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RISK SHARING IN LIFE INSURANCE AND PENSIONS
within and across generations

Ragnar Norberg
ISFA Université Lyon 1/London School of Economics
Email: ragnar.norberg@univ-lyon1.fr
Homepage: http://isfa.univ-lyon1.fr/∼norberg/

LIFE INSURANCE WITH NO ECONOMIC-DEMOGRAPHIC RISK

Life insurance policy issued at time 0 and terminating at time T
State of policy at time t is $Z_t \in \{0, \ldots, n\}$, $Z_0 = 0$, $\mathbf{F}^Z = (\mathcal{F}_t^Z)_{t \in [0,T]}$
Payments (benefits less premiums) in $[0, t]$ total B_t:

$$dB_t = \sum_j I_t^j \, b_t^j \, dt + \sum_{j \neq k} b_t^{jk} \, dN_t^{jk}$$

$I_t^j = 1_{[Z_t = j]}$, state indicators
$N_t^{jk} = \sharp\{s;\, s \leq t,\, Z_{s-} = j,\, Z_s = k\}$, $j \neq k$, counting processes
b_t^j rate of annuity payable in state j at time t, deterministic
b_t^{jk} sum assured payable upon transition $j \to k$ at time t, deterministic
Transition probabilities $p^{jk}(s,t) = \mathbf{P}[Z(t) = k \,|\, Z(t) = j]$ (Z is Markov)
Rates of transition: $\mu_t^{jk} = \lim_{h \searrow 0} p^{jk}(t, t+h)/h$, $j \neq k$, deterministic
Rate of interest at time t: r_t, deterministic

Principle of *equivalence*:

$$\mathbf{E}\left[\int_0^T e^{-\int_0^\tau r} dB_\tau\right] = 0 \tag{1}$$

or

$$\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau)\left(b_\tau^j + \sum_{k;k\neq j} b_\tau^{jk}\,\mu_\tau^{jk}\right) d\tau = 0 \tag{2}$$

Rationale: In a large portfolio of N identical policies

$$\frac{1}{N}\sum_{i=1}^N \int_0^T e^{-\int_0^\tau r} dB_\tau^{(i)} \to \mathbf{E}\left[\int_0^T e^{-\int_0^\tau r} dB_\tau\right] \tag{3}$$

Equivalence is a benchmark: as $N \to \infty$, infinite surplus if premiums are on the safe side and infinite loss if premiums are insufficient.

Prospective reserve in state j at time t:

$$V_t^j = \mathbf{E}\left[\int_t^n e^{-\int_t^\tau r} dB_\tau \,\middle|\, Z_t = j\right]$$
$$= \int_t^n e^{-\int_t^\tau r} \sum_g p^{jg}(t,\tau) \left(b_\tau^g + \sum_{h;h\neq g} b_\tau^{gh} \mu_\tau^{gh}\right) d\tau \qquad (4)$$

Reserve (insurer's debt to insured) should be non-negative for any sensible insurance product. Equivalence $V_0^0 = 0$.

Thiele's differential equations

$$\frac{d}{dt} V_t^j = r_t V_t^j - b_t^j - \sum_{k;\,k\neq j} R_t^{jk} \mu_t^{jk} \qquad (5)$$

$$R_t^{jk} = b_t^{jk} + V_t^k - V_t^j \quad \text{(sum at risk)} \qquad (6)$$

LIFE INSURANCE WITH ECONOMIC-DEMOGRAPHIC RISK

$Y_t = (r_t, \mu_t^{jk})$, $t \in [0,T]$ unknown at time 0, hence "stochastic",
$\mathbf{F}^Y = (\mathcal{F}_t^Y)_{t\in[0,T]}$
Z_t, $t \in [0,T]$, follows model above conditional on $\mathcal{F}_T^Y$
Instead of (3) we have

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^T e^{-\int_0^\tau r}dB_\tau^{(i)} \to \mathbf{E}\left[\int_0^T e^{-\int_0^\tau r}dB_\tau \,|\, \mathcal{F}_T^Y\right]$$

Equivalence must now mean

$$\mathbf{E}\left[\int_0^T e^{-\int_0^\tau r}dB_\tau \,\middle|\, \mathcal{F}_T^Y\right] = 0 \qquad (7)$$

which is (2) now with r, μ^{jk} (hence p^{jk}) stochastic.

Thus, B must be adapted not only to $\mathbf{F}^Z$, bu to $\mathbf{F}^Y \vee \mathbf{F}^Z$.

WITH PROFIT

Payments $\bar{b}_t^j$ and $\bar{b}_t^{jk}$ *guaranteed* at time 0. Designed by equivalence principle using prudent *technical basis* with elements $\bar{r}_t$ and $\bar{\mu}_t^{jk}$. Technical surplus by time t is retrospective reserve based on realized elements less prospective reserve based on technical elements:

$$S_t = -e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\bar{b}_\tau^j + \sum_{k;k\neq j} \bar{b}_\tau^{jk}\, \mu_\tau^{jk} \right) d\tau - \sum_j p^{0j}(0,t)\, \bar{V}_t^j$$

Differentiate using Thiele (5) for the $\bar{V}_t^j$ and Kolmogorov forward for the $p^{0j}(0,t)$,

$$\frac{d}{dt} p^{0j}(0,t) = \sum_{g;g\neq j} p^{0g}(0,t)\, \mu_t^{gj} - p^{0j}(0,t) \sum_{g;g\neq j} \mu_t^{jg}$$

$$dS_t = S_t\, r_t\, dt + \sum_j p^{0j}(0,t)\, c_t^j\, dt \tag{8}$$

where c_t^j is the rate at which technical surplus emerges in state j at time t

$$c_t^j = \bar{V}_t^j (r_t - \bar{r}_t) + \sum_{k;\, k \neq j} \bar{R}_t^{jk}\, (\bar{\mu}_t^{jk} - \mu_t^{jk}) \tag{9}$$

Technical interest to the safe side if $\bar{r}_t \leq r_t$.
Technical rate of transition $j \to k$ to the safe side if $\text{sign}(\bar{\mu}_t^{jk} - \mu_t^{jk}) = \text{sign}\bar{R}_t^{jk}$.

Integrate (8), using side condition $S_0 = 0$, to recast

$$S_t = e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau)\, c_\tau^j\, d\tau \tag{10}$$

Technical surpluses are to be paid back as *bonuses*, assume here as annuity dividends at rate δ_t^j in state j at time t and assurance dividends δ_t^{jk} upon transition $j \rightarrow k$ at time t.
The *net surplus* at time t is

$$\begin{aligned} W_t &= S_t - e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\delta_\tau^j + \sum_{k;k\neq j} \delta_\tau^{jk} \, \mu_\tau^{jk} \right) d\tau \\ &= e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(c_\tau^j - \delta_\tau^j - \sum_{k;k\neq j} \delta_\tau^{jk} \, \mu_\tau^{jk} \right) d\tau \end{aligned}$$

This is the balance of the account after accumulated dividends have been deducted from the technical surplus (10).

Dividends are not stipulated in the contract: they are controlled by the pension fund/insurance company.
They need to be non-negative and to satisfy

$$W_t \geq 0, \ \ \forall t$$

and $W_T = 0$, which means equivalence reestablished with factual rates:

$$\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(\bar{b}_\tau^j + d_\tau^j + \sum_{k;k\neq j} (\bar{b}_\tau^{jk} + \delta_\tau^{jk})\, \mu_\tau^{jk} \right) d\tau = 0$$

Remark 1: No model assumptions for r and μ^{jk}

Remark 2: Solvency for sure: No “longevity risk” (if implemented with sufficient prudence)

AUTOMATIC BALANCING MECHANISMS

Policy issued at time 0 specifies baseline rates $\bar{r}_t$, $\bar{\mu}_t^{jk}$, baseline payments $\bar{b}_t^j$, $\bar{b}_t^{jk}$, and contractual payments b_t^j and b_t^{jk} adapted to the realized indices r_t, μ_t^{jk}:

$$b_t^j = \frac{e^{-\int_0^t \bar{r}}\, \bar{p}^{0j}(0,t)}{e^{-\int_0^t r}\, p^{0j}(0,t)}\, \bar{b}_t^j \qquad b_t^{jk} = \frac{e^{-\int_0^t \bar{r}}\, \bar{p}^{0j}(0,t)\, \bar{\mu}_t^{jk}}{e^{-\int_0^t r}\, p^{0j}(0,t)\, \mu_t^{jk}}\, \bar{b}_t^{jk}$$

$$\begin{aligned} &\int_0^T e^{-\int_0^\tau r} \sum_j p^{0j}(0,\tau) \left(b_\tau^j + \sum_{k;k\neq j} b_\tau^{jk}\, \mu_\tau^{jk} \right) d\tau \\ &= \int_0^T e^{-\int_0^\tau \bar{r}} \sum_j \bar{p}^{0j}(0,\tau) \left(\bar{b}_\tau^j + \sum_{k;k\neq j} \bar{b}_\tau^{jk}\, \bar{\mu}_\tau^{jk} \right) d\tau \\ &= 0 \ \text{ by choice of baseline elements at time 0} \end{aligned}$$

PENSIONS

A pension scheme is introduced at time 0. At any time $t > 0$, every person aged x pays the amount $a_t(x)$ per time unit:

$$a_t(x) = \begin{cases} c_t(x)\,, & \text{if } x < z \quad \text{(contribution)} \\ -b_t(x)\,, & \text{if } x \geq z \quad \text{(benefit)} \end{cases}$$

z is retirement age.
$\ell_t(x)\,dx$ members at age $(x, x + dx)$ at time t.
The total fund at time t is U_t with dynamics

$$dU_t = U_t\,r_t\,dt + \left(\int_{x=0}^{z} c_t(x)\,\ell_t(x)\,dx - \int_{x=z}^{T} b_t(x)\,\ell_t(x)\,dx \right) dt$$

starting from $U_0 \geq 0$.

Some obvious facts can be read out of the formula. For instance, with time-independent contributions and benefits functions $c(x)$, $b(x)$, and z, we see that a sufficiently big drop in the number of entrants will in due course lead to negative net payment into the scheme, and the same goes for a sufficiently big drop in the mortality rates at ages $x > z$.

Life annuities in private insurance: Pension products offered on an individual and voluntary basis by the life insurance offices must be based on the "principle of (individual) equivalence". This means that, for given mortality and interest rates, expected discounted contributions must cover expected discounted benefits so that balance is obtained in a large portfolio. The reason for this is that enrollment in the scheme is voluntary and cannot be anticipated; if the company would seek to cover a deficit by charging premiums in excess of the equivalence rate, then potential new customers would be deterred and existing customers would cancel their policies. The problem with such schemes, with contributions and benefits set out in the contract at the outset, is that the solvency of the scheme depends on non-diversifiable indices. This problem is tackled in with-profit schemes.

There is no initial fund, so

$$U_t = e^{\int_0^t r} \int_0^t e^{-\int_0^\tau r} \int_0^\tau a_\tau(x)\, \ell_\tau(x)\, dx\, d\tau$$

Change variables $s = \tau - x$, $u = \tau$ to see payments per generation:

$$U_t = e^{\int_0^t r} \int_{s=0}^t \int_{u=s}^t e^{-\int_0^\tau r} a_u(u-s)\, \ell_u(u-s)\, du\, ds$$

Equivalence on an individual basis (or for each generation) means

$$\int_s^\infty e^{-\int_0^\tau r} a_u(u-s)\, \ell_u(u-s)\, du = 0$$

Since $a_u(u-s)$ is first positive for $u < s+z$ (contributions) and then negative for $u \geq s+z$ (benefits),

$$\int_s^t e^{-\int_0^\tau r} a_u(u-s)\, \ell_u(u-s)\, du > 0,$$

and so $U_t > 0$.

Occupational pension schemes. Entries into such schemes are due to employment (occupational schemes) We list two commonly used occupational schemes:

Defined benefits: The functions $c_t(x)$, $b_t(x)$, and z are fixed for a certain time period. Contributions must be set sufficiently high to ensure $U_t \geq 0$ for all likely developments of r, μ, and φ over the period.

Defined contributions: $c_t(x)$ is fixed for a certain time period. Benefits $b_t(x)$ may be regulated currently to ensure $U_t \geq 0$ over the period.

Usually contributions and benefits are related to income on an individual basis, and in principle they are set such that the present value of contributions less benefits is 0 on the average. This means that there is a positive reserve at any time, that is, $U_t > 0$. State pensions do not necessarily guarantee a certain level of benefits. Benefits are controlled such that the U_t stays positive, and solvency is not an issue. This is all different in occupational pension schemes and for annuity products offered by private life insurers.

State pensions. Surpluses and losses may be transferred across groups of participants and also across generations, and it is up to political and governmental bodies to decide when and how in view of experience from the past and predictions about the future. The main characteristics of a given scheme are the functions $c_t(x)$, $b_t(x)$, and z and, in particular, the extent to which they can be controlled and adapted to the development of the uncontrollable processes r_t, $\mu_t(x)$.

Pay-as-you-go: The functions $c_t(x)$, $b_t(x)$, and z are currently chosen such that contributions match benefits at any time. Thus, for all t, $dA_t = 0$ or

$$\int_{x=0}^{z} c_t(x)\ell_t(x)\,dx = \int_{z}^{\infty} b_t(x)\ell_t(x)\,dx$$

Thus, if $U_0 = 0$, then $U_t = 0$ for all t so there is no savings element in the scheme. There is no need to predict r and μ (or even to know what they are today).

INTERGENERATIONAL RISK SHARING IN PENSIONS

$Y_t = (r_t, \mu_t(x), x \in (0,T)$, $t \geq 0$ (t calendar time, x age)
$h_s\, ds$ new entrants in $(s, s+ds)$, all aged 0 (say)
The number of members at age $(x, x+dx)$ at time t is $\ell_t(x)\, dx$, where
$\ell_t(x) = h_{t-x}\, e^{-\int_{t-x}^{t} \mu_u(u-(t-x))\, du}$
Net surplus by time t is W_t:

$$\begin{aligned} dW_t &= W_t\, r_t\, dt + \int_{s=0}^{t} h_s\, ds\, e^{-\int_s^t \mu_u(u-s)\, du} \times \\ &\quad \left[(r_t - \bar{r} + \mu_t(t-s) - \bar{\mu}(t-s))\, \bar{V}_{t-s} - \delta_t(t-s) \right] dt \end{aligned}$$

HOW TO POSE THE PROBLEM ?

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