-0.21	0.40	0.35		0.46
compression	-0.01	1.47	1.09	1.22	1.34	0.09		5.19
model error	-0.24	-0.05	0.08	0.24	0.39	0.36		0.78
75th perc. of age at death	82.72	82.57	84.23	85.01	85.98	88.12	88.99	6.27
Δ due to: postponement	-0.03	0.21	-0.32	-0.35	0.64	0.57		0.71
compression	-0.01	1.57	1.11	1.19	1.25	0.08		5.20
model error	-0.11	-0.11	-0.01	0.12	0.26	0.21		0.37
90th perc. of age at death	87.79	87.79	89.55	90.13	90.71	92.62	93.41	5.62
Δ due to: postponement	-0.05	0.30	-0.45	-0.48	0.85	0.75		0.92
compression	-0.01	1.48	1.02	1.07	1.09	0.07		4.72
model error	0.07	-0.03	0.01	-0.01	-0.02	-0.03		-0.02
95th perc. of age at death	90.66	90.73	92.52	92.94	93.28	94.99	95.75	5.09
Δ due to: postponement	-0.06	0.36	-0.53	-0.55	0.96	0.86		1.03
compression	-0.01	1.38	0.95	0.98	0.98	0.06		4.34
model error	0.15	0.05	0.01	-0.08	-0.23	-0.16		-0.28

Changes in percentile bands of theoretical distribution of age-at-death

USA Male	1959	1969	1979	1989	1999	2009	2019	Total Δ
50th perc. less 25th perc.	5.94	5.75	6.24	6.40	6.62	6.90	7.23	1.29
Δ due to: compression	0.00	0.40	0.25	0.21	0.15	0.01		1.02
postponement	-0.01	0.07	-0.11	-0.12	0.22	0.20		0.25
model error	-0.18	0.01	0.03	0.13	-0.09	0.12		0.02
75th perc. less 50th perc.	6.16	6.27	6.42	6.20	5.91	5.94	6.01	-0.15
Δ due to: compression	0.00	0.10	0.03	-0.03	-0.09	-0.01		0.00
postponement	-0.02	0.10	-0.14	-0.15	0.24	0.22		0.25
model error	0.13	-0.06	-0.09	-0.12	-0.13	-0.15		-0.42
90th perc. less 75th perc.	5.08	5.23	5.31	5.12	4.73	4.50	4.42	-0.66
Δ due to: compression	0.00	-0.09	-0.09	-0.13	-0.16	-0.01		-0.48
postponement	-0.02	0.09	-0.13	-0.13	0.21	0.18		0.20
model error	0.17	0.08	0.03	-0.14	-0.28	-0.25		-0.39
GMA less 90'th perc.	16.61	16.43	15.53	13.78	12.09	11.99	12.84	-3.77
Δ due to: compression	0.01	-1.48	-1.02	-1.07	-1.09	-0.07		-4.72
postponement	-0.12	0.55	-0.71	-0.63	0.96	0.89		0.94
model error	-0.07	0.03	-0.01	0.01	0.02	0.03		0.01

Other countries

Males aged 65	USA	UK	France	Australia	Japan
75th percentile less 50th percentile	-0.15	-0.12	-0.16	-0.59	0.08
Δ due to: compression	0	-0.31	-0.24	-0.43	-0.52
postponement	0.25	0.6	0.93	0.35	1.05
model error	-0.42	-0.43	-0.87	-0.51	-0.45
90th percentile less 75th percentile	-0.66	-0.55	-0.64	-1.04	-0.48
Δ due to: compression	-0.48	-0.73	-0.58	-0.91	-0.83
postponement	0.2	0.52	0.8	0.26	0.88
model error	-0.39	-0.35	-0.86	-0.36	-0.5
Females aged 65	USA	UK	France	Australia	Japan
75th percentile less 50th percentile	-0.31	-0.32	-0.57	-0.82	-0.72
Δ due to: compression	-0.43	-0.59	-0.73	-0.86	-1.12
postponement	0.46	0.75	0.98	0.56	1.12
model error	-0.34	-0.51	-0.83	-0.51	-0.73
90th percentile less 75th percentile	-0.58	-0.5	-0.66	-0.79	-0.89
Δ due to: compression	-0.56	-0.68	-0.75	-0.94	-1.14
postponement	0.37	0.61	0.76	0.44	0.87
model error	-0.41	-0.45	-0.68	-0.31	-0.63

Analysis by cohort: US males born 1879-1939

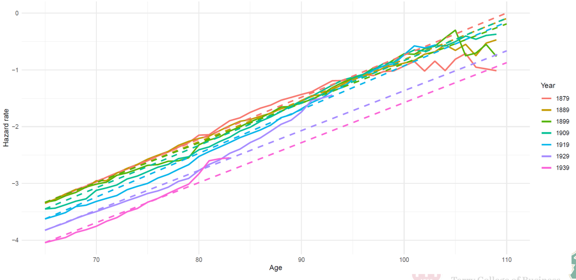
- Repeat the analysis by birth cohort
- (Intend to use the methodology of McCarthy and Wang (2023, PLoS One) to estimate parameters for incomplete cohorts (Bayesian analysis using change in parameters of completed cohorts as a prior for parameter changes in incomplete cohorts), still working on this, results shown where we have estimates)

USA Male	1879	1889	1899	1909	1919*	1929*
λ (intercept)	-3.33	-3.35	-3.33	-3.45	-3.63	-3.82
δ (slope)	0.07416	0.07237	0.06989	0.07468	0.07707	0.07023
Γ (GMA)	104.50	105.69	106.90	105.77	106.78	113.69
x^* (critical percentile)	0.757	0.760	0.777	0.726	0.670	0.656
K-S statistic	0.01	0.01	0.04	0.05	0.05	0.04

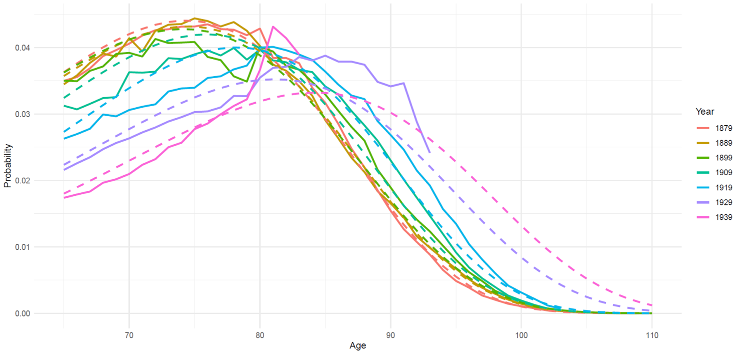
Note: * indicates incomplete cohort: 1919 > 95th percentile; 1929 > 90th percentile

US male population: by birth cohort 1879-1939

- Virtually no compression between 1879 and 1899 incl. (but some postponement)
- Compression and postponement between 1899 and 1939
- Increasing Gompertzian error (as in period data), may be biasing slope estimates, even with a strong prior
- Covid-19 represented a significant set-back, eliminated a decade or more of mortality improvements



Theoretical distribution of age-at-death shows picture more clearly



US male population: by birth cohort 1879-1929

USA Male	1879	1889	1899	1909	1919*	1929*	Total Δ
Expected age at death	77.74	77.99	78.06	78.57	79.73	82.07	4.33
Δ due to: postponement	0.16	0.16	-0.16	0.17	1.23		1.56
compression	0.09	-0.09	0.67	0.99	1.14		2.80
model error	-0.12	0.49	0.25	0.02	0.37		1.01
Median age at death	76.39	76.36	76.98	77.97	79.50	81.34	4.95
Δ due to: postponement	0.12	0.12	-0.13	0.14	1.05		1.30
compression	0.10	-0.11	0.80	1.19	1.37		3.35
model error	-0.25	0.61	0.32	0.20	-1.42		-0.54
75th perc. of age at death	82.52	82.62	83.72	84.64	86.07	87.91	5.39
Δ due to: postponement	0.25	0.24	-0.25	0.26	1.88		2.38
compression	0.11	-0.12	0.87	1.24	1.38		3.48
model error	-0.26	0.97	0.30	-0.08	-1.42		-0.49
90th perc. of age at death	87.59	88.21	89.06	89.87	91.31	92.59	5.00
Δ due to: postponement	0.38	0.37	-0.37	0.37	2.64		3.39
compression	0.11	-0.11	0.84	1.16	1.26		3.26
model error	0.14	0.59	0.35	-0.09	-2.62		-1.63
95th perc. of age at death	90.55	91.39	92.16	92.65	94.02		
Δ due to: postponement	0.45	0.45	-0.45	0.43	3.07		3.95
compression	0.10	-0.11	0.79	1.08	1.16		3.02
model error	0.29	0.43	0.15	-0.14			

Composition of changes in percentile bands

USA Male	1879	1889	1899	1909	1919*	1929*	Total Δ
50th perc. less 25th perc.	5.82	5.78	6.25	6.44	7.01	7.61	1.79
Δ due to: compression	0.03	-0.03	0.24	0.30	0.29		0.83
postponement	0.08	0.08	-0.09	0.09	0.66		0.82
model error	-0.16	0.42	0.04	0.18	-0.35		0.13
75th perc. less 50th perc.	6.13	6.26	6.74	6.68	6.57	6.58	0.45
Δ due to: compression	0.01	-0.01	0.07	0.05	0.01		0.13
postponement	0.13	0.13	-0.12	0.12	0.83		1.09
model error	0.00	0.37	-0.02	-0.28	-0.84		-0.77
90th perc. less 75th perc.	5.07	5.59	5.34	5.23	5.25	4.68	-0.39
Δ due to: compression	0.00	0.00	-0.04	-0.08	-0.12		-0.24
postponement	0.13	0.13	-0.12	0.11	0.76		1.01
model error	0.40	-0.38	0.05	-0.01	-1.21		-1.15
GMA less 90th perc.	16.91	17.48	17.84	15.89	15.46	21.10	4.19
Δ due to: compression	-0.11	0.11	-0.84	-1.16	-1.26		-3.26
postponement	0.81	0.84	-0.76	0.64	4.27		5.8
model error	-0.14	-0.59	-0.35	0.09	2.62		1.63

Conclusion

- The Gompertz law can be used to divide mortality changes between compression and postponement in a rigorous way; it implies that percentile bands of age-at-death will widen unambiguously in the case of postponement, but percentile bands in the case of compression may narrow or widen (or both), depending on where in the distribution they are relative to a critical percentile x^*
- Over much of the period, even higher percentile bands (eg P75 - P50) lie in the region which widens due to compression, so widening percentile bands are not always a guide to whether compression is taking place in the sense of Frees (1980)
- The Gompertz law appears increasingly inapplicable, both in period and cohort data: some deaths are being shifted (but the GMA is not, at least not yet), so this is not postponement in the sense modeled here

Proof of mathematical results

$$\frac{\partial P(x)}{\partial \lambda} = log(1 - \frac{log(2/3)-\lambda}{\Gamma - x_0} e^{-\lambda} log(1-x)) \frac{\Gamma - x_0}{(log(2/3)-\lambda)^2}$$
$$+ \frac{e^{-\lambda}(1 + log(2/3) - \lambda)\frac{log(1-x)}{log(2/3)-\lambda}}{1 - \frac{log(2/3)-\lambda}{\Gamma - x_0} e^{-\lambda} log(1-x)}.$$

The confidence interval (u, l) will widen if $\frac{dP(u)}{d\lambda}$ and $\frac{dP(l)}{d\lambda}$. We calculate

$$\frac{\partial^2 P(x)}{\partial \lambda \partial x} = \frac{ab}{(1-x)(1-alog(1-x))} - \frac{c}{(1-x)(1-alog(1-x))^2}, \tag{3}$$

where

$$a = \frac{log(2/3)-\lambda}{\Gamma - x_0} e^{-\lambda}, b = \frac{\Gamma - x_0}{(log(2/3)-\lambda)^2}, c = \frac{e^{-\lambda}(1 + log(2/3) - \lambda)}{log(2/3)-\lambda}. \tag{4}$$

Proof of mathematical results continued

Note that $a, b, c > 0$, so the first term is greater than 0, and the second term less than zero. The sign of their sum is thus indeterminate. It can be shown that:

$$\frac{\partial^2 P(x)}{\partial\lambda\partial x} > 0 \iff x > 1 - exp(-(\Gamma - x_0)e^{\lambda}). \tag{5}$$

Percentile ranges than therefore be divided into three groups:

- the first group has the higher end of the range less than $x^*(\Gamma, \lambda) = 1 - exp(-\Gamma - x_0)e^{\lambda})$
- the second group has the lower end of the range higher than x^*, and
- the third contains groups with the lower end of the range below x^* and the upper end above x^*.

Numerical example