

A Relational Model Approach to deal with the Longevity Risk

Ninth International Longevity Risk and Capital Markets
Solutions Conference

Sep 7, 2013

Hsin Chung Wang, Department of Statistics and Actuarial
Science, Aletheia University, Taiwan.

Jack C. Yue, Department of Statistics, National Chengchi
University, Taiwan.

Outline

- Introduction(The main results and past works)
- Motivation
- Relational Model
- Mortality Prediction for the Elderly
- Pricing on Annuity Products
- Conclusion

Introduction

- The discount sequence ratio is a new relational model for correcting the mortality variation in age.
- The model is simple and transparent for making forecasts from past data.
 - by estimating a time series model for parameter.
 - or
 - by combining with block bootstrap method.
- The model can be used to forecasting elderly mortality or long-term forecasting.

Introduction

- Extend past works

1. Wang, H. C. and Yue, C. J. (2010), *Longevity Risk and Regular Discount Sequence with Application*, 14 International Congress on Insurance: Mathematics and Economics, University of Toronto.
2. Wang, H. C. and Yue, C. J. (2011), *Using Regular Discount Sequence to Model Elderly Mortality*, Journal of Population Studies vol. 43, pp37-70

Introduction

Reviews

- Discount Sequence (bandit problems)

→ Defined a random variable N with

$$\alpha_n = P(N \geq n), \quad \gamma_n = \sum_{i=n}^{\infty} \alpha_i.$$

The sequences $(\alpha_1, \alpha_2, \dots)$ and $(\gamma_1, \gamma_2, \dots)$ are regular, if the following conditions are satisfied

$$\alpha_n \cdot \alpha_{n+2} \leq (\alpha_{n+1})^2 \quad \text{or} \quad \gamma_n \cdot \gamma_{n+2} \leq (\gamma_{n+1})^2 \quad (1)$$

Introduction

Reviews(conti.)

● Let the survival time T and α_x as the survival function $S(x) = P(T \geq x)$. Then, the relationship between Discount Sequence and mortality is

$$l_n \cdot l_{n+2} \leq (l_{n+1})^2 \quad \text{or} \quad \frac{l_n \cdot l_{n+2}}{(l_{n+1})^2} \leq 1 \quad (2)$$

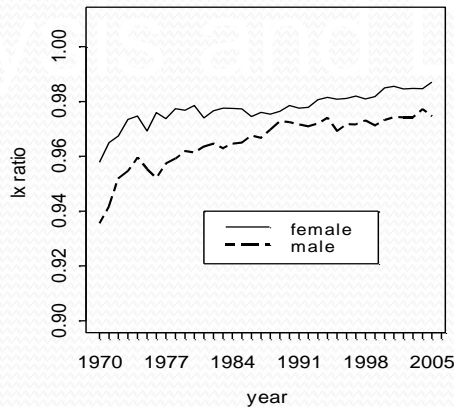
where l_n is the number of survivors at age n in life table.

Introduction

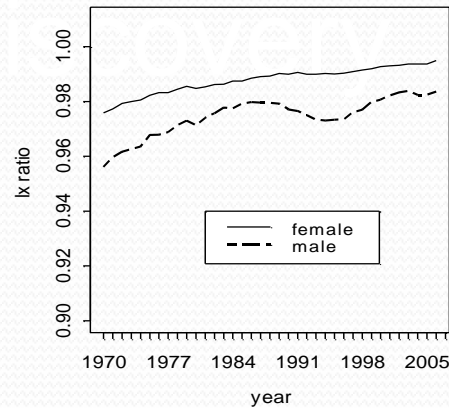
Reviews(conti.)

For the elderly(age 55-59, 60-64 and 65-69), the surviving numbers and life expectancy ratios satisfy the regularity condition.

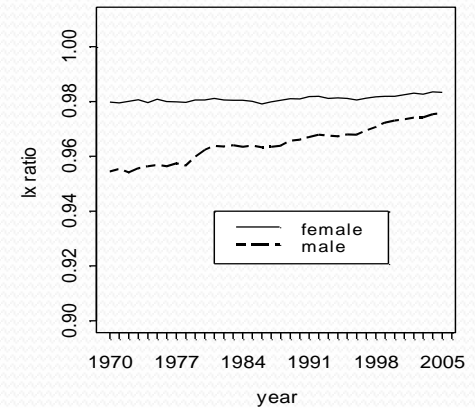
Taiwan (1970-2008)



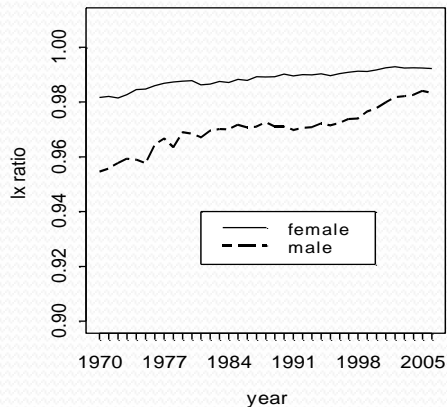
Japan (1970-2007)



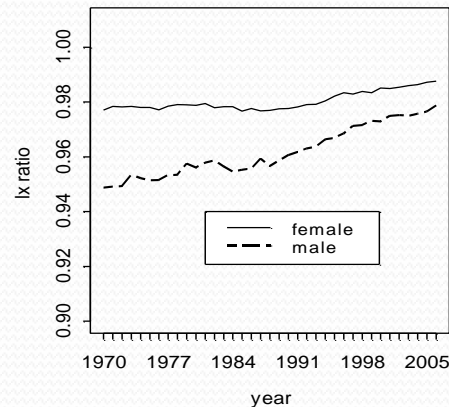
USA (1970-2006)



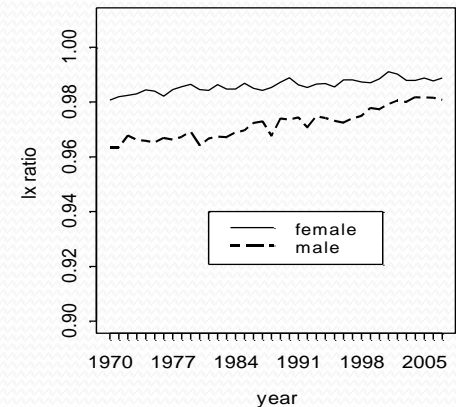
France (1970-2006)



England & Wales (1970-2006)



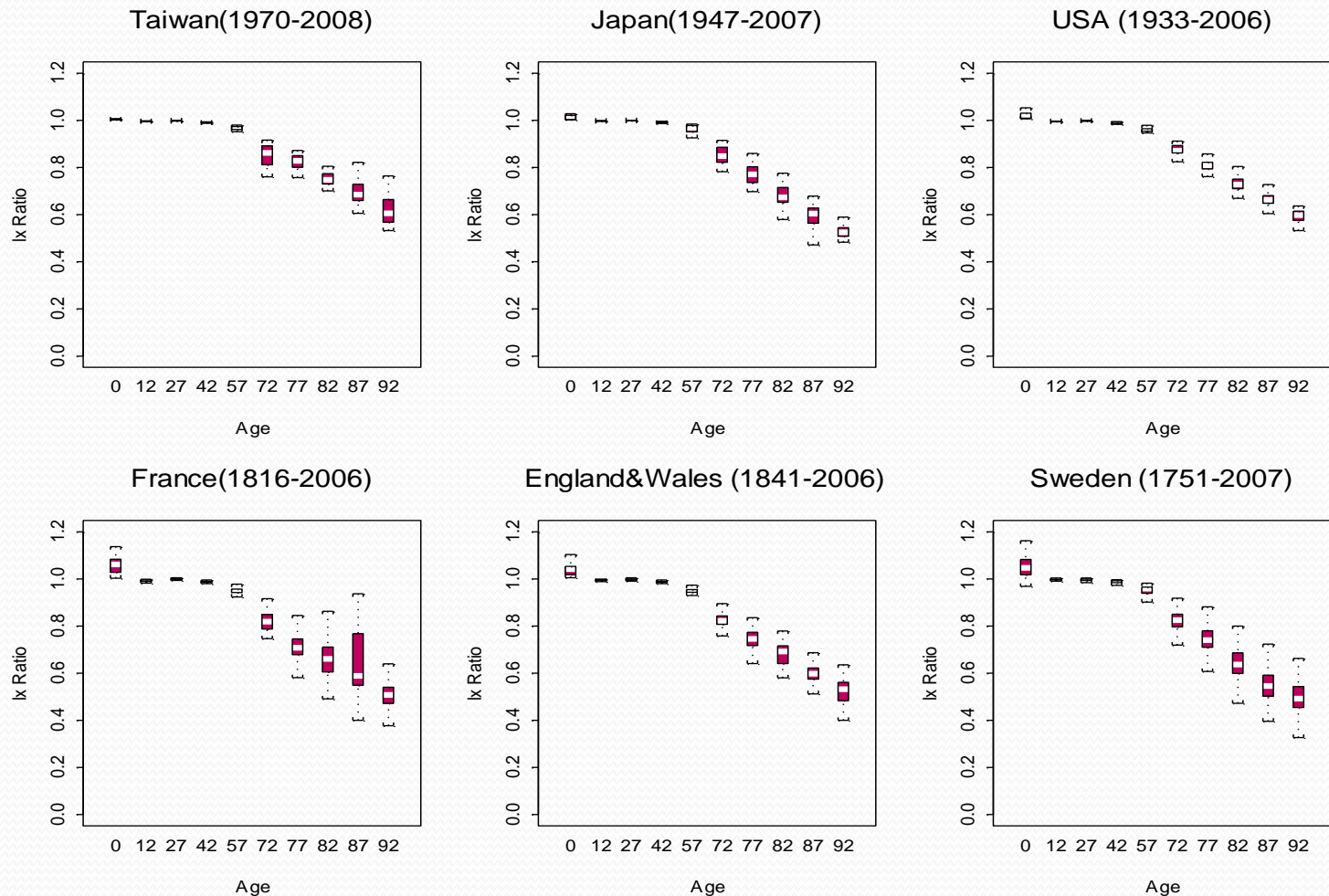
Sweden (1970-2007)



Introduction

Reviews(conti.)

The boxplots show similar results. A common trend: exponential decrease and stable convergence.



Introduction

Reviews(conti.)

- The discount sequence approach can cover
 - 1) Gompertz model
 - 2) Coale-Kisker (CK) model
 - 3) Uniform distribution of death (UDD)
 - 4) Constant Force (CF) assumption
 - 5) Hyperbolic assumption

Introduction

Reviews(conti.)

- Modeling the Discount Sequence Ratio:
- Brownian Motion Stochastic Differential Equations

Model (BMSDE)

$$dr_x(t) = k(\theta - r_x(t)) \cdot dt + \sigma \cdot dz \quad , \quad dz \sim N(0, dt)$$

k : is the strength, strong or weak, for $r_x(t)$

returning to the average level θ

θ : the long-term average level of $r_x(t)$.

σ : Diffusion coefficient

Introduction

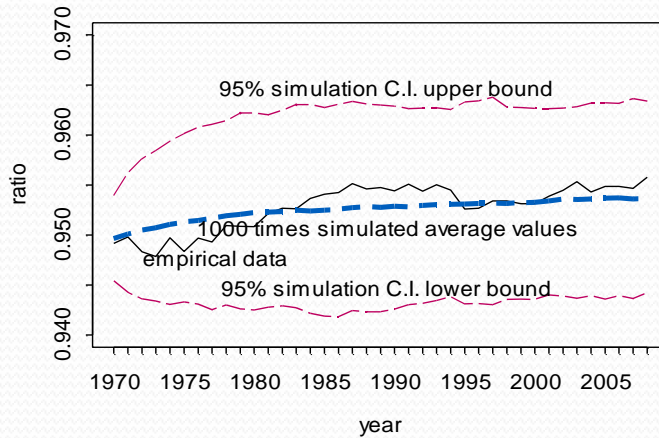
Reviews(conti.)

- Modeling the Discount Sequence Ratio:
- Example: The life expectancy and The number of survivors for the Taiwan males (5-age group):
 - we take the sample mean from 1000 simulations as the central tendency of the life expectancy and the number of survivors ratio and compare with the trend of the historical data.
- It meets the character of toward a certain fixed value and approaching stabilization and all of the simulations fall in the 95% confidence interval.

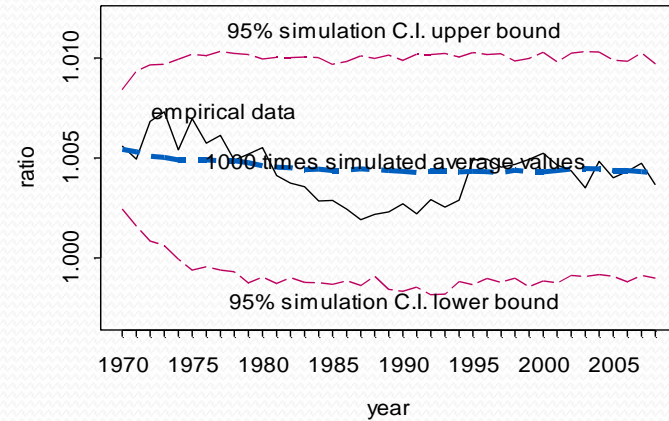
Introduction

Reviews(conti.)

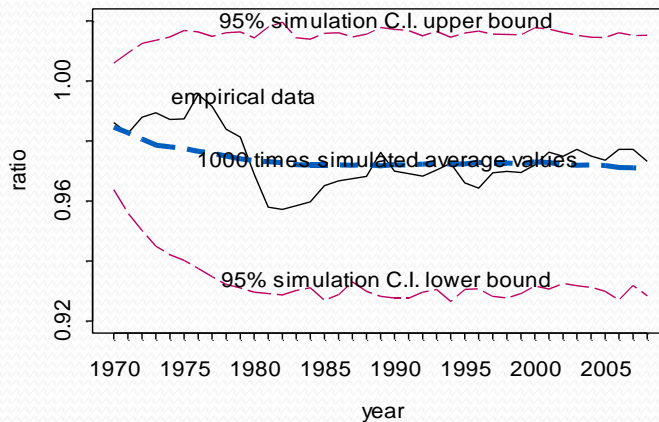
male age 0 ex ratio



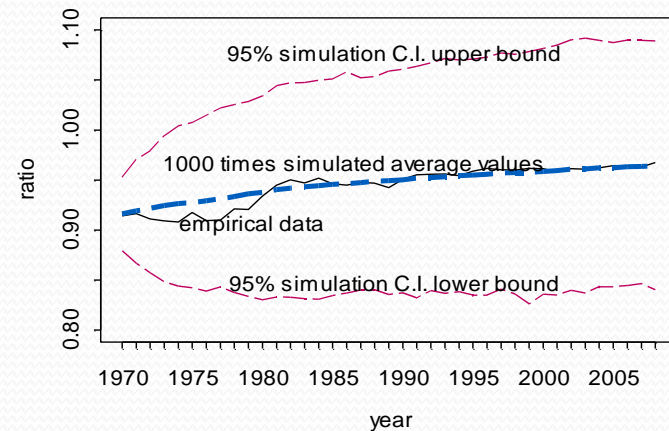
male age 0 lx ratio



male ages 60~64 ex ratio



male ages 60~64 lx ratio



Introduction

Reviews(conti.)

- Modeling the Discount Sequence Ratio
- Elderly Mortality Calculation (age 66~85)

Comparison of MAPE: Taiwan

		Lee-Carter	BMSDE
1970-2003	Male	5.95%	7.39%
	Female	5.47%	7.30%
2004-2008	Male	7.40%	5.69%
	Female	10.94%	13.63%

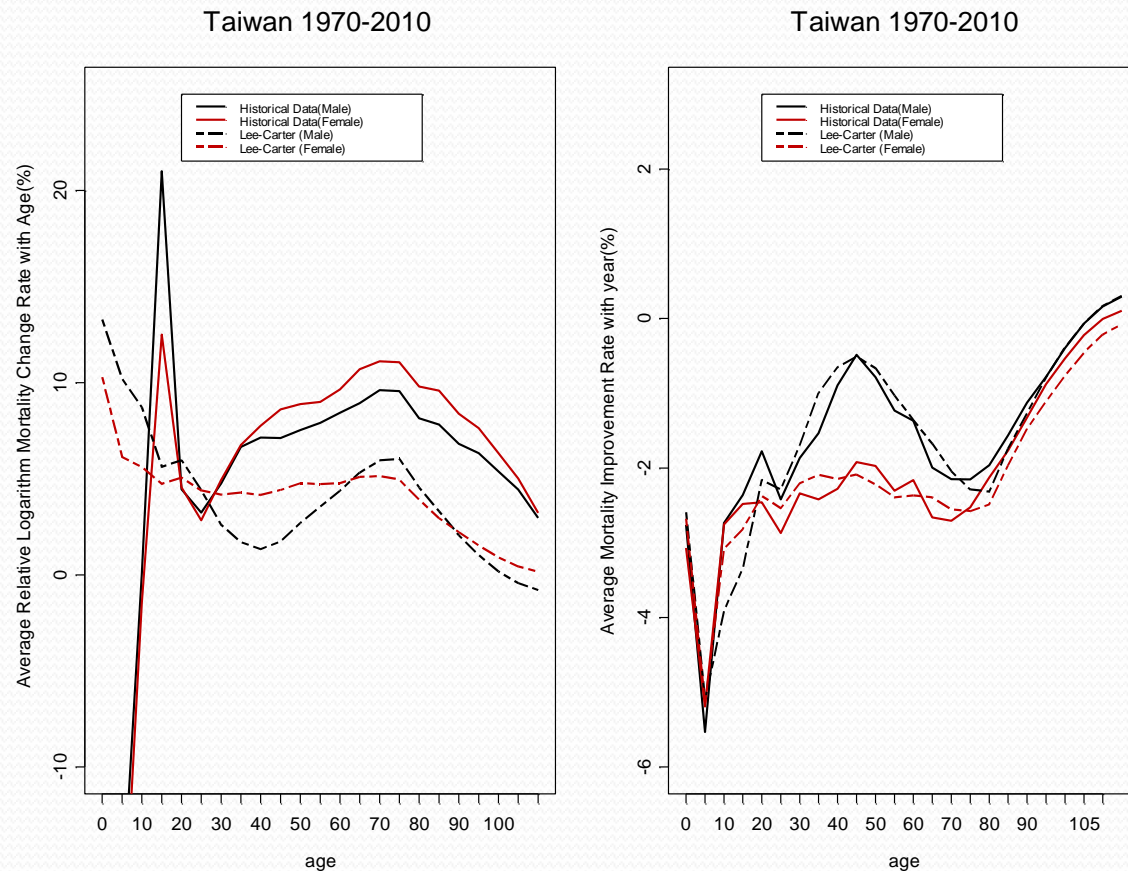
The BMSDE method by PCA clustering method indeed can be used for the fitness and forecast.

Motivation

- Non-dynamic Mortality Model, such as Gompertz's Model, does not consider the mortality improvement over the time.
- The stochastic mortality model, such as Lee-Carter (LC) model, uses time and age as the variables to consider the overall trend of the mortality changed over time and the age changes.
- LC model is perhaps the most popular mortality model, used in many countries. But it has some limitations.
-for example: The rates of mortality improvement are fixed for every age/age-group.

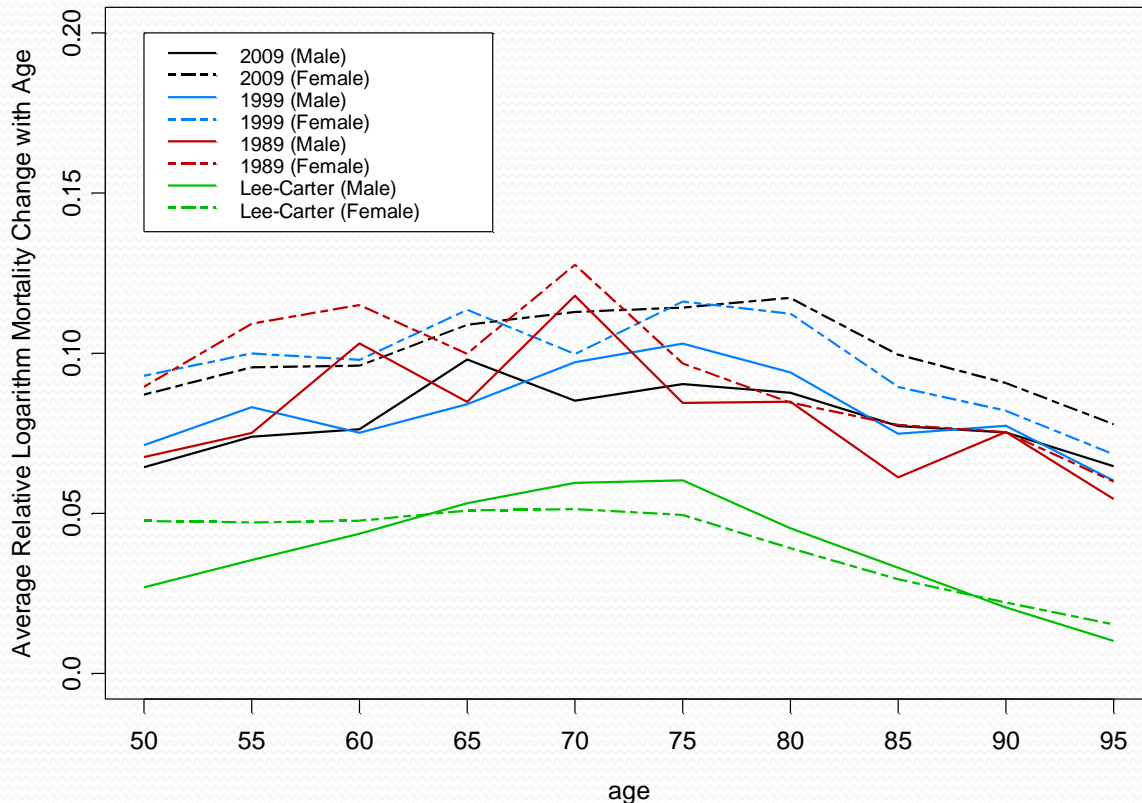
Motivation

- Lee-Carter model can follow the year variation trend(graph right). But it has a large gap to catch the age variation. (graph left)



Motivation

- Lee-Carter model cannot explain the mortality accelerating or decelerating phenomenon advisably.



Motivation

- The advantage of the relational approach is that the complexity of the mortality age pattern is captured in the standard mortality rates.
 - in constructing complete annual U. S. life tables (1999) used the formula $q_x = q_{x-1} \cdot e^{kx}$ for ages over 85 years.
 - probabilities of death at each age 100 and older were calculated by means of the formula $q_x = 1.05 \cdot q_{x-1}$ (Actuarial Study No. 116).

Motivation

- Bandit Problem is to choose between two decisions:
 - Discount sequences in Bandit Problem are used to model the possibility of future rounds.
 - Similar to the survival function, the discount sequence is a non-increasing function of time.
 - If the discount sequence satisfies certain regularity condition, choosing the optimal decision would become straightforward.
- A simple measure, the life-table aging rate (LAR), has proven to be a powerful tool for detecting age variation in mortality patterns (mortality acceleration or deceleration)

Relational Model

- By deducing a theorem to connect the regular discount sequence, the force of mortality rate and LAR.
- For a discrete random variable X , we define the relational mortality model (i.e. Discount Sequence Ratio) of age x

as $r_x = \frac{p_{x+1}}{p_x}$.

We have

$$1 - r_x \cong -\frac{S''(x)}{S'(x)} \cong -\frac{f'(x)}{f(x)} = 1 - \frac{p_{x+1}}{p_x} .$$

$$2. \frac{d}{dx} \log \mu_x = \mu_x + r_x - 1 = \text{LAR}.$$

Mortality Prediction for the Elderly

- Proposed method for Prediction : predict the mortality rates for ages 65 to 95
- ◆ Cross-Validation: Separating methods into control group and experimental group .
- Experimental Group:

Model DS: Suppose we know the true $q_{64}(t)$.

Model DS1: $\hat{q}_{64}(t)$ is estimated by the LC model.

Model DS2: Median of 1,000 Block Bootstrap simulation runs for both h_{xt} and r_{xt} , where central mortality rate is separated into $m_{xt} = h_{xt} \times r_{xt}$.

Mortality Prediction for the Elderly

◆ Cross-Validation

- Control Group:

LC model: the logarithm of central mortality rate satisfies

$$\ln(m_{xt}) = \alpha_x + \beta_x \kappa_t + \varepsilon_{xt}$$

Parameters estimate: SVD (singular value decomposition) method.

The parameters α_x and β_x are treated as constants of time and predict the future κ_t :

Model LC1: Combining OLS and AR (1)

Model LC2: OLS without time series trend

Model LC3. Median of 1,000 Block Bootstrap simulation runs

Mortality Prediction for the Elderly

- Mortality Projection Comparison:
- ◆ Cross-validation: separating the data into training period (or in-sample) and testing period (or out-sample).
- Short-term Forecast Data Description

	Taiwan	Japan
Training Period	1970-2002	1970-2002
Testing Period	2003-2008	2003-2008

- Long-term Forecast Data Description

	England & Wales	Japan	West Germany	U.S.A.
Training Period	1950-1981	1947-1982	1956-1983	1933-1981
Testing Period	1982-2006	1983-2007	1984-2008	1982-2006

Mortality Prediction for the Elderly

● Mortality Projection Comparison:

The criteria for forecast accuracy: Mean Absolute Percentage Error

(MAPE): $MAPE = \frac{1}{n} \sum_{i=1}^n \frac{|\varepsilon_i|}{Y_i} \times 100\%$, where Y_i and ε_i are the observed value and residual of observation \mathbf{i} , $i = 1, 2, \dots, n$. (Lewis, 1982)

● Short-term Forecast results

1. The LC model has a better predicting accuracy in Japan, than in Taiwan.
2. For the proposed method, knowing the true mortality rate does have a smaller MAPE and can provide very accurate prediction.
3. If we use double BB method, i.e., model DS2, the proposed discount sequence approach is very competitive and is a possible alternative for the LC model.

Mortality Prediction for the Elderly

- Short-term Forecast(MAPE)%

Model		Taiwan		Japan		Average
		Male	Female	Male	Female	
DS		7.1	4.6	4.9	11.0	6.9
DS	1	8.8	5.0	4.6	11.2	7.4
	2	7.9	4.9	4.7	10.8	7.1
LC	1	14.8	15.9	4.0	17.3	13.0
	2	12.8	10.5	3.1	5.3	7.9
	3	14.6	6.9	3.0	6.2	7.7

Mortality Prediction for the Elderly

- Long-term(at least 25 years) Forecast results
 1. using the idea of DS with perfect information (known $q_{64}(t)$) has the smallest MAPE.
 2. Both the discount sequence approach and the LC model can provide a fairly accurate prediction consistent with the results in short-term forecast.

Mortality Prediction for the Elderly

● Long-term Forecast (MAPE)%

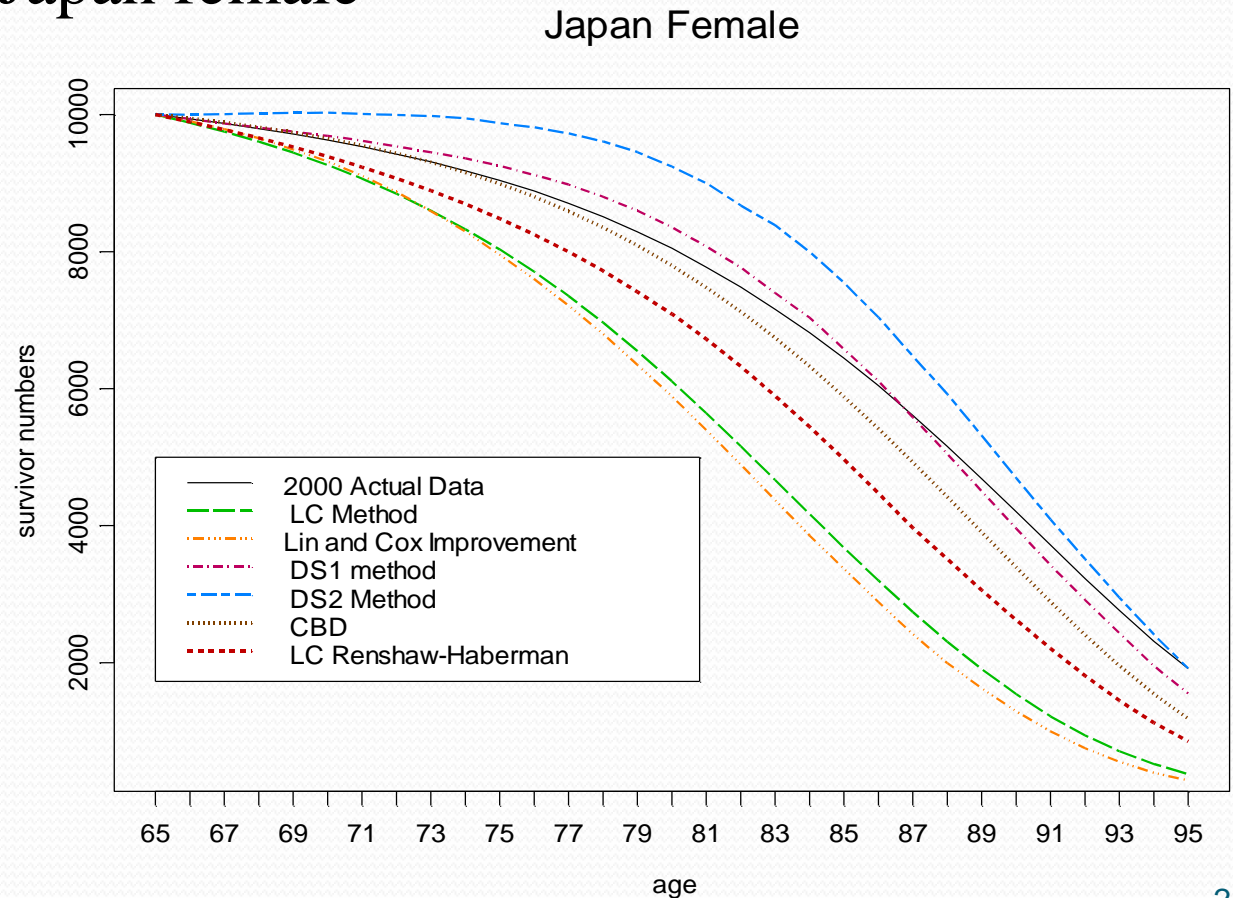
		Japan		USA		England and Wales		West Germany		Average
		M	F	M	F	M	F	M	F	
DS		4.8	12.7	6.0	12.1	8.4	5.7	15.4	5.2	8.8
DS	1	5.7	11.6	9.0	18.5	15.4	4.8	16.6	7.9	11.2
	2	18.9	27.9	7.5	19.3	15.4	5.9	13.5	8.8	14.6
LC	1	35.1	74.5	19.2	23.4	21.7	17.3	19.1	16.3	28.3
	2	16.9	31.7	14.5	7.3	18.7	6.2	21.7	10.8	16.0
	3	21.1	41.0	16.0	7.9	17.8	7.6	18.3	4.0	16.8

Pricing on Annuity Products

- We will use the idea of longevity risk securitization by Lin and Cox (2005) but emphasize mainly on the accuracy of mortality prediction.
- The calculation is based on the survivors of ages 65~95 in 1970 from HMD (Human Mortality Database).
- Mortality improvements methods : Lin and Cox (2005) 、 LC1 model 、 DS1 and DS2 、 Renshaw and Haberman (2006) 3-factor model (or RH model) 、 CBD (2006) model and Actual Data
- 10,000 persons age 65.

Pricing on Annuity Products

- Year 2000 Survival Curve Comparison
- For example: Japan female

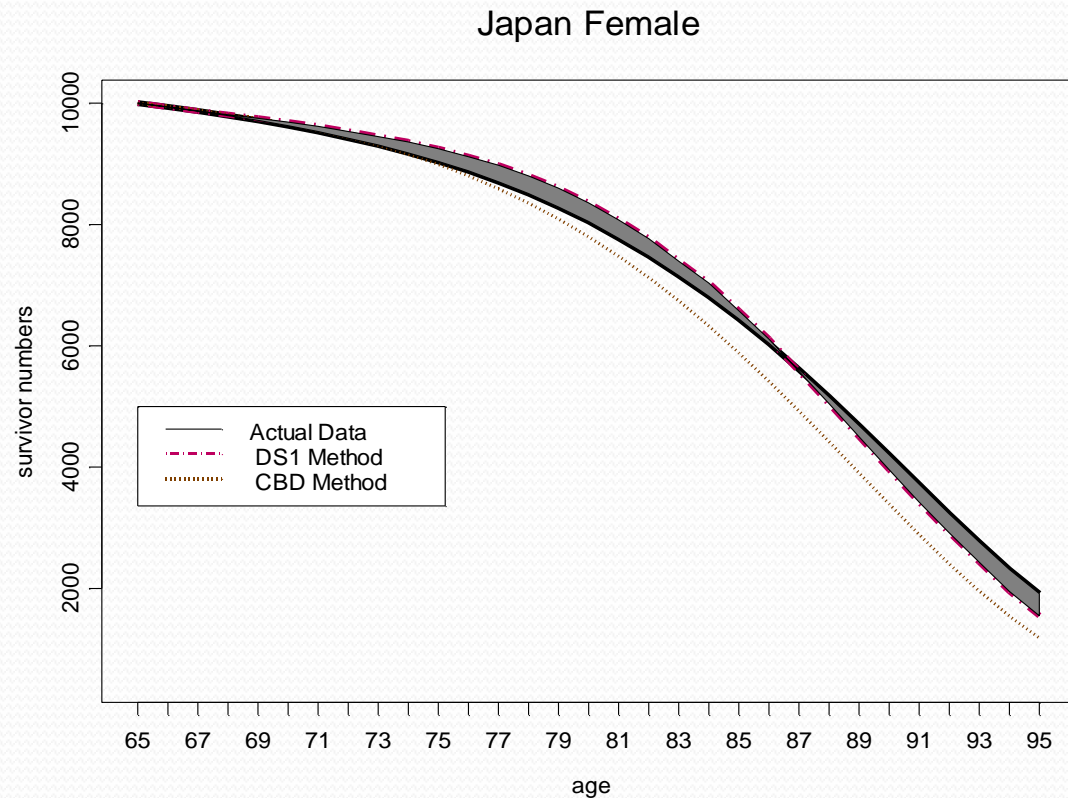


Pricing on Annuity Products

- Year 2000 Survival Curve Comparison: Japan(Conti.)
- Japanese females experienced a big mortality improvement from 1970 to 2000.
- The predicted values of the DS1 method (and CBD model) are very close to the true values in 2000. .
- LC method and Lin & Cox model are relatively conservative and underestimate the true numbers of survivors

Pricing on Annuity Products

- For example, the grey areas as follow are the differences between the true survival curve and estimated curve via the DS1 method.



Pricing on Annuity Products

- Define two measurements to Describe the difference between the true survival curve and estimated curve via mortality model: gross error and net error (Brown, 1997).
- The net error is to use the area over the true curve to deduct the area under the true curve, divided by the total area of true survival curve.
--a positive/negative net error indicates over/under-estimate in annuity products.
- The gross error as the discrepancy (or total area of grey areas as above Figure) between the true and estimated curves. The gross area is always non-negative.

Pricing on Annuity Products

● Mean Percentage Net Error of Survival Curve (2000-2006)

Model	Japan		U.S.A.		England and Wales		West Germany		Average
	Male	Female	Male	Female	Male	Female	Male	Female	
DS1	3.3%	2.2%	-12.4%	10.5%	-15.8%	-0.1%	-13.1%	1.6%	-3.0%
DS2	17.1%	12.7%	-4.9%	15.1%	-12.7%	0.0%	-8.7%	4.4%	2.9%
LC1	-18.5%	-23.7%	-16.1%	-10.1%	-20.9%	-11.4%	-14.9%	-8.0%	-15.5%
LC2	-7.4%	-10.9%	-12.7%	2.5%	-18.3%	-4.5%	-16.7%	-5.3%	-9.2%
RH	5.9%	-13.6%	-3.2%	-4.2%	-16.2%	-16.2%	-10.0%	-4.5%	-7.7%
CBD	0.0%	-5.0%	-9.0%	4.9%	-17.1%	-3.3%	-15.0%	-3.7%	-6.0%

Pricing on Annuity Products

- Mean Percentage Gross Error of Survival Curve (2000-2006)

Model	Japan		U.S.A.		England and Wales		West Germany		Average
	Male	Female	Male	Female	Male	Female	Male	Female	
DS1	3.3%	3.3%	12.4%	10.5%	15.8%	0.8%	13.1%	1.7%	7.6%
DS2	17.1%	12.7%	4.9%	15.1%	12.7%	0.7%	8.7%	4.4%	9.6%
LC1	18.5%	23.7%	16.1%	10.1%	20.9%	11.4%	14.9%	8.0%	15.5%
LC2	7.4%	10.9%	12.7%	2.5%	18.3%	4.5%	16.7%	5.3%	9.8%
RH	5.9%	13.6%	3.2%	4.3%	16.2%	16.2%	10.0%	4.5%	9.2%
CBD	1.9%	5.1%	9.0%	4.9%	17.1%	3.3%	15.0%	3.7%	7.5%

Pricing on Annuity Products

- For the net error, which can be used for annuity evaluation (i.e., assuming zero interest), the discount sequence approach (DS1 and DS2) has the best performance in 5 out of 8 scenarios (2 genders & 4 countries).
 - The DS type approaches also have the smallest average net errors.
 - The CBD and RH models have larger errors but still are better than the LC model.
 - The LC model (LC1 and LC2) almost always has negative errors, meaning that the annuity products are under-priced. → matches the LC model is conservative in mortality prediction.

Pricing on Annuity Products

- The results of gross error are similar and the discount sequence approach also has the best performance in 5 out of 8 scenarios.
- Among all models, the CBD and DS1 have the best performance.
- The role of gross error is similar to the MAPE. If we adapt the criterion of Lewis (1982), all models are fairly accurate in predictions and all mortality models considered in this study are possible candidates for dealing with mortality improvement.

Conclusion

- Longevity bonds, mortality bonds, and mortality swaps are some examples of securitization. However, the mortality index is still one of the key elements in these products and choosing a reliable mortality model plays an important role.
- From theoretical and simulation verification, we constructed the relationships between force of mortality, discount sequence ratio and LAR, these functions are important respectively to mortality research.
- The discount sequence ratio is a new relational model for elder mortality estimate and long-term mortality prediction.

Conclusion

- Conclusion (Conti.)
- For short-term and long-term forecasts, the proposed discount sequence approach has competitive errors (MAPE) in forecasting mortality rates than the LC model.
---This suggests that the proposed model can serve as an alternative mortality model to the LC model.
- The proposed approach is competitive with the CBD and RH models, two popular models for the elderly mortality.
- On average, the proposed approach is the best if judging by the net errors (i.e., pricing the annuity).



Thank you for your
attention.