A Common Age Effect Model for the Mortality of Multiple Populations

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Introduction

- We consider the mortality rates of I countries (or populations)
- $\tilde{m}_i(x,t) = \log(D/E)$ denotes the log mortality rate in country $i = 1, \dots, I$ at age $x = x_1, \dots, x_n$ in year $t = t_1, \dots, t_T$
- We use data from the Human Mortality Database
- Average log mortality rate for a life aged x in population i:

$$\bar{m}_i(x) = \frac{1}{T}\sum_{k=1}^{T} \tilde{m}_i(x, t_k).$$

We consider the centralised log mortality rates:

$$m_i(x,t) = \tilde{m}_i(x,t) - \bar{m}_i(x).$$

Individual Model of order p (Lee-Carter model)

$$m_i = \Big(m_i(x,t)\Big)_{n\times T} = \beta_i \kappa_i + \varepsilon_i$$

where

$$\beta_i = \begin{pmatrix} \beta_i^{(1)}(x_1) & \cdots & \beta_i^{(p)}(x_1) \\ \vdots & & \vdots \\ \beta_i^{(1)}(x_n) & \cdots & \beta_i^{(p)}(x_n) \end{pmatrix} \qquad (n \times p \text{ matrix})$$

$$\kappa_i = \begin{pmatrix} \kappa_i^{(1)}(t_1) & \cdots & \kappa_i^{(1)}(t_T) \\ \vdots & & \vdots \\ \kappa_i^{(p)}(t_1) & \cdots & \kappa_i^{(p)}(t_T) \end{pmatrix} \qquad (p \times T \text{ matrix})$$

$\varepsilon_i = \Big(\varepsilon_i(x,t)\Big)$ is a $n \times T$ error matrix That is:

$$m_i(x,t) = \beta_i^{(1)}(x)\kappa_i^{(1)}(t) + \ldots + \beta_i^{(p)}(x)\kappa_i^{(p)}(t) + \varepsilon_i(x,t) \qquad (1)$$

Individual Model of order p (Lee-Carter model)

It is well known that estimates for $\beta_i^{(j)}$ for any individual population i can be obtained by a principle component analysis using a singular value decomposition of the matrix m_i

$$m_i = \beta_i L_i U_i'$$

where

- L_i is a $p \times p$ diagonal matrix and
- U_i is a $T \times p$ matrix with mutually orthonormal columns, that is, $U_i' U_i = I$ (p-dimensional identity matrix).

Individual Model of order p (Lee-Carter model)

Equivalently, estimates for β_i can also be obtained from computing the eigenvectors of $m_i m_i'$:

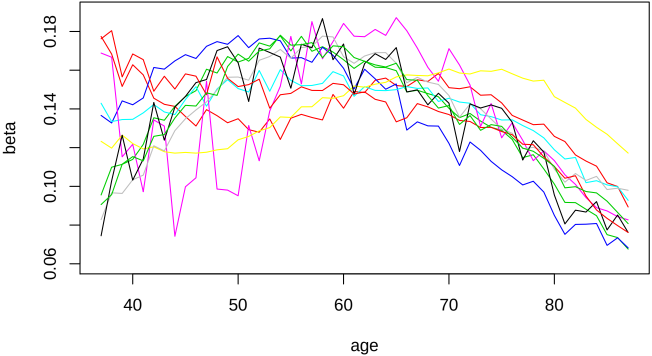
$$Q_i = m_i m_i' = \beta_i \kappa_i \kappa_i' \beta_i' = \beta_i \Lambda_i \beta_i'$$

since $(m_i = \beta_i L_i U_i')$

$$m_i m_i' = \beta_i L_i U_i' U_i L_i' \beta_i' = \beta_i L_i L_i' \beta_i' = \beta_i \Lambda_i \beta_i' \text{ with } \Lambda = LL'$$

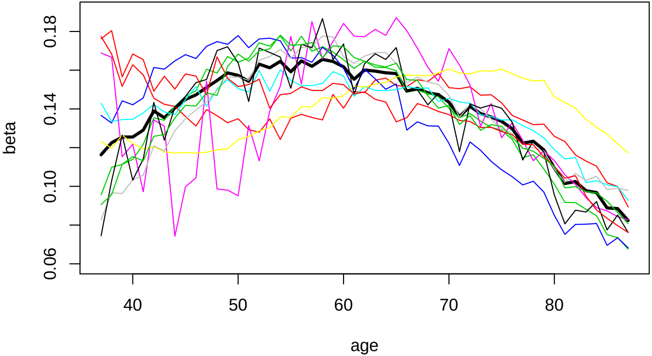
The eigenvalues of $m_i m_i'$ are on the diagonal of the matrix Λ_i, and the first age effect β_1^i is then the eigenvector corresponding to the largest eigenvalue of $m_i m_i'$.

Individual Model of order p (Lee-Carter model)



$\beta_i^{(1)}$ for ten countries (males).
Based on rates from 1948 to 2007.

Individual Model of order p (Lee-Carter model)



$\beta_i^{(1)}$ for ten countries (males).
Based on rates from 1948 to 2007.

Common age-effect model

We will now consider more than one population, $I > 1$, with the aim to reduce the overall number of parameters.
Our common age-effect model has the same structure as the individual model, but we now assume that the impact of age captured by β_i is independent of the population i.

$$m_i = \beta\kappa_i^c + \varepsilon_i \qquad i = 1, \ldots, I,\ t = t_1, \ldots, t_T. \tag{2}$$

$$m_i(x,t) = \beta^{(1)}(x)\kappa_i^{(1)}(t) + \ldots + \beta^{(p)}(x)\kappa_i^{(p)}(t) + \varepsilon_i(x,t) \tag{3}$$

where β is a $n \times p$ matrix. We call the model in (2) common age effect model of order p.
Note that in this model the period effects κ_i^c are still depending on the specific population.

Estimation of common age effects

- For the estimation of the common age effect β in (2) we use a common principal component analysis (cPCA) first introduced by Flury (1984).
- Instead of using the estimators proposed by Flury (1984) which are based on Maximum Likelihood estimation, we use here a modification based on least squares estimation.
- The numerical algorithm to obtain estimates of β is the F-G-algorithm, see Flury and Constantine (1985) with a modification by Clarkson (1988).

Estimation of common age effects

Our aim is to find an orthogonal matrix β and diagonal matrices Λ_i such that

$$Q_i := m_i m_i' = \beta \Lambda_i \beta' \qquad \forall\, i = 1, \ldots, I.$$

But this is equivalent to finding an orthogonal matrix β such that

$$\beta' Q_i \beta = \Lambda_i$$

is a diagonal matrix for all $i = 1, \ldots, I$.
This is, in general, not possible.

Estimation of common age effects

However, our estimate $\hat{\beta}$ for the common age-effect matrix β is the orthogonal matrix that makes all matrices $\beta' Q_i \beta$ as close to diagonal matrices as possible.
We denote by $||A||^2 = \sum_{i,j} a_{ij}^2$ the Frobinus-norm of a matrix A.
We now estimate β by minimizing the function

$$T(\beta) \;=\; \sum_{i=1}^{I} ||\beta' Q_i \beta - \mathrm{diag}(\beta' Q_i \beta)||^2 = \sum_{i=1}^{I} \sum_{j \neq l} (\beta' Q_i \beta)_{jl}^2$$

which is the sum of the squares of the off-diagonal elements of $\beta' Q_i \beta$.
Our estimate $\hat{\beta}$ is then

$$\hat{\beta} = \arg\min_{\beta} T(\beta)$$

where the minimum is taken over all orthogonal matrices β.

Estimation of common age effects

We also obtain an estimate for Λ_i

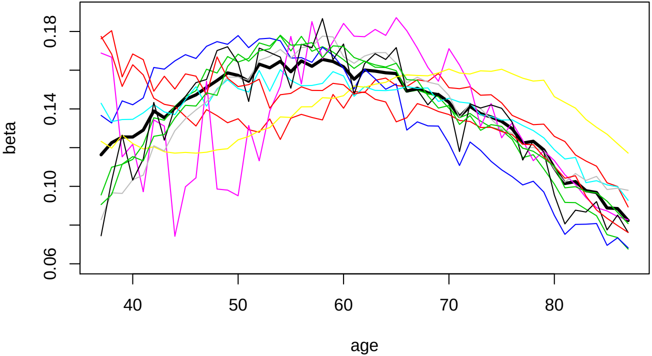
$$\hat{\Lambda}_i = diag(\hat{\beta}' Q_i \hat{\beta})$$

The modified F-G-Algorithm by Clarkson (1988) is used to obtain the estimates for β numerically.
As in the individual model, we only take the first p columns of $\hat{\beta}$ to obtain a common age-effect model of order p, that is

$$\hat{\beta}(p) = (\hat{\beta}_1, \ldots, \hat{\beta}_p)$$

The first p columns are here the columns that correspond to the largest p values in the diagonal of Λ_1.
Problem: The order of elements in the diagonal of Λ_1 might be different from the order in other matrices Λ_i.

Estimation of common age effects



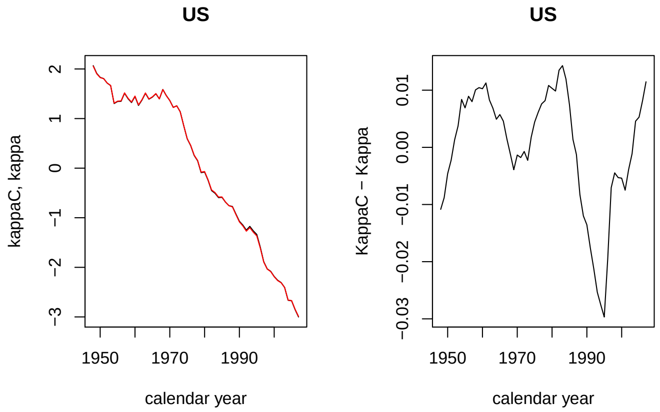
$\beta_i^{(1)}$ and $\beta^{(1)}$ for ten countries (males).
Based on rates from 1948 to 2007.

Estimation of common age effects

After obtaining estimates $\hat{\beta}(p)$ for the common age effects β, we estimate κ_i in the usual way treating $\hat{\beta}(p)$ as given. Since $\hat{\beta}(p)$ is an orthogonal matrix, we obtain $\hat{\kappa}^i$ as

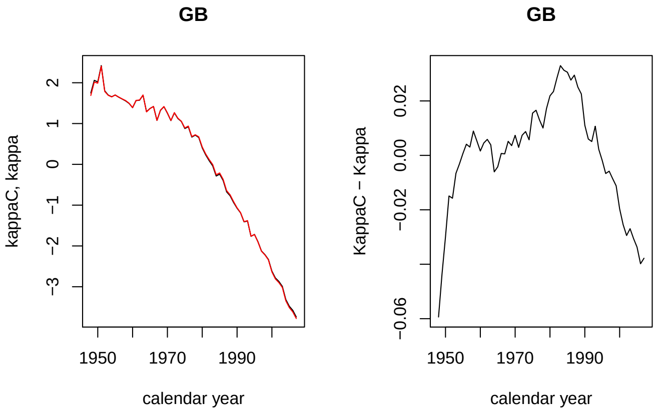
$$\hat{\kappa}^i(t) = \hat{\beta}(p)' m_i(t)$$

Empirical results and model comparison



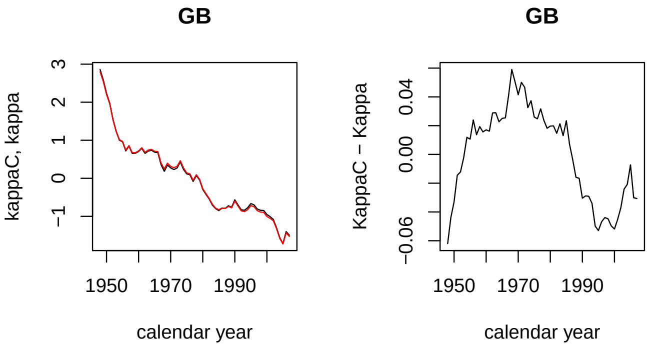
$\kappa_{US}^{(1)}$ for the US (common age effect - black, individual model - red). ages: 47-87

Empirical results and model comparison



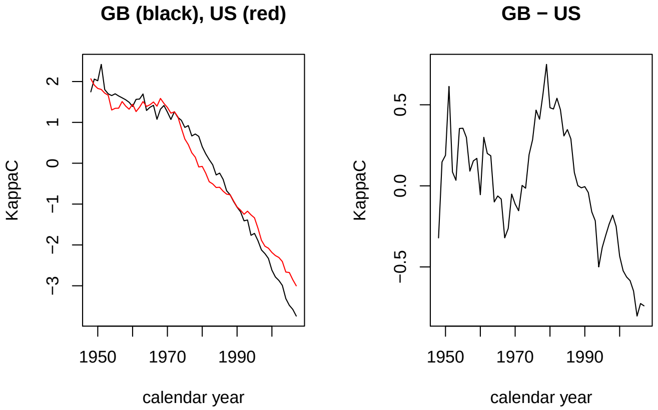
$\kappa_{GB}^{(1)}$ for the United Kingdom (common age effect - black, individual model - red). ages: 47-87

Empirical results and model comparison



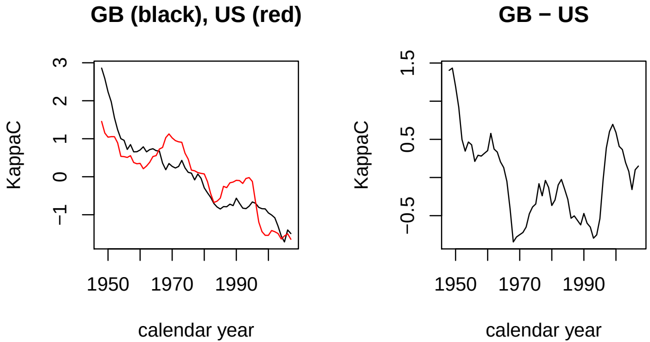
$\kappa_{GB}^{(1)}$ for the United Kingdom (common age effect - black, individual model - red). ages: 17-47

Empirical results and model comparison



$\kappa_i^{(1)}$ for the UK and the US (UK - black, US - red). ages: 47-87

Empirical results and model comparison



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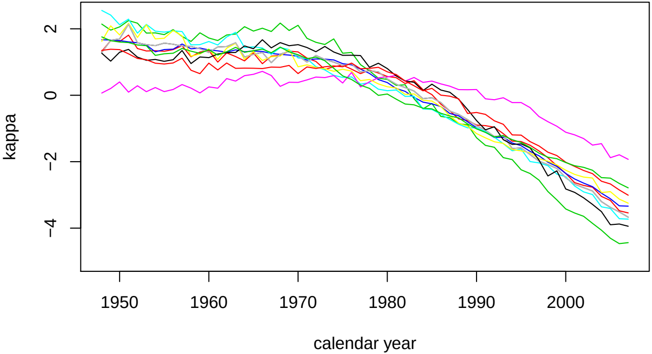
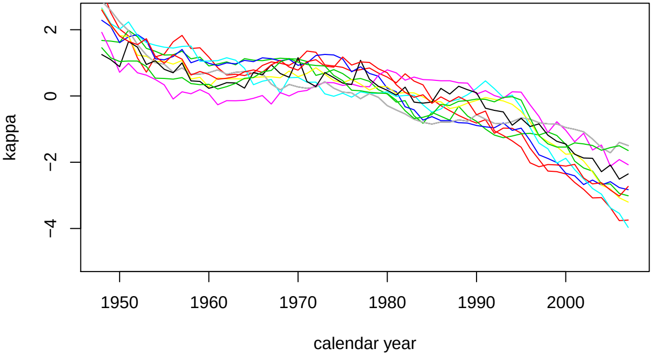


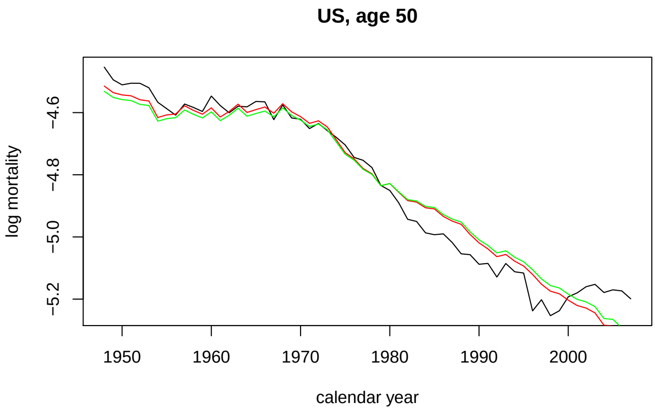
$\kappa_i^{(1)}$ for all ten countries based on common age effect (Denmark - pink). ages: 47-87

Empirical results and model comparison



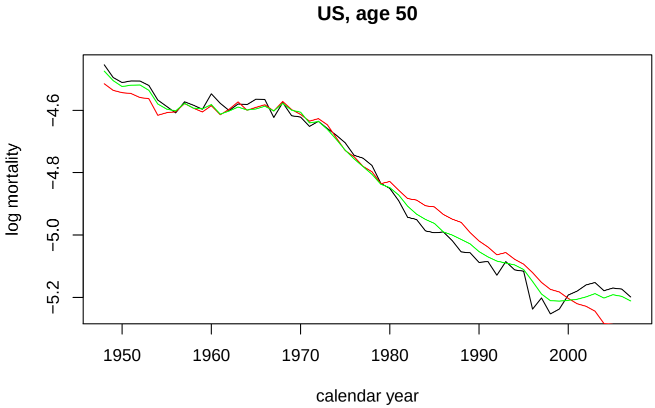
$\kappa_i^{(1)}$ for all ten countries based on common age effect (Denmark - pink). ages: 17-47

Goodness of Fit



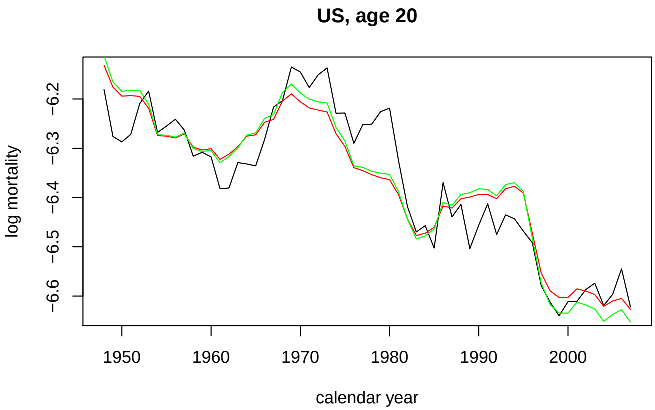
Observed (black) log mortality rates in the US with fitted individual model (red) and fitted common age effect model (green), $p = 1$, ages: 47-87

Goodness of Fit



Observed (black) log mortality rates in the US with fitted individual model (red, $p = 1$) and fitted common age effect model (green, $p = 2$), ages: 47-87

Goodness of Fit



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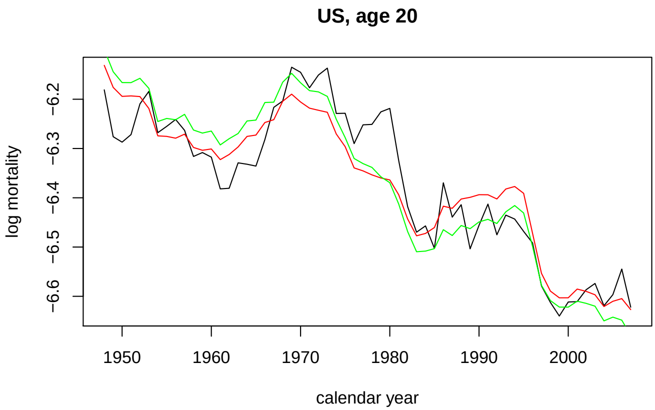

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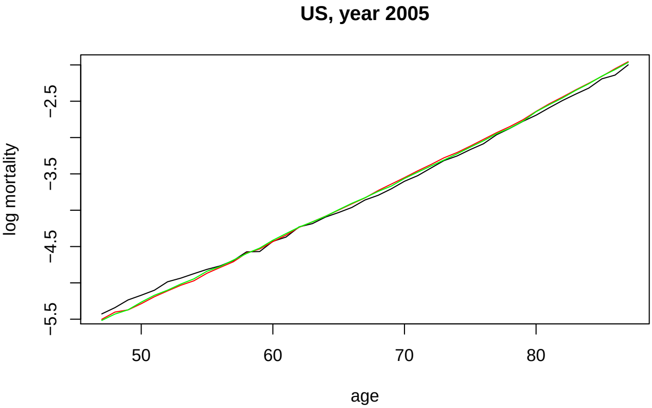

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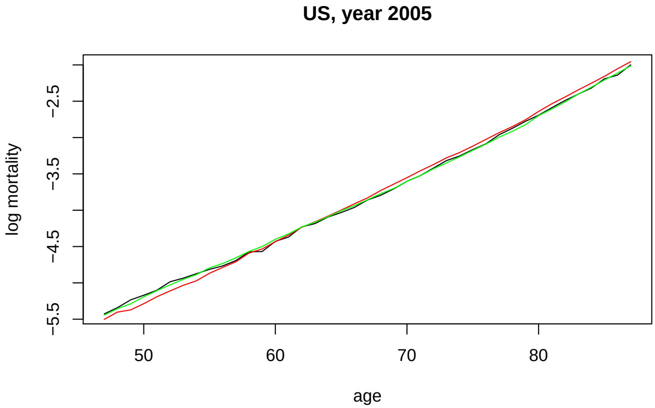

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Goodness of Fit

Observed (black) log mortality rates in the US with fitted individual model (red, $p = 1$) and fitted common age effect model (green, $p = 2$), ages: 17-47

Goodness of Fit

$$\sum_i \sum_x \sum_t \Big(m_i(x,t) - \hat{m}_i(x,t) \Big)^2$$

ages	17 - 47		47 - 87	
	$p=1$	$p=2$	$p=1$	$p=2$
Ind. Mod.	5.07	4.34	1.54	0.73
CAE Mod.	5.37	4.46	2.11	1.11

Conclusions

$$m_i(x,t) = \beta^{(1)}(x)\kappa_i^{(1)}(t) + \ldots + \beta^{(p)}(x)\kappa_i^{(p)}(t) + \varepsilon_i(x,t)$$

- Country-specific age effects seem to have a small impact on estimated period effects ($\kappa_i \approx \kappa_i^c$).
- A "second order" age-period effect improves the goodness of fit of a CAE model of order $p = 1$ more than country-specific age effects.
- The period effects of different populations in a CAE model are better comparable with each other since the impact of different age effects is eliminated. This could be relevant for modelling basis risk and forecasting the mortality of multiple populations.

Clarkson, D. B., 1988. Remark as r71: A remark on algorithm as 211. the f-g diagonalization algorithm. Journal of the Royal Statistical Society. Series C (Applied Statistics) 37 (1), 147–151. URL `http://www.jstor.org/stable/2347513`

Flury, B. N., 1984. Common principal components in k groups. Journal of the American Statistical Association 79 (388), 892–898.
URL `http://www.jstor.org/stable/2288721`

Flury, B. N., Constantine, G., 1985. Algorithm as 211: The f-g diagonalization algorithm. Journal of the Royal Statistical Society. Series C (Applied Statistics) 34 (2), 177–183.
URL `http://www.jstor.org/stable/2347375`