

Modeling Infectious Mortality Risk with Application to Mortality Security Pricing

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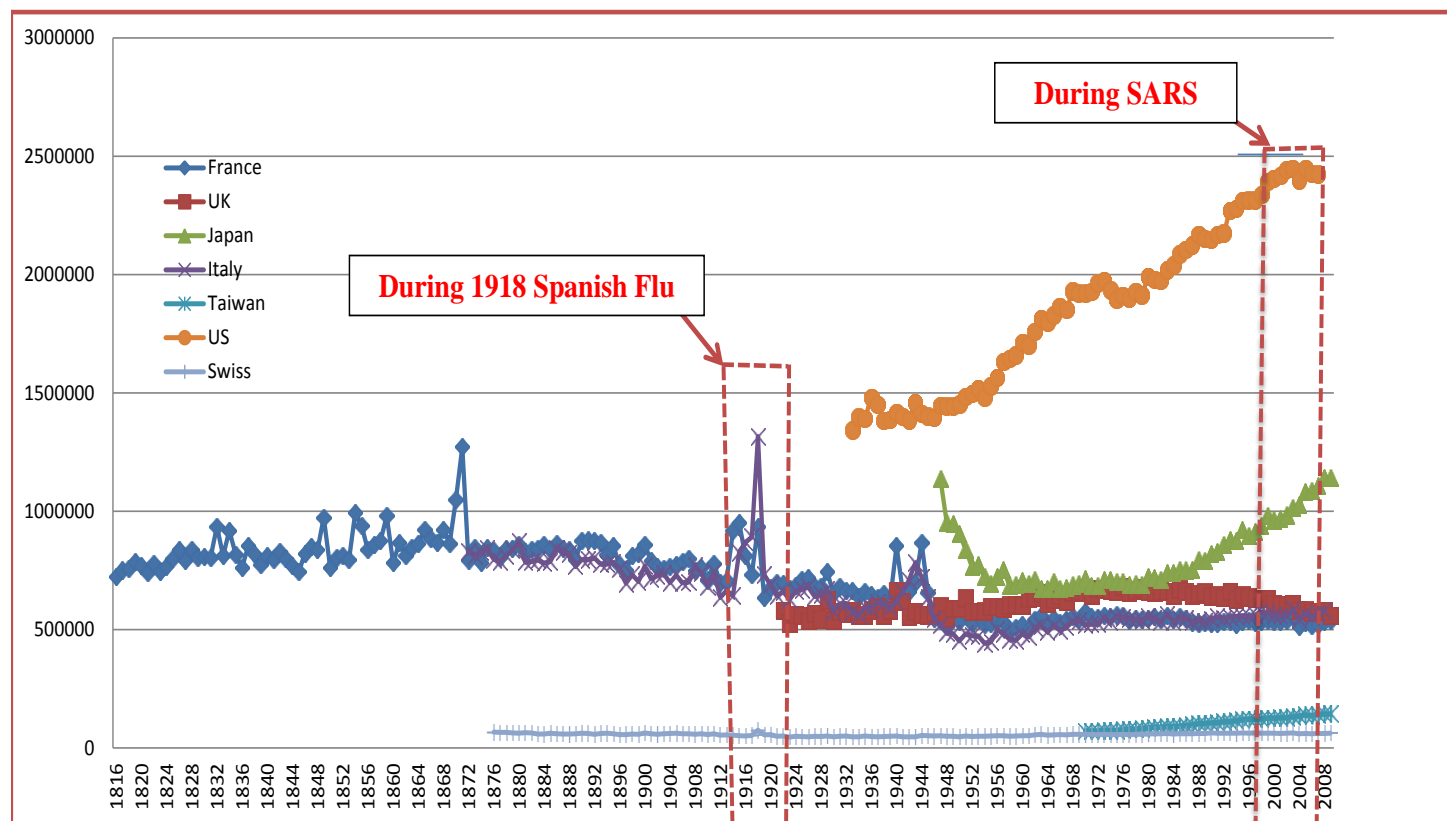
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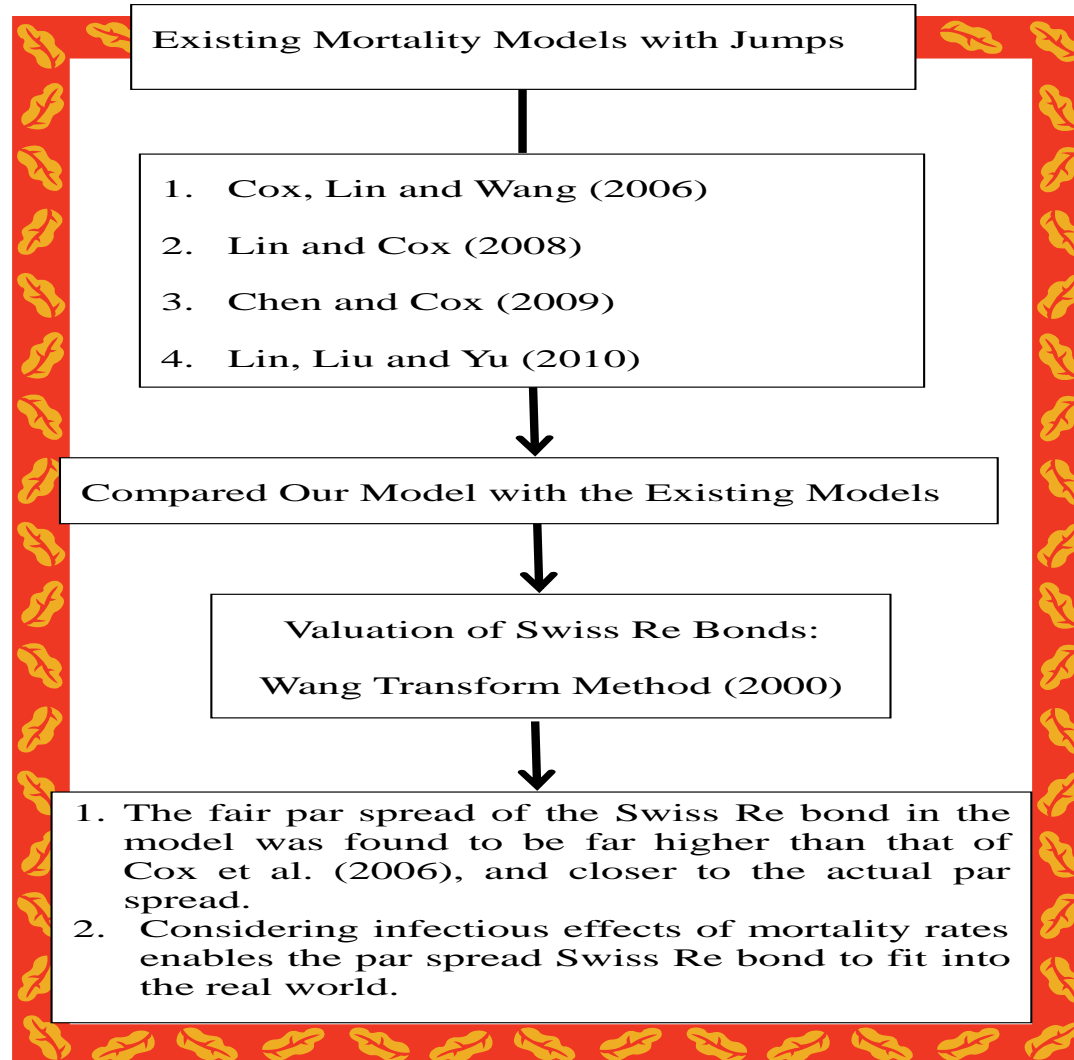
Figure 1: The Deaths in the World from 1816 to 2008



Purposes and Contribution

- The actual mortality rates show that when an event occurs, mortality rates do not significantly co-move in the world until major numbers of deaths occur. We call it "infectious mortality risk" in this paper.
- Despite the fact, no existing literature presents any model to describe the phenomenon. This paper attempts to model infectious mortality risk and investigate its effect in pricing mortality-linked securities.
- Using the Swiss Re mortality bond as an example, we derive a closed-form solution using Wang's transform (2000) and the model parameters of the Swiss Re bond are estimated by means of calibration method.

Framework



Literature Review (1)

Cox, Lin and Wang (2006):

$$\frac{dq_{US,t}}{q_{US,t}} = (\mu_{US} - \Gamma_{US}k_{US} - \Gamma_{intel}k_{intel})dt + \sigma_{US}dW_{US,t} + (\pi_{US} - 1)dN_{US,t} + (\pi_{intel} - 1)dN_{intel,t},$$

$$\frac{dq_{UK,t}}{q_{UK,t}} = (\mu_{UK} - \Gamma_{UK}k_{UK} - \Gamma_{intel}k_{intel})dt + \sigma_{UK}dW_{UK,t} + (\pi_{UK} - 1)dN_{UK,t} + (\pi_{intel} - 1)dN_{intel,t},$$

with $E[\pi_{US} - 1] = k_{US}$, $E[\pi_{UK} - 1] = k_{UK}$, $E[\pi_{intel} - 1] = k_{intel}$,

$N_{US,t} \sim Poisson(\Gamma_{US}dt)$, $N_{UK,t} \sim Poisson(\Gamma_{UK}dt)$, $\ln(\pi) \sim N(u_{\pi}, \sigma_{\pi})$.

Estimation of Model Parameters

Model Parameters	US.	UK.
μ	-0.01	-0.0033
σ	0.0308	0.0237
Γ	0.00001	0.8533
u_{π}	-0.005	-0.0114
σ_{π}	0.00001	0.06
k	-0.005	-0.0096

Based on Vital Statistics of the United States (VSUS) and Human Mortality Database (HMD), the market risk price of Swiss Re bond is 0.83, and par spread of Swiss Re bond is 0.45%.

Literature Review (2)

Lin and Cox (2008):

$$\frac{dq_{US,t}}{q_{US,t}} = (\mu_{US} - \Gamma_{US} k_{US}) dt + \sigma_{US} dW_{US,t} + (\pi_{US} - 1) dN_{US,t}$$

Estimation of Model Parameters

Model Parameters	US.
μ	-0.01
σ	0.0302
Γ	0.00536
u_{π}	-0.0233
σ_{π}	0.1028
k	-0.0178

Based on Vital Statistics of the United States (VSUS), the market risk price of Swiss Re bond without jumps is 1.2523, and that of Swiss Re bond with jumps is 0.8657.

Literature Review (3)

Chen and Cox (2009) allow the time-varying factor to jump in Lee-Carter (1992) model and use it to value Swiss Re bonds. The jumps are divided into permanent part and transitory part. The jump sizes of two parts are the same. Additionally, the market risk price of Swiss Re bonds is 1.5.

Literature Review (4)

Lin, Liu and Yu (2010):

$$\frac{dq_{US,t}}{q_{US,t}} = \mu_{US} dt + \sigma_{US} dW_{US,t} + \sum_{s=1}^2 (u_{US, int el}^s + \sigma_{US, int el}^s Z^s) dN_{int el, t}$$

⋮

$$\frac{dq_{swiss,t}}{q_{swiss,t}} = \mu_{swiss} dt + \sigma_{swiss} dW_{swiss,t} + \sum_{s=1}^2 (u_{swiss, int el}^s + \sigma_{swiss, int el}^s Z^s) dN_{int el, t}$$

Based on HMD, the market risk price of Swiss Re bond is 1.21.

Model Formulation (1)

Suppose N_t stands for the total number of deaths in the world at time t , also denoted as

$$N_t = \sum_{i=1}^m \sum_{j=1}^{n_i} D_{i,j}(t)$$

with $D_{i,j}(t) = \begin{cases} 1, & \text{if } \tau_{i,j} \leq t \\ 0, & \text{o.w} \end{cases}$, $i = 1, 2, 3, \dots, m$; $j = 1, 2, 3, \dots, n_i$.

$N_t^{(i)}$ is the total number of deaths except for the i^{th} country,

equally, $N_t^{(i)} \equiv \sum_{k \neq i=1}^m \sum_{j=1}^{n_k} D_{k,j}(t)$; $V_t^{(i)} = \frac{N_t^{(i)}}{N_t}$ is the ratio of the deaths

except the i^{th} country relative to the total deaths in the world,

and follows the geometric Brownian motion as follows.

$$\frac{dV_t^{(i)}}{V_t^{(i)}} = \mu_{v,i} dt + \sigma_{v,i} dW_{v,t}, \quad (1)$$

in which $\mu_{v,i}$ and $\sigma_{v,i}$ denote the drift term and volatility, respectively; $W_{v,t}$ is a one dimensional standard Brownian motion under the original probability measure, P .

Model Formulation (2)

If $V_t^{(i)}$ is higher than the threshold a , then the death rate except for the i^{th} country can affect the death rates of other countries. Suppose the jump number of the impacts of the death rate except the i^{th} country on the death rates of the other countries follows Poisson distribution with a density of $\lambda_{i,t}$ at

time t , also denoted as
$$I(t) \equiv \int_0^t D_s^{(i)} ds \sim \text{Poisson}(\lambda_{i,t}).$$

Model Formulation (3)

Let $q_{i,t}$ represent the population mortality index of the i^{th} country at time t and follow the dynamic process as below.

$$\frac{dq_{i,t}}{q_{i,t}} = \mu_i dt + \sigma_i dW_{i,t} + (\Lambda_i - 1)d\Gamma_{i,t} + (\pi_i - 1)dI_{i,t}, \quad (2)$$

μ_i , σ_i : constants;

$W_{i,t}$: one dimensional standard Brownian motion under the original probability measure, P , and $\text{corr}(dW_{v,t}, dW_{i,t}) = \rho_{v,i}$;

$\pi_i - 1$: the random variable percentage in the mortality index of the i^{th} country resulting from common jumps of deaths in other countries, and $\ln \pi_i \sim N(u_{\pi_i}, \sigma_{\pi_i}^2)$, $\pi_i > 0$, $i = 1, 2, 3, \dots, m$;

$\Lambda_i - 1$: the percentage in the mortality index of the i^{th} country resulting from specific jumps in deaths of the i^{th} county, and $\ln \Lambda_i \sim N(u_{\Lambda_i}, \sigma_{\Lambda_i}^2)$, $\Lambda_i > 0$, $i = 1, 2, 3, \dots, m$,

$dI_{i,t}$ is independent of $d\Gamma_{i,t}$.

Model Formulation (4)

In the same vein, when $V_t^{(k)}$ is higher than the threshold (a), the death rate except in the k^{th} country can affect the death rates of other countries. Suppose the jump number of the impacts of the death rate except in the k^{th} country on the death rates of the other countries follows Poisson distribution with a density of $\lambda_{k,t}$ at time t , also denoted as

$$I_{k,t} \equiv \int_0^t D_s^{(k)} ds \sim \text{Poisson}(\lambda_{k,t}), \quad i \neq k.$$

Model Formulation (5)

Let $q_{k,t}$ represent the population mortality index of the k^{th} country at time t and follows the dynamic process as below.

$$\frac{dq_{k,t}}{q_{k,t}} = \mu_k dt + \sigma_k dW_{k,t} + (\Lambda_k - 1)d\Gamma_{k,t} + (\pi_k - 1)dI_{k,t}, \quad (3)$$

μ_k , σ_k : constants;

$W_{k,t}$: one dimensional standard Brownian motion under the original probability

measure, P , and $\text{corr}(dW_{k,t}, dW_{i,t}) = \rho_{v,k}$ and $\text{corr}(dW_{v,t}, dW_{k,t}) = \rho_{v,k}$ for $i, k = 1, 2, 3, \dots, m$, $i \neq k$;

$\pi_k - 1$: the random variable percentage in the mortality index of the k^{th} country resulting from common jumps of deaths in other countries, and $\ln \pi_i \sim N(u_{\pi_i}, \sigma_{\pi_i}^2)$, $\pi_k > 0$, $k = 1, 2, 3, \dots, m$;

$\Lambda_k - 1$: the percentage in the mortality index of the k^{th} country resulting from specific jumps in deaths of the k^{th} county, and $\ln \Lambda_k \sim N(u_{\Lambda_k}, \sigma_{\Lambda_k}^2)$, $\Lambda_i > 0$, $k = 1, 2, 3, \dots, m$,

$dI_{k,t}$ is independent of $d\Gamma_{k,t}$.

Model Formulation (6)

From equations (2) and (3), $\ln \pi_i$ and $\ln \pi_k$ are also called the effects of infectious mortality. When the threshold (a) is infinite, the model can be reduced to Lin and Cox (2008).

Description of Swiss Re Bond (1)

- Swiss Re bond was a mortality security which can transfer mortality risk into investors in the capital market. The bond was issued by Swiss Reinsurance Company in 2003 and matured on January 1, 2007. It was a three-year deal. The principal was exposed to mortality risk.
- The mortality risk was defined in terms of an index based on the average annual population death rates in the US, UK, France, Italy, and Switzerland. If the index exceeded 130% of the actual 2002 level, then investors had have a reduced principal payment at maturity (T) as follows.

Description of Swiss Re Bond (2)

$$B_T = \text{Max}(1 - \sum_{i=1}^3 L_{t_i}, 0), \quad (4)$$

$$\text{with } \sum_{i=1}^3 L_{t_i} = \frac{\text{Max}(Y_{\text{max}} - 1.3Y_{t_0}, 0) - \text{Max}(Y_{\text{max}} - 1.5Y_{t_0}, 0)}{0.2Y_{t_0}},$$

$$Y_{\text{max}} = \text{Max}(Y_{t_1}, Y_{t_2}, Y_{t_3}) \quad \text{and} \quad Y_t = (q_{1,t}^{a_1} q_{2,t}^{a_2} \dots q_{5,t}^{a_5})^{\frac{1}{a_1 + a_2 + \dots + a_5}}.$$

Y_{t_0} , Y_{t_1} , Y_{t_2} and Y_{t_3} : the geometric average population death rates in the US, UK

France, Italy, and Switzerland in 2002, 2003, 2004, 2005,

and 2006, respectively.

a_1, a_2, \dots, a_5 : the weights of population mortality indices for the US, UK, France,

Italy, and Switzerland, respectively.

Valuation of Swiss Re Bond

The fair price of the bond is shown in equation (5).

$$B_0 = 4000000000 \times e^{-rT} E^Q [B_T], \quad (5)$$

in which r is the riskless rate; $E_t^Q(\cdot)$ denotes the expectation value

under the risk neutral probability measure, Q , at time t .

Equation (5) can be generally rewritten as

$$B_0 = 4000000000 \times e^{-rT} E^Q \left[\text{Max} \left(1 - \frac{\text{Max}(Y_{\max} - K_1, 0) - \text{Max}(Y_{\max} - K_2, 0)}{K_2 - K_1}, 0 \right) \right], \quad (7)$$

$$K_1 = 1.3Y_{t_0} \quad \text{and} \quad K_2 = 1.5Y_{t_0} .$$

Wang transform (2000)

In incomplete markets, Wang transform (2000) is a popular pricing methodology based on the following transformation: For a risk with CDF $F(x)$ under the original probability measure, P , the risk-adjusted CDF $F^*(x)$ under the risk-neutral probability measure, Q for the pricing of the risk is given by

$$F^*(x) = \Phi(\Phi^{-1}(F(x)) + \theta), \quad (8)$$

where θ is a constant risk premium. In this section, this paper proposes a methodology to solve equation (7).

The Fair Price of Swiss Re Bond (1)

Consequently, the fair par spread of Swiss Re bond is shown in equation (16).

$$B_0 = 400000000 e^{-rT} \left\{ \begin{array}{l} E^Q \left[(1 - S_T) 1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_1} \right] P_r^Q(Y_{\max} = Y_{t_1}) \\ + E^Q \left[(1 - S_T) 1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_2} \right] P_r^Q(Y_{\max} = Y_{t_2}) \\ + E^Q \left[(1 - S_T) 1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_3} \right] P_r^Q(Y_{\max} = Y_{t_3}) \end{array} \right\}, \quad (16)$$

$$\text{with } E^Q \left[1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_1} \right] = 1 - \Phi(\Phi^{-1}(F_{Y_{t_1}}(K_1)) + \theta_1),$$

$$E^Q \left[1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_2} \right] = 1 - \Phi(\Phi^{-1}(F_{Y_{t_2}}(K_1)) + \theta_2),$$

$$E^Q \left[1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_3} \right] = 1 - \Phi(\Phi^{-1}(F_{Y_{t_3}}(K_1)) + \theta_3),$$

$$E^Q \left[S_T 1_{\{S_T < 1\}} \middle| Y_{\max} = Y_{t_i} \right] = 1 - \Phi(\Phi^{-1}(F_{Y_{t_i}}(K_2)) + \theta_i)$$

$$+ \Phi(\Phi^{-1}(F_{Y_{t_i}}(K_2)) + \theta_i) \times \left\{ \Phi(\Phi^{-1}(F_{Y_{t_i}}(K_2)) + \theta_i) - \Phi(\Phi^{-1}(F_{Y_{t_i}}(K_1)) + \theta_i) \right\},$$

$$\times \left\{ \frac{1}{K_2 - K_1} \left[\sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} Y_{t_0} e^{\mu_y (t_i - t_0) + s(u_z + \sigma_z^2) + \frac{1}{2} \sigma_y^2 (t_i - t_0)} + \theta_i \sqrt{\text{Var}^P(Y_{t_i})} \right] - \frac{K_1}{K_2 - K_1} \right\}$$

$$i = 1, 2, 3,$$

The Fair Price of Swiss Re Bond (2)

$$P_r^P(Y_{\max} = Y_{t_1}) = \Phi(d_1, d_2, \rho_{1,2}), P_r^P(Y_{\max} = Y_{t_2}) = \Phi(d_3, d_4, \rho_{3,4}),$$

$$P_r^P(Y_{\max} = Y_{t_3}) = \Phi(d_5, d_6, \rho_{5,6}),$$

$$d_1 = \frac{-\mu_y(t_2 - t_1)}{\sigma_y \sqrt{|t_2 - t_1|}}, d_2 = \frac{-\mu_y(t_3 - t_1)}{\sigma_y \sqrt{|t_3 - t_1|}}, \rho_{1,2} = \text{corr}\left(\frac{W_{t_2-t_1}}{\sqrt{|t_2 - t_1|}}, \frac{W_{t_3-t_1}}{\sqrt{|t_3 - t_1|}}\right),$$

$$d_3 = \frac{\mu_y(t_2 - t_1)}{\sigma_y \sqrt{|t_1 - t_2|}}, d_4 = \frac{-\mu_y(t_3 - t_2)}{\sigma_y \sqrt{|t_3 - t_2|}}, \rho_{3,4} = \text{corr}\left(\frac{W_{t_2-t_1}}{\sqrt{|t_2 - t_1|}}, \frac{W_{t_3-t_2}}{\sqrt{|t_3 - t_2|}}\right),$$

$$d_5 = \frac{\mu_y(t_3 - t_1)}{\sigma_y \sqrt{|t_3 - t_1|}}, d_6 = \frac{\mu_y(t_3 - t_2)}{\sigma_y \sqrt{|t_3 - t_2|}}, \rho_{5,6} = \text{corr}\left(\frac{W_{t_3-t_1}}{\sqrt{|t_3 - t_1|}}, \frac{W_{t_3-t_2}}{\sqrt{|t_3 - t_2|}}\right),$$

$$E^P(Y_{t_i}) = \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} Y_{t_0} e^{\mu_y(t_i-t_0) + s(u_z + \sigma_z^2) + \frac{1}{2}\sigma_y^2(t_i-t_0)},$$

$$\text{Var}^P(Y_{t_i}) = \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} Y_{t_0}^2 e^{2\mu_y(t_i-t_0) + 2s u_z + 2\sigma_z^2(t_i-t_0)} - \left\{ \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} Y_{t_0} e^{2\mu_y(t_i-t_0) + 2s(u_z + \sigma_z^2) + \frac{1}{2}\sigma_y^2(t_i-t_0)} \right\}^2,$$

The Fair Price of Swiss Re Bond (3)

$$E^Q [Y_{t_i}] = \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} Y_{t_0} e^{\mu_y(t_i-t_0) + s(u_z + \sigma_z^2) + \frac{1}{2}\sigma_y^2(t_i-t_0)} + \theta_i \sqrt{\text{Var}^P(Y_{t_i})}, \quad i=1, 2, 3,$$

$$F_{Y_{t_i}}(K_1) = \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} \Phi\left(\frac{\ln \frac{K_1}{Y_{t_0}} - \mu_y(t_i-t_0) - su_z}{\sqrt{\sigma_y^2(t_i-t_0) + s\sigma_z^2}}\right),$$

$$F_{Y_{t_i}}(K_2) = P_r^P(Y_{t_i} \leq K_2) = \sum_{s=0}^{\infty} \frac{e^{-\lambda_t} (\lambda_t)^s}{s!} \Phi\left(\frac{\ln \frac{K_2}{Y_{t_0}} - \mu_y(t_i-t_0) - su_z}{\sqrt{\sigma_y^2(t_i-t_0) + s\sigma_z^2}}\right), \quad i=1, 2, 3.$$

Estimation of Model Parameters (1)

A calibration approach was adopted to estimate the variables $(\mu_i, \sigma_i, u_{\pi_i}, \sigma_{\pi_i}, \mu_{v,i}, \sigma_{v,i})$ for the US, UK, France, Italy, and Switzerland. The term calibration indicates the task of estimating best-fitted parameters in a parametric model in comparison with a chosen observable quantity.

Estimation of Model Parameters (2)

The detailed procedure is described in the following steps.

- (1) Collect the actual log returns of the population mortality indexes of the US, UK,

France, Italy, and Switzerland. $\hat{d}(\ln q_{i,t})$ was the model log returns of the five countries from equation (11), $d(\ln q_{i,t})$ the observed log returns of the population mortality index of each country. The differences of $d(\ln q_{i,t}) - \hat{d}(\ln q_{i,t})$ was a function of the values taken by $\Theta = (\mu_i, \sigma_i, u_{\pi_i}, \sigma_{\pi_i}, \mu_{v,i}, \sigma_{v,i})$.

- (2) Given the initial values of $(\mu_i, \sigma_i, u_{\pi_i}, \sigma_{\pi_i}, \mu_{v,i}, \sigma_{v,i})$, the parameter vector Θ was found to solve the nonlinear the sum of the squared errors during Period I and Period II as follows:

$$SSE = \min_{\Theta} \sum_{j=1}^n \left| \varepsilon_j[\Theta] \right|^2$$

Results under Calibration Approach (1)

Table 1 Parameter Estimation of Dynamic Processes of the Mortality Index of the 5 Countries

	US	UK	France	Italy	Switzerland
μ_i	-0.007635698	-0.0019212853	-0.0023350499	-0.0022268845	-0.0017903048
σ_i	0.0353039713	0.0303390187	0.0208596021	0.0346345882	0.0041059500
u_{π_i}	-0.4080070179	-0.0685499403	-0.0480721838	-0.0797427948	-0.0545940218
σ_{π_i}	0.1727495194	0.0287313469	0.0201543124	0.00328499675	0.0227485367

Note that $i=US, UK, France, Italy, \text{ and } Switzerland, a_1 = 0.7, a_2 = 0.15, a_3 = 0.025, a_4 = 0.05, a_5 = 0.075.$
















































Table 2 Parameter Estimation of Dynamic Processes of the Ratio of the Deaths Except in the i^{th} country

	US	UK	France	Italy	Switzerland
$\mu_{v,i}$	-0.0002826201	-0.0018850476	-0.0021297657	-0.0018799084	-0.0004465767
$\sigma_{v,i}$	0.0184113032	0.0126177338	0.0093617168	0.0124682592	0.0122047626

Note that $i=US, UK, France, Italy, \text{ and } Switzerland, a_1 = 0.7, a_2 = 0.15, a_3 = 0.025, a_4 = 0.05, a_5 = 0.075.$

Analysis of Comparative Statics (1)

Table 3 Impacts of Various Important Model Parameters on Swiss Re Bond

Parameter	$\theta = 0.83$	$\theta = 0.8657$	$\theta = 1.21$	$\theta = 1.5$
u_z	Panel A: u_z changes			
-0.001	0.5163	0.5196	0.5897	0.6125
-0.003	0.4987	0.5011	0.5734	0.5813
-0.005	0.4593	0.4972	0.5539	0.5712
-0.007	0.4886	0.5313	0.5618	0.5896
-0.009	0.5098	0.5478	0.5715	0.5947
σ_z	Panel B: σ_z changes			
 0.1	 0.9125	 0.9237	 0.9358	 0.9399
0.2	0.8143	0.8168	0.8915	0.9141
0.3	0.7759	0.7825	0.8598	0.8611
0.4	 0.5647	 0.5998	 0.6315	 0.7014
0.5	 0.3325	 0.3985	 0.4918	 0.5481
a	Panel C: a changes			
 0.01	 0.8169	 0.8198	 0.8245	 0.8266
 0.02	 0.8256	 0.8267	 0.8309	 0.8351
0.03	0.8321	0.8357	0.8401	0.8416
0.04	0.8395	0.8400	0.8415	0.8423
 0.05	 0.8411	 0.8425	 0.8438	 0.8509
λ_i	Panel D: λ_i changes			
 0.01	 0.6458	 0.6511	 0.6715	 0.6798
 0.02	 0.6135	 0.6212	 0.6598	 0.6613
0.03	0.5123	0.5237	0.5997	0.6011
0.04	 0.4978	 0.5198	 0.5498	 0.5599
 0.05	 0.4569	 0.4986	 0.5058	 0.5149

Analysis of Comparative Statics (2)

Given that the risk premiums for Swiss Re bond presented by Cox et al. (2006), Lin and Cox (2008), Chen and Cox (2009) and Lin, Liu and Yu (2010) were 0.83, 0.8657, 1.5, and 1.21, respectively, the impacts of mean and volatility on the magnitudes of infectious mortality, the threshold values (a), and jump intensities on the par spread of Swiss Re bond are demonstrated in Table 3.

Conclusion (1)

- Actual data of mortality report from when an event occurs show that mortality rates do not significantly co-move in the world until large deaths occur. There is no existing literature to model the real phenomenon of mortality rates. This paper fills the gap by offering a fresh look at the infectious effects of mortality rates on the valuation of mortality securities.

Conclusion (2)

- From empirical results, the fair par spread of the Swiss Re bond in the model was found to be far higher than that of Cox et al. (2006), and closer to the actual par spread. This shows that considering infectious effects of mortality rates enables the par spread of Swiss Re bond to fit into the real world. This is helpful to price mortality securities for insurance issuers.

**THANK YOU FOR YOUR
ATTENTION.**
