Bequest Motives and Demand for Deferred Annuities

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Research Background

Annuity Puzzle

- Yaari (1965) shows that without a bequest motive and under actuarially fair pricing, individuals should fully invested in the "actuarial note".
- Annuity market in developed countries remain small.
- Bequest motive is one of the natural explanation for the low annuity demand, e.g., Yaari (1965); Friedman and Warshawsky (1988); d'Albis and Thibault (2012); Lockwood (2012, 2018); Zhang et al. (2021).

Logics on Bequest versus Annuity

- Desire to leave a bequest induce agents to decrease annuity demand to have wealth liquid to leave as a bequest.
- Annuitizing savings reduces short-lifespan bequests and increases wealth in old age, some of which is spent on consumption

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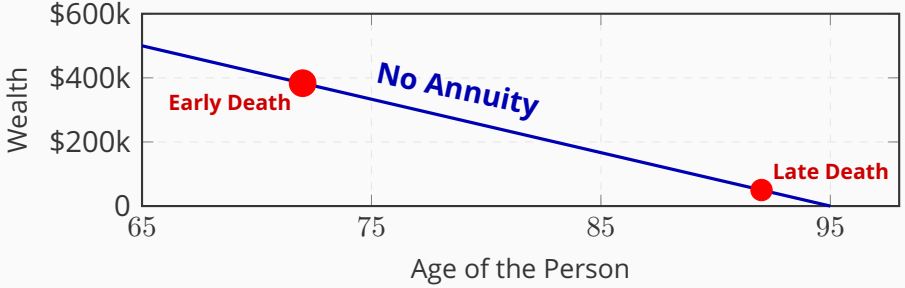

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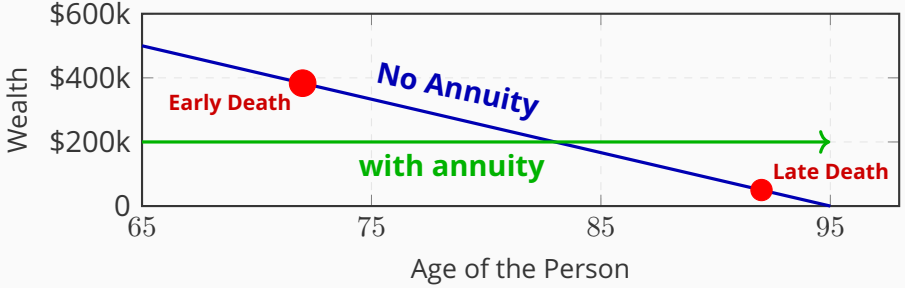

Research Question

For a perfectly altruism person that is risk averse about the bequest amount, will s/he purchase annuity to safeguard the uncertainty of future bequest.

What kind of annuity product should be purchased (e.g., whole life, term life, deferred life, etc)?

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Deferred annuity is optimal.

Einav and Finkelstein (2026) also demonstrate the optimality of deferred annuity in the design of social security.

Problem Formulation

Model Setup

- Denote T as the remaining lifetime of a person currently age x_0 with limiting age $x_0 + \omega$.
- Denote W_t as the wealth process with initial wealth $v > 0$. The person consume at a fixed consumption (nominal) rate of $c > 0$ and invest at a risk-free rate of $r \geq 0$.
- Insurance/annuity products are available at time 0 with present value random variable of $I(T)$, and associated lump-sum premium of $\pi(I)$.
- The present value of the bequest is therefore

$$B_I(T) := e^{-rT} \times \left(v - \pi(I) - c\bar{a}_{\overline{T}|} + I(T)\right)$$

Subjective Mortality Belief

- Objective survival probability

$$ {}_tp_{x_0} = \exp\left(-\int_0^t \mu_{x_0+s}\,\mathrm{d}s\right),\ f(t) = {}_tp_{x_0} \times \mu_{x_0+t},\ 0 < t < \omega $$

- Subjective survival probability

$$ {}_t\tilde{p}_{x_0} = \exp\left(-\int_0^t \tilde{\mu}_{x_0+s}\,\mathrm{d}s\right),\ \tilde{f}(t) = {}_t\tilde{p}_{x_0} \times \tilde{\mu}_{x_0+t},\ 0 < t < \omega $$

- Define the mortality likelihood ratio

$$ \ell(t) := \frac{f(t)}{\tilde{f}(t)},\ 0 < t < \omega $$

  and we focus on $\ell(t)$ being an non-decreasing function of t.

Technical Assumptions

- We consider expected premium principle such that

$$\pi(I) = (1+\eta)\mathbb{E}_f[I(T)]$$

  where η is the loading and f is the density function of T.

- The admissible set for $I(\cdot)$ is defined

$$\mathcal{A} = \left\{ I \in L^1(f) : I \geq 0 \ f\text{-a.e.}, \pi(I) \leq \pi_0 \right\}$$

  where π_0 is exogenous upper bound satisfies $v - c\bar{a}_{\overline{\omega}|} - \pi_0 > 0$.

Optimization Problem

Consider an utility function u satisfies the Inada conditions,

$$\max_{I \in \mathcal{A}} J(I) := \mathbb{E}_{\tilde{f}} \left[u(B_I(T)) \right]$$

Theorem

The optimization problem admits a unique optimal solution $I^* \in \mathcal{A}$, f-a.e.

Our Findings

No Annuitization or Full Annuitization

- **No Annuity**: Denote

$$\bar{\eta} = \operatorname*{ess\,sup}_{0<t<\omega} \frac{u'(v - c\bar{a}_{\overline{t}|})}{\mathbb{E}_{\tilde{f}}[u'(v - c\bar{a}_{\overline{T}|})]} - 1,$$

  then $I^* \equiv 0$ if and only if $\eta \geq \bar{\eta}$.

- **Whole Life Annuity**: Suppose that $\eta > 0$, then

$$\int_{\{t:I^*(t)=0\}} f(t)\,\mathrm{d}t > 0.$$

  That is, whole life annuity is never optimal.

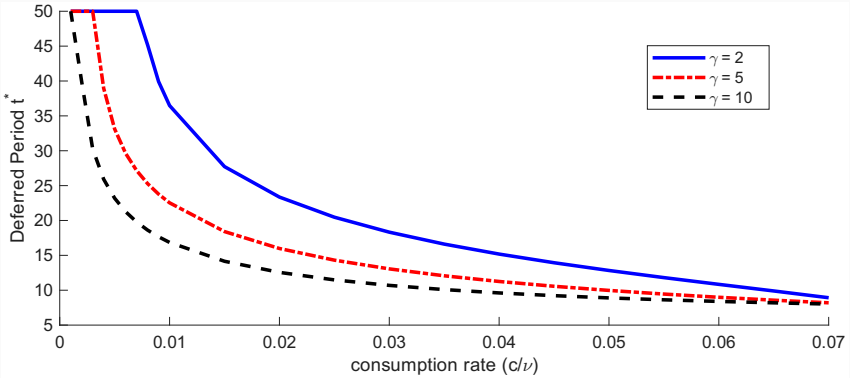
When Belief is Homogeneous $\ell \equiv 1$

The optimal indemnity is

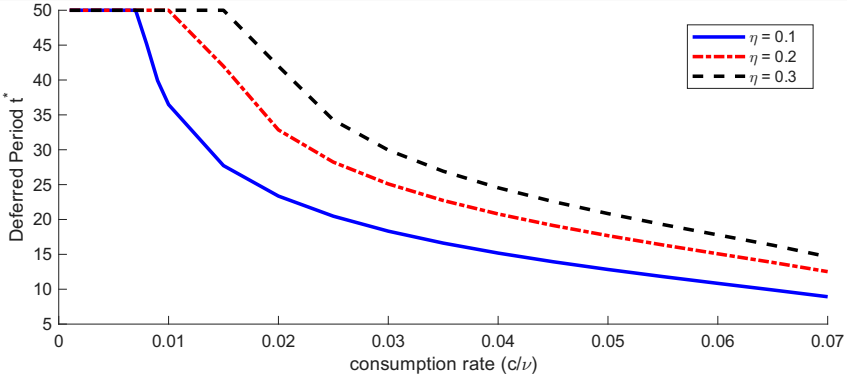
$$I^*(t) = \begin{cases} 0, & 0 < t \leq t^* \\ c \cdot \left(\bar{a}_{\overline{t}|} - \bar{a}_{\overline{t^*}|} \right), & t^* < t < \omega \end{cases}$$

which is the present value random variable for a t^*-year deferred annuity with payment of c.

Graphical Illustration, Power Utility and Effect of γ



Graphical Illustration, Effect of η



When Belief is Heterogeneous $\ell \not\equiv 1$

Denote $g(\cdot) = (u')^{-1}$, and denote $a^*(t)$ as the annuity payment at time t, then

$$a^*(t) = \begin{cases} 0, & 0 < t \leq t^*, \\ c + e^{rt} \times \lambda^* \times (1+\eta) \times g'(\lambda^* \times (1+\eta) \times \ell(t)) \times \ell'(t), & t^* < t < \omega, \end{cases}$$

where $\lambda*$ is, and moreover, we have

$$a^*(t) \geq c, \quad \text{for } a.e.\ t \in (t^*, \omega).$$

Illustrative Example, Gompertz and Esscher Tilt

Consider Gompertz Law such that

$$\mu_{x_0+t} = B_g \times c_g^{x_0+t},$$

then $a^*(t)$ is an increasing function of t.

Further consider the Esscher Tilt

$$\tilde{f}(t) = \frac{e^{\beta t} f(t)}{M(\beta)}, \qquad \beta > 0,$$

then

$$a^*(t) = c + \mathcal{K} \times \exp\left(\left(r + \frac{\beta}{\gamma}\right) t\right),$$

with $\frac{\mathrm{d}t^*}{\mathrm{d}\beta} < 0$.

Illustrative Example, Gompertz and Proportional Hazard

Consider proportional hazard

$$\tilde{\mu}_{x_0+t} = \theta \times \mu_{x_0+t}, \qquad 0 < \theta < 1,$$

then under Gompertz law

$$a^*(t) = c + \mathcal{K} \times \exp\left((r + \ln c_g)t + \mathcal{D}\left(c_g^t - 1\right)\right),$$

with $\frac{\mathrm{d}t^*}{\mathrm{d}\theta} > 0$.

Conclusion and Future Work

We demonstrate that an perfectly altruist who is not longevity pessimistic would purchase deferred life annuity.

Our study has several limitations that should be further investigated:

- We expressed everything in the present value sense.
- Consumption is exogenously given.
- Other shocks, in particularly the health shock, are important.
- We only consider a longevity-optimistic person.

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