Modeling the evolution of the dependence structure between two lifetimes

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Outline

INTRODUCTION

- Framework :
  - Two lifetimes (T, U),
  - $T =$ lifetime of a man,
  - $U =$ lifetime of his wife.
- T and U are not independent.
- Censoring : some couples quit observation before dying.
- Truncation : observation only for people who lived long enough to enter the study.

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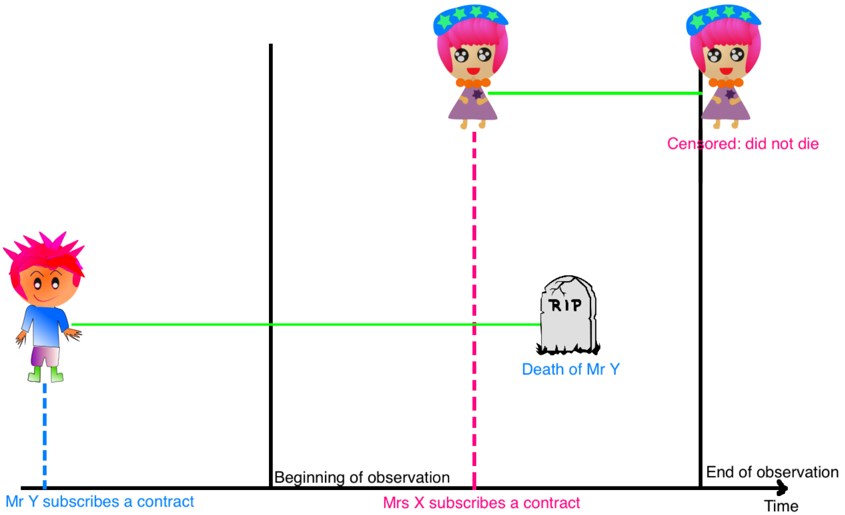
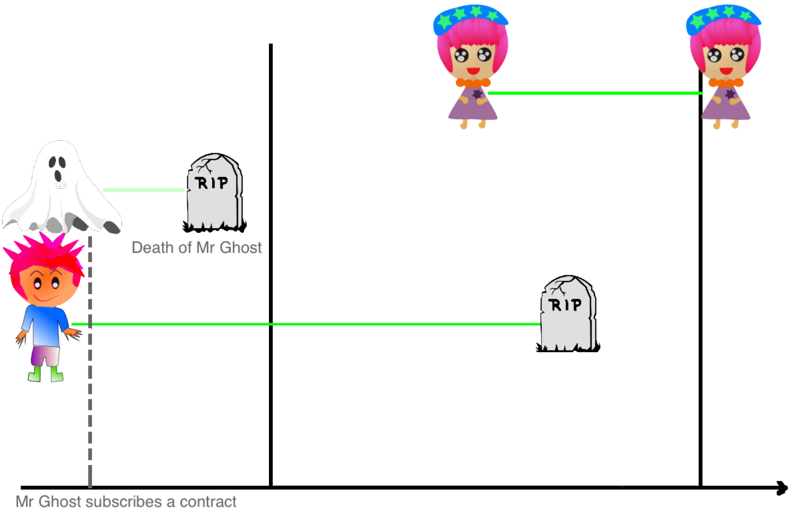
Censoring and truncation



CENSORING AND TRUNCATION



Observations

Bivariate right-censored and left-truncated data

We observe n i.i.d. copies $(Y_i, Z_i, \mu_i, \nu_i, \delta_i, \gamma_i)_{1\leq i\leq n}$, with

$$\begin{cases} Y_i = \inf(T_i, C_i), \\ Z_i = \inf(U_i, D_i), \end{cases}$$

where C_i and D_i are censoring variables, and

$$\begin{cases} \delta_i = \mathbf{1}_{T_i \leq C_i}, \\ \gamma_i = \mathbf{1}_{U_i \leq D_i}, \end{cases}$$

where $Y_i \geq \mu_i$ and $Z_i \geq \nu_i$.

- We assume that :
  - (T_i, U_i) is independent from (C_i, D_i).
  - (μ_i, ν_i) is independent from (T_i, U_i, C_i, D_i).

COVARIATES

BIVARIATE RIGHT-CENSORED AND LEFT-TRUNCATED DATA

We observe n i.i.d. copies $(Y_i, Z_i, \mu_i, \nu_i, \delta_i, \gamma_i, X_i)_{1\leq i\leq n}$, with

$$\begin{cases} Y_i & = & \inf(T_i, C_i), \\ Z_i & = & \inf(U_i, D_i), \end{cases}$$

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where $Y_i \geq \mu_i$ and $Z_i \geq \nu_i$.

- We assume that :
  - (T_i, U_i) is independent from (C_i, D_i) conditionally to X_i
  - (μ_i, ν_i) is independent from (T_i, U_i, C_i, D_i) conditionally to X_i.

NATURE OF THE COVARIATES

- In the following, we focus on $X_i = (X_i^{(1)}, X_i^{(2)})$, where :
  - $X_i^{(1)}$ = birth date of the husband
  - $X_i^{(2)}$ = birth date of his wife
- May be extended to any kind of covariates :
  - salary
  - size of the family
  - ...
- One constraint : since we are using smoothing techniques, we assume the covariates to be continuous.

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1. INTRODUCTION
2. COPULAS
   - Definition
   - Estimation in absence of censoring and truncation
3. ESTIMATION OF THE CONDITIONAL COPULA UNDER CENSORING AND TRUNCATION
   - Estimation of the conditional distribution function
   - Estimation of the conditional copula parameter

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1. INTRODUCTION

2. COPULAS
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3. ESTIMATION OF THE CONDITIONAL COPULA UNDER CENSORING AND TRUNCATION
   - Estimation of the conditional distribution function
   - Estimation of the conditional copula parameter

Sklar's Theorem

Sklar's Theorem - distribution functions

Let (T, U) be absolutely continuous variables with d.f. F, $F_T(t) = \mathbb{P}(T \leq t)$, $F_U(u) = \mathbb{P}(U \leq u)$. There exists a unique copula function $\mathfrak{C}$ such that

$$F(t, u) = \mathfrak{C}(F_T(t), F_U(u)).$$

Sklar's Theorem - survival functions

Let (T, U) be absolutely continuous variables with survival function S_F, $S_T(t) = \mathbb{P}(T > t)$, $S_U(u) = \mathbb{P}(U > u)$. There exists a unique copula function $\mathfrak{C}_S$ such that

$$S_F(t, u) = \mathfrak{C}_S(S_T(t), S_U(u)).$$

Moreover,

$$\mathfrak{C}_S(u, v) = u + v - 1 + \mathfrak{C}(1 - u, 1 - v).$$

Parametric copula families, notations

- We first consider that there is no covariate.
- (T_i, U_i) are i.i.d. with the same distribution as (T, U), whose dependence structure is defined by $\mathfrak{C}$.
- Assume that $\mathfrak{C} \in \{\mathfrak{C}_\theta : \theta \in \Theta\}$, where $\Theta \subset \mathbb{R}^k$.
- Let c_θ denote the copula density corresponding to $\mathfrak{C}_\theta$, that is

$$\mathfrak{c}_\theta(u, v) = \partial_u \partial_v \mathfrak{C}_\theta(u, v).$$

Estimation, margins known, no censoring nor truncation, no covariate

- Let $I = F_T(T)$ and $J = F_U(U)$.
- I and J are uniformly distributed over $[0, 1]$.
- Their joint distribution function if $\mathfrak{C}$.
- Maximum likelihood estimation :

$$\begin{aligned} \theta^* &= \arg\max_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \log \mathfrak{c}_\theta(I_i, J_i) \\ &= \arg\max_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \log \mathfrak{c}_\theta(F_T(T_i), F_U(U_i)). \end{aligned}$$

ESTIMATION, MARGINS UNKNOWN, NO CENSORING NOR TRUNCATION, NO COVARIATE

- Estimation of the margins by $\hat{F}_T$, $\hat{F}_U$.
- Various way of estimating the margins :
  - empirical distribution function
  - parametric models
  - semiparametric models (proportional hazard models when it comes to regression)
- Pseudo maximum likelihood estimation :

$$\begin{aligned}\theta^* &= \arg\max_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \log \mathfrak{c}_\theta(\hat{I}_i, \hat{J}_i) \\ &= \arg\max_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^{n} \log \mathfrak{c}_\theta(\hat{F}_T(T_i), \hat{F}_U(U_i)).\end{aligned}$$

CONDITIONAL COPULA

- The dependence structure depends on the value of the covariates X.
- We will denote $\mathfrak{C}(\cdot,\cdot|x)$ the copula of the conditional distribution of $(T,U)|X=x$.
- Nonparametric estimation : Abegaz, Gijbels, and Veraverbeke (2012), Gijbels, Omelka, Veraverbeke (2012), Acar, Craiu, and Yao (2011)...
- Here, we focus on a parametric model :

$$\mathfrak{C}(\cdot,\cdot|x)=\mathfrak{C}_{\theta(x)}(\cdot,\cdot).$$

- Our aim is then to estimate the function $\theta(x)$.

Estimation, margins UNKNOWN, no censoring nor truncation, COVARIATE

- Estimation of the conditional distributions of the margins by $\hat{F}_T(\cdot|X)$, $\hat{F}_U(\cdot|X)$.
- Pseudo maximum likelihood estimation :

$$\theta^*(x) \quad = \quad \arg\max_{\theta\in\Theta} \sum_{i=1}^{n} w_{i,n}(x) \log \mathfrak{c}_\theta(\hat{F}_T(T_i|X_i), \hat{F}_U(U_i|X_i)),$$

where

$$w_{i,n}(x) = \frac{K\left(\frac{X_i - x}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h}\right)},$$

introducing a kernel K and a bandwidth $h > 0$.

OUTLINE

ESTIMATION OF THE DISTRIBUTION FUNCTION OF (T, U)

- Recall that
$$\frac{1}{n}\sum_{i=1}^{n}\phi(T_i, U_i) = \int \phi(t, u)d\mathbb{P}_n(t, u),$$
where $\mathbb{P}_n$ is the empirical distribution function of (T, U).
- Recall that
$$\sum_{i=1}^{n} w_{i,n}(x)\phi(T_i, U_i) = \int \phi(t, u)d\mathbb{P}_n(t, u|x),$$
where $\mathbb{P}_n$ is a kernel estimator of
$F(t, u|x) = \mathbb{P}(T \leq t, U \leq u|X = x)$.
- How to estimate (non parametrically) the distribution of (T, U) under censoring and truncation ?

PROPERTIES OF $F(t) = \mathbb{P}(T \leq t)$ (NO TRUNCATION)

- Usual definition (absolutely continuous variables) :

$$\hat{F}(t) = 1 - \prod_{Y_i \leq t} \left(1 - \frac{\delta_i}{\sum_{j=1}^{n} \mathbf{1}_{Y_j \geq Y_i}} \right).$$

- Definition as a sum (no truncation) :

$$\hat{F}(t) = \sum_{i=1}^{n} W_{in} \mathbf{1}_{Y_i \leq t},$$

where $G(t) = \mathbb{P}(C \leq t)$ and $\hat{G}$ its Kaplan-Meier estimator,

$$W_{in} = \frac{\delta_i}{n[1 - \hat{G}(Y_i -)]}.$$

Properties of the Kaplan-Meier estimator of $F(t) = \mathbb{P}(T \leq t)$ (no truncation)

- Assume that T is independent from C.
- Define

$$\begin{aligned} H_1(t) &= \mathbb{P}(Y \leq t, \delta = 1), \\ 1 - H(t) &= \mathbb{P}(Y > t) = [1 - G(t)][1 - F(t)]. \end{aligned}$$

- F is the solution of the equation :

$$dF(t) = \frac{[\int_t^\infty dF(s)]dH_1(t)}{1 - H(t)}.$$

Properties of the Kaplan-Meier estimator of $F(t) = \mathbb{P}(T \leq t)$ (no truncation)

- Assume that T is independent from C.
- Empirical version :

$$\hat{H}_1(t) = n^{-1} \sum_{i=1}^{n} \delta_i \mathbf{1}_{Y_i \leq t},$$

$$1 - \hat{H}(t) = n^{-1} \sum_{i=1}^{n} \mathbf{1}_{Y_i > t} = [1 - \hat{G}(t)][1 - \hat{F}(t)].$$

- $\hat{F}$ is the solution of the equation :

$$d\hat{F}(t) = \frac{[\int_t^\infty d\hat{F}(s)] d\hat{H}_1(t)}{1 - \hat{H}(t)}.$$

A PARTIAL DERIVATIVE EQUATION IN THE GENERAL CASE

- Assume that (T, U) is independent from (C, D), and (μ, ν) are independent from the other variables.

P.D.E. (LOPEZ (2012), *Insurance : Mathematics and Economics*)

$F(t, u) = \mathbb{P}(T \leq t, U \leq u)$ is the solution of

$$dF(t, u) = \frac{[\int_t^\infty \int_u^\infty dF(t', u')]dV(t, u)}{H(t, u)},$$

where

$$\begin{aligned} H(t, u) &= \mathbb{P}(Y \geq t, Z \geq u | Y \geq \mu, Z \geq \nu), \\ V(t, u) &= \mathbb{P}\left(\mu \leq t \leq Y, \nu \leq u \leq Z, \delta = \gamma = 1 | Y \geq \mu, Z \geq \nu\right). \end{aligned}$$

EMPIRICAL VERSION OF THE P.D.E.

- For all function ψ,

$$\frac{1}{n}\sum_{i=1}^{n}\psi(Y,Z,\delta,\gamma)\rightarrow_{n\rightarrow\infty}E[\psi(Y,Z,\delta,\gamma)|Y\geq\mu,Z\geq\nu]\ p.s.$$

- Estimate $H(t,u)=\mathbb{P}(Y\geq t,Z\geq u|Y\geq\mu,Z\geq\nu)$ par

$$\hat{H}(t,u)=\frac{1}{n}\sum_{i=1}^{n}\mathbf{1}_{Y_i\geq t,Z_i\geq u}.$$

- For $V(t,u)=\mathbb{P}\left(\mu\leq t\leq Y,\nu\leq u\leq Z,\delta=\gamma=1|Y\geq\mu,Z\geq\nu\right)$,

$$\hat{V}(t,u)=\frac{1}{n}\sum_{i=1}^{n}\delta_i\gamma_i\mathbf{1}_{\mu_i\leq t\leq Y_i,\nu_i\leq u\leq Z_i}.$$

- $\hat{F}$ is defined as the solution of

$$d\hat{F}(t,u)=\frac{[\int_t^\infty\int_u^\infty d\hat{F}(t',u')]d\hat{V}(t,u)}{\hat{H}(t,u)}.$$

Empirical version of the p.d.e.

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- For $V(t,u)=\mathbb{P}\left(\mu\leq t\leq Y,\nu\leq u\leq Z,\delta=\gamma=1|Y\geq\mu,Z\geq\nu\right)$,

$$\hat{V}(t,u)=\frac{1}{n}\sum_{i=1}^{n}\delta_i\gamma_i\mathbf{1}_{\mu_i\leq t\leq Y_i,\nu_i\leq u\leq Z_i}.$$

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EMPIRICAL VERSION OF THE P.D.E.

- We are back to a finite dimensional problem :

$$\hat{F} \leftrightarrow \mathbf{W} = (W_{1,n}, W_{2,n}, \ldots W_{n,n}, W_{\infty,n}),$$

where $W_{i,n}$ is the weight assigned to observation i where $W_{\infty,n}$ represents some residual mass.

PROPOSITION

There exists an unique solution $\mathbf{W}$ of the previous equation satisfying $\sum W_{i,n} = 1$. Moreover, all the weights are positive.

SOLVING THE EQUATION

- We need to solve

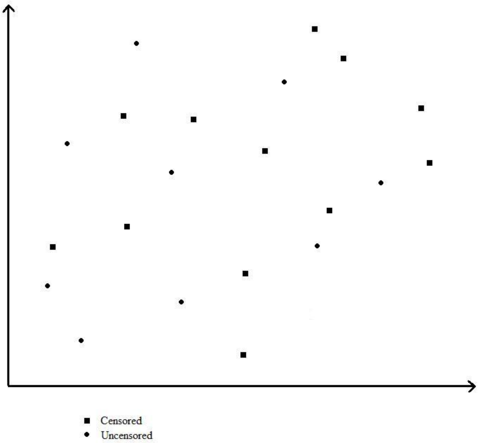
$$\mathbf{W} = A\mathbf{W}.$$

- Observe (after sorting the observations with respect to increasing values of Y)

$$A = \begin{pmatrix} 0 & * & * & 0 \\ \vdots & \ddots & * & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}.$$

- From that, one deduces unicity of the solution.

SOLVING THE EQUATION - GEOMETRICAL POINT OF VIEW



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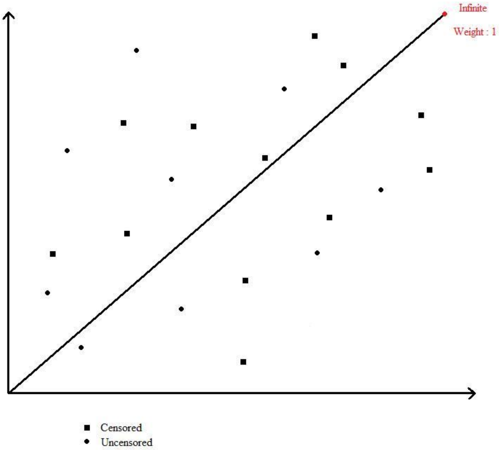

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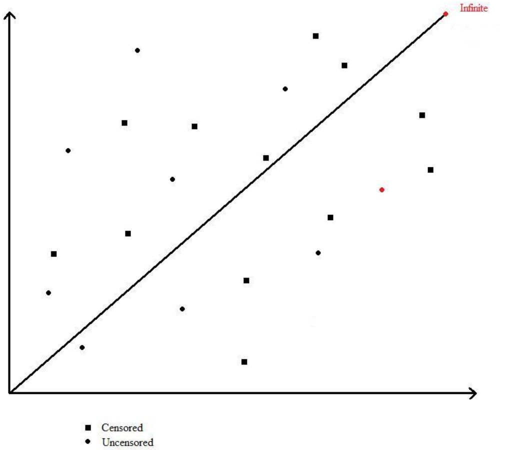
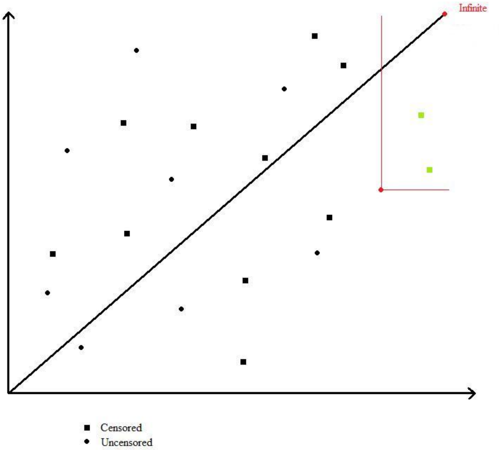


Solving the equation - geometrical point of view



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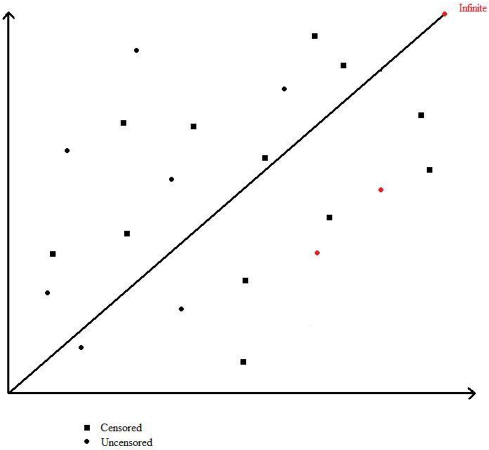

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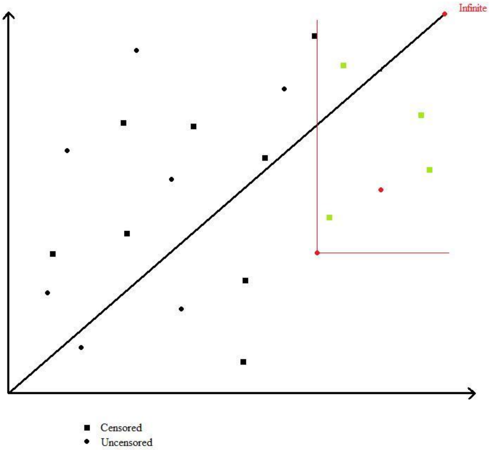

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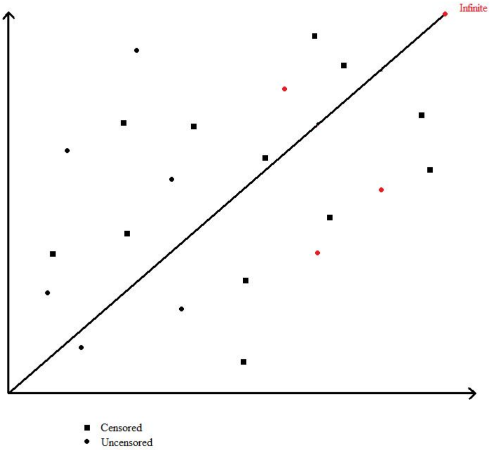

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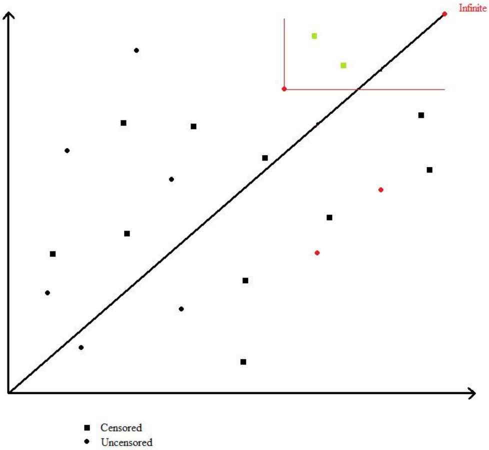

EQUIVALENT EXPRESSION OF THE P.D.E.

- Define

$$W^*_{i,n} = \frac{\delta_i \gamma_i}{n} \frac{\int_{Y_i}^{\infty} \int_{Z_i}^{\infty} dF(t,u)}{H(t,u)}.$$

- Second expression : for all function ϕ,

$$E\left[nW^*_{i,n}\phi(Y_i, Z_i) | Y_i \geq \mu_i, Z_i \geq \nu_i\right] = E[\phi(T_i, U_i)].$$

- Define $F^*(t,u) = \sum_{i=1}^{n} W^*_{i,n} \mathbf{1}_{Y_i \leq t, Z_i \leq u}$, then, for all ϕ,

$$\int \phi(t,u) dF^*(t,u) = \sum_{i=1}^{n} W^*_{i,n} \phi(Y_i, Z_i) \rightarrow_{p.s.} E[\phi(T,U)].$$

- One can show that $W_{i,n}$ and $W^*_{i,n}$ are close.

EQUIVALENT EXPRESSION OF THE P.D.E.

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EQUIVALENT EXPRESSION OF THE P.D.E.

- Define

$$W_{i,n}^* = \frac{\delta_i \gamma_i}{n} \frac{\int_{Y_i}^{\infty} \int_{Z_i}^{\infty} dF(t,u)}{H(t,u)}.$$

- Second expression : for all function ϕ,

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- One can show that $W_{i,n}$ and $W_{i,n}^*$ are close.

Conditional version

- Assume that, conditionally on (X^1, X^2), (T, U) is independent from (C, D), and (μ, ν) is independent from (T, U, C, D). Moreover assume that all variables are continuous.

P.D.E. (similar to Lopez, 2012)

$F(t, u|x) = \mathbb{P}(T \le t, U \le u | X = x)$ is solution of

$$dF(t, u|x) = \frac{[\int_{t^+}^{\infty} \int_{u^+}^{\infty} dF(t', u'|x)] dV(t, u|x)}{H(t, u|x)},$$

where

$$H(t, u|x) = \mathbb{P}(Y \ge t, Z \ge u | Y \ge \mu, Z \ge \nu, X = x),$$
$$V(t, u|x) = \mathbb{P}\left(Y \le t, Z \le u, \delta\gamma = 1 | Y \ge \mu, Z \ge \nu, X = x\right).$$

Estimation of H (estimation of V is similar)

- Let K denote a bivariate probability density with compact support.
- Define, for $x = (x^1, x^2)$, and $X_i = (X_i^1, X_i^2)$,

$$w_{i,n}(x) = \frac{K\left(\frac{X_i - x}{h}\right)}{\sum_{j=1}^{n} K\left(\frac{X_j - x}{h}\right)},$$

  where h is a bandwidth tending to 0 as n tends to infinity.
- Define

$$\hat{H}(t, u|x) = \sum_{i=1}^{n} w_{i,n}(x) \mathbf{1}_{Y_i \geq t, Z_i \geq u}.$$

- The bias of $\hat{H}$ is of order h^2, the variance of order $n^{-1/2}h^{-1}$.

Computation of an estimator $\hat{F}(\cdot|x)$

- The computation is similar to the case where no covariate is present.
- $\hat{F}(\cdot,\cdot|x) \leftrightarrow \mathbf{W}(x) = (W_{1,n}(x), ..., W_{n,n}(x), W_{\infty}(x))$ is the solution of an equation of the type

$$\mathbf{W} = A(x)\mathbf{W},$$

  where the matrix A now depends on x.
- The convergence rate is modified ($h^2 + n^{-1/2}h^{-p/2}$ rather than $n^{-1/2}$, where p is the dimension of X).

Estimation of $\theta(x)$

- Recall that

$$\hat{F}(t,u|x)=\sum_{i=1}^{n}W_{i,n}(x)\mathbf{1}_{Y_i\leq t, X_i\leq x}.$$

- Estimation of $\theta(x)$:

$$\hat{\theta}(x)=\arg\max_{\theta\in\Theta}\sum_{i=1}^{n}W_{i,n}(x)\log(\mathfrak{c}_\theta(\hat{F}_T(Y_i)),\hat{F}_U(U_i)).$$

ASYMPTOTIC THEORY

- Assumptions :
  - the copula function stays in the same copula family when x changes
  - $\sup_{t,u,x} |\partial_x H(t,u|x)| + |\partial_x^2 H(t,u|x)| < \infty$
  - conditions on the tail of the distributions of (T, U)
  - standard conditions on the kernel

ASYMPTOTIC THEORY

Under appropriate assumptions,

$$\sup_{x\in\mathcal{X}} \|\hat{\theta}(x) - \theta(x)\| = O_{a.s.}\left(h^2 + \frac{1}{n^{1/2}h^{p/2}}\right),$$

where p is the dimension of x, and where $\mathcal{X}$ is a compact on which the density of X is bounded away from zero.

CONCLUSION, PERSPECTIVES

- Practical choice of the bandwidth.
- Dimension of X : dimension reduction assumption to compensate the poor convergence rate when p is high.
- Goodness-of-fit, testing the assumption on the copula family...