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Cass Business School
Faculty of Finance
106 Bunhill Row
London EC1Y 8TZ

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Vitali Alexeev, Giovanni Urga and Wenying Yao

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Vitali Alexeev[a,*], Giovanni Urga[b,c], Wenying Yao[d]

[a] *Finance Discipline Group, UTS Business School, University of Technology Sydney, Sydney NSW 2007, Australia*
[b] *Centre for Econometric Analysis, Faculty of Finance, Cass Business School, 106 Bunhill Row, EC1Y 8TZ, London, UK*
[c] *Department of Management, Economics and Quantitative Methods, University of Bergamo, Bergamo, 24127, Italy*
[d] *Department of Economics, Faculty of Business and Law, Deakin University, Burwood VIC 3125, Australia*

Abstract

We evaluate the impact of extreme market shifts on equity portfolios and study the implications of the difference in negative and positive sensitivities to market jumps for portfolio risk management. We investigate to what extent the portfolio exposure to the downside and upside jumps can be mitigated by contrasting the results for individual stocks with the results for portfolios as the number of holdings increases. Varying the jump identification threshold, we show that the asymmetry is more prominent for more extreme events and that the number of holdings required to stabilise portfolios' sensitivities to negative jumps is higher than under positive jumps. Ignoring this asymmetry results in under-diversification of portfolios and increased exposure to extreme negative market shifts.

Keywords: Systematic risk, extreme events, jumps, high frequency, downside beta

JEL Classification: C58, G11, C61

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*Corresponding author

Email addresses: `Vitali.Alexeev@uts.edu.au; Phone: +61 2 9514 7781` (Vitali Alexeev), `G.Urga@city.ac.uk` (Giovanni Urga), `Wenying.Yao@deakin.edu.au` (Wenying Yao)

I. Introduction

This paper contributes to the literature on portfolio risk management by evaluating the impact of extreme market shifts on equity portfolios as the number of holdings increases. An important feature explored in our study is the asymmetry in portfolios' behaviour during extreme market downturns versus extreme market upsurges. Sudden large market shifts are rare events, but have substantially higher impacts than the diffusive price movements (Patton and Verardo, 2012). Mitigating the effects of these extreme shifts is difficult, unless the portfolios are large enough to diversify away such risks. Recent studies by Bollerslev et al. (2008), Jacod and Todorov (2009) and Mancini and Gobbi (2012), based on high-frequency data, have all argued for the presence of common jump arrivals across different assets, thus possibly inducing stronger dependencies in the "extreme". However, Bajgrowicz et al. (2016) argue that no co-jump affects all stocks simultaneously, suggesting jump risk is diversifiable.

In this paper, we consider the importance of the asymmetry effects of extreme events on asset prices and the presence of common jumps across assets. We evaluate the impact of extreme negative and positive market shifts on equity portfolios as the number of holdings in these portfolios increase, and to what extent the downside and upside jump risk can be diversified away. In particular, we address the following questions: How many stocks should investors hold on average, in order to reduce the level of sensitivity to market jumps to a certain level? How does the recommended portfolio size changes with the magnitude of extreme events? Are there any differences in recommended portfolio sizes for investors seeking to diversify only against negative extreme events? We expect that our findings will depend on the magnitude as well as the sign of extreme events, in that more stocks will be required to stabilise portfolio sensitivities to extreme negative market jumps than to extreme positive ones.[1] In addition, the more we focus on the tails of distributions, the larger the difference we anticipate in their behaviours. When defining extreme shifts, we explore several thresholds and consider the asymmetry effects for different levels of extreme market movements. We expect that the difference in recommended portfolio sizes needed to diversify negative jumps versus positive jumps will be larger in the presence of more extreme jumps. Investors seeking stability in their portfolio sensitivities to such events may find that ignoring asymmetry results in under-diversification of portfolios and increases exposure to extreme negative market shifts.

Our analysis relies on the developments from two strands of literature: modelling of extreme events and jump identification. In the past two decades, modelling extreme events has become mainstream in risk management practice.[2] Regulators are attentive to market conditions during a crisis because they are concerned with the protection of the financial system against catastrophic events. The model in Bates (2008) formalizes the intuition that investors can treat extreme events differently than they treat more common and frequent ones. The increased availability of high-frequency data amplified the interest in the analysis of tail events.[3] Modelling rare and extreme events often explains the high observed equity risk premia by taking into account the premia for rare events, provided that these events are sufficiently severe (Rietz, 1988; Barro, 2006; Bates, 2008). However, Julliard and Ghosh (2012),

[1]This is in line with the previous literature that shows that correlations among securities tend to increase during turbulent market conditions, which in then makes it harder to diversify portfolios. The increase in correlation coefficients during periods of distress is well recognized in the literature (see, for example, Forbes and Rigobon, 2002; Fry et al., 2010; Rigobon, 2016).

[2]Francois Longin, program chair of ESSEC Business School Conference on Extreme Events in Finance held on December 15-17, 2014, writes: "When I started to study extreme events in finance after the stock market crash of October 1987, academic studies considered extreme events as outliers and such data were usually discarded from empirical work. A few decades later I am more than happy to organize an international conference on extreme events in finance." (http://extreme-events-finance.net/scientific-committee/). Extreme value theory and tail behaviour, as well as its applications in finance and insurance, are extensively discussed in Embrechts et al. (1997).

[3]For example, Bollerslev and Todorov (2011) develop a new framework for estimating jump tails, and, using 1-minute data, find strong evidence for richer and more complex dynamic dependencies in the jump tails than previously entertained in the literature.

using U.S. and international data on the history of economic disasters, argue that rare events cannot explain the equity premium puzzle, unless one assumes that disasters occur every 6-10 years.[4]

The jumps in the high-frequency literature may be rare events when considered spatially, but often appear too frequently in calendar time to be considered extreme or disastrous. Consider the study of Liu et al. (2005) on the asset pricing implication of imprecise knowledge about rare events. It concludes that, in fact, the equilibrium equity premium has three components: the diffusive- and jump-risk premia, both driven by risk aversion; and the "rare-event premium", driven exclusively by uncertainty aversion. The need for a refined classification of jumps according to their magnitude and its association with extreme events is apparent. Mounting empirical evidence in the high-frequency literature suggests that jumps occur, on average, on 4%-13% of days per year (e.g., Andersen et al., 2007; Patton and Verardo, 2012; Alexeev et al., 2015, among others). It can be argued, however, that events with such frequency of occurrence can hardly be classified as "extreme".[5] The jump identification literature offers a number of methods that can sieve out less extreme events by varying the threshold used in its detection (e.g., Mancini, 2001; Mancini and Renò, 2011; Davies and Tauchen, 2015). Thus, in our empirical application, we focus on the results obtained by setting the threshold high enough to allow only the most severe occasions. In line with Christensen et al. (2014), we detect far fewer jumps than usually found in the literature, at a rate of 2.27% and 2.36% for the negative and positive jumps respectively. More importantly, it is only when we focus on more severe and rare events, we begin to observe the asymmetry in portfolio sensitivities to market shifts.

[4] Another plausible explanation of the equity premium puzzle could be that investors tend to overweight the probability of rare, extreme events. This probability weighting can independently generate a large equity premium (Giorgi and Legg, 2012; Barberis, 2013).

[5] Christensen et al. (2014) using ultra-high frequency data argue that jumps in financial asset prices are often erroneously identified and are not nearly as common as generally thought.

Investors typically perceive downside and upside extreme events differently.[6] It is believed that the fear of large negative shocks is a component that drives asset prices, because investors expect compensation for the risk that such a rare event occurs. It is not only the occurrence of rare events but also the very fear of them that influences investors' behaviour and market prices. Following Fama and MacBeth (1973), we explore pricing implications for jump risk in periods of extreme downside losses as opposed to extreme upside gains, and check if investors demand additional compensation for holding stocks with high sensitivities to these movements. Bollerslev and Todorov (2011) also show that, although the behaviour of the two tails is clearly related, the contributions to the overall risk premium are far from symmetric. Further evidence on asymmetric effects of jump risk measures can be found in Guo et al. (2015); Audrino and Hu (2016). The findings provided in the existing literature highlight the importance of considering the asymmetry effects of extreme events in portfolio risk management.

The remainder of the paper is organised as follows. Section II sets up the model framework. The data used in our empirical investigation are detailed in Section III. We discuss our pricing implication for cross section of individual assets in Section IV, and investigate the behaviours of systematic negative and positive jump risk factors in portfolios of assets in Section V. Section VI concludes.

[6] It has long been recognised that investors care differently about downside losses versus upside gains. Ang et al. (2006) show that the cross section of stock returns reflects a downside risk premium of 6% per annum. The reward for bearing downside risk is not simply compensation for regular market beta, nor it is explained by common stock market return predictors. Bollerslev and Todorov (2011); Audrino and Hu (2016) find similar results but for downside and upside extreme events. The economic intuition underlying downside risk is simple: Agents require a premium not only for securities the more their returns co-vary with the market return, but also, and even more so, when securities co-vary more with market returns conditional on low market returns.

II. Model Setup

We start with a panel of N assets over a fixed time interval $[0, T]$. Following the convention in the high-frequency financial econometrics literature, we assume the log-price $p_{i,t}$ of the i^{th} asset follows a semi-martingale plus jumps process in continuous time. In turn, the log-return of any asset, $r_{i,t}$, has the following representation:

$$r_{i,t} \equiv \mathrm{d}p_{i,t} = b_{i,t}\,\mathrm{d}t + \sigma_{i,t}\,\mathrm{d}W_{i,t} + \kappa_{i,t}\,\mathrm{d}\mu_{i,t}, \qquad t \in [0, T], \quad i = 1, 2, \ldots, N, \tag{1}$$

where $b_{i,t}$ is a locally bounded drift term, $\sigma_{i,t}$ denotes the non-zero spot volatility, $W_{i,t}$ is a standard Brownian motion for asset i. The $\kappa_{i,t}\,\mathrm{d}\mu_{i,t}$ term represents the jump component. The jump measure $\mathrm{d}\mu_{i,t}$ is such that $\mathrm{d}\mu_{i,t} = 1$ if there is a jump in $r_{i,t}$ at time t, and $\mathrm{d}\mu_{i,t} = 0$ otherwise. The size of jump at time t is represented by $\kappa_{i,t}$. In fact, $\kappa_{i,t}$ can be defined as $\kappa_{i,t} = p_{i,t} - p_{i,t-}$ in general, where $p_{i,t-} = \lim_{s\uparrow t} p_{i,s}$ for s increasing in value and approaching t. It follows immediately that $\kappa_{i,t} = 0$ for $t \in \{t : \mathrm{d}\mu_{i,t} = 0\}$ under this definition, meaning that at times when jump does not occur, the size of jump is zero.

Return on the market portfolio $r_{0,t}$ can be decomposed in a way similar to (1):

$$r_{0,t} = b_{0,t}\,\mathrm{d}t + \sigma_{0,t}\,\mathrm{d}W_{0,t} + \kappa_{0,t}\,\mathrm{d}\mu_{0,t}. \tag{2}$$

We assume that the jumps in processes (1) and (2) have only finite activity. Processes with finite activity jumps, as opposed to the infinite activity, have only a finite number of jumps in $[0, T]$. Since we only focus on "big" jumps with sizes bounded away from zero, this assumption is not overly restrictive. For a detailed discussion on finite versus infinite activity in jumps see Aït-Sahalia and Jacod (2012) among others.

A. Jump Regression

In studying jump dependence between equity portfolios and the broad market index, we use high-frequency observations focusing on segments of data on the fringes of return distributions. Thus, we only consider a few outlying observations that, at the time, are informative for the jump inference. In particular, we study the relationship between jumps of a process for a portfolio of assets and an aggregate market factor, and we analyse the co-movement of the jumps in these two processes. By increasing the threshold used to identify jumps, we focus on the most pronounced (extreme) events. For instance, we present in Figure 1 the frequency of positive and negative jump occurrence for each year from 2003 to 2011. For more liberal truncation thresholds, more than 100 jumps can be identified in most years.

We study the dependence between the jump components of individual assets (or portfolios) and that of the market return, utilising the methodology proposed by Li et al. (2015, 2016). Let $\mathcal{T}$ be the collection of jump times for the market portfolio $r_{0,t}$, i.e. $\mathcal{T} = \{\tau : \mathrm{d}\mu_{0,\tau} = 1, \tau \in [0,T]\}$. The set $\mathcal{T}$ has finite elements almost surely given the assumption of finite activity jumps. Given the predominance of factor models in asset pricing applications, we use a linear relationship to assess the sensitivity of jumps in portfolios to jumps in the market. In parallel with the classical one-factor market model, we set the linear factor model for jumps in the following form

$$\kappa_{i,\tau} = \beta_i^d \kappa_{0,\tau} + \epsilon_{i,\tau}, \qquad \tau \in \mathcal{T}, \quad i = 1, 2, \ldots, N, \tag{3}$$

where the superscript d stands for discontinuous (or jump) beta, and $\epsilon_{i,\tau}$ is the residual series. We only consider the jump times of the market portfolio $\mathcal{T}$ because β_i^d is not identified elsewhere. Therefore, β_i^d only exists if there is at least one jump in $r_{0,t}$ in $[0,T]$. Model (3) implicitly assumes that β_i^d is constant over the interval $[0, T]$.

The jump beta β_i^d in (3) has a similar interpretation to the market beta in the CAPM model. It allows us to assess the sensitivity of an asset (or a portfolio of assets) to extreme market fluctuations. Lower β_i^d would signify a resistance of an asset or portfolio to move as much as a market during extreme event (jump defensive assets), while higher β_i^d values represent high sensitivity of an asset exacerbating the effect of the market moves during the extreme event (jump sensitive assets).

B. Isolating Jumps from the Brownian Component

Under discrete-time sampling, neither the jump times $\mathcal{T}$ nor jump sizes $\kappa_{i,\tau}$ are directly observable from the data. Suppose the price and return series are observed every Δ interval, i.e. we obtain return series $r_{i,\Delta}$, $r_{i,2\Delta}$, ..., $r_{i,m\Delta}$, where $m = \lfloor T/\Delta \rfloor$, for $i = 0, 1, \ldots, N$.

Our first step in constructing the jump regression model (3) is to identify the discrete-time returns on the market portfolio $r_{0,j\Delta} = p_{0,j\Delta} - p_{0,(j-1)\Delta}$ that contain jumps, $j = 1, 2, \ldots, m$. We use the truncation threshold proposed by Mancini (2001) for this purpose (see also Mancini, 2009; Mancini and Renò, 2011). The threshold, denoted by $u_{0,m}$, is a function of the sampling interval Δ, and hence the sampling frequency m. The most widely used threshold is

$$u_{0,m} = \alpha \Delta^{\varpi}, \qquad \text{with} \quad \alpha > 0 \quad \text{and} \quad \varpi \in (0, 1/2). \tag{4}$$

Taking into account the time-varying spot volatility of the return series, the constant α is usually different for different assets, and could vary over time (see, for example, Jacod, 2008). One example is to set α to be dependent on the estimated continuous volatility of the given asset.

As $\Delta \to 0$ and $m \to \infty$, the condition $|r_{0,j\Delta}| > u_{0,m} = \alpha \Delta^{\varpi}$ eliminates the continuous diffusive returns on the market portfolio asymptotically, and hence only keeps returns that contain jumps. We collect the indices of these discrete-time intervals where the market return exceeds the truncation level, and denote this set as

$$\mathcal{J}_m = \{j : 1 \le j \le m, |r_{0,j\Delta}| > u_{0,m}\}. \tag{5}$$

We denote the collection of interval returns for $\mathcal{J}_m$ by $\{r_{0,j\Delta}\}_{j\in\mathcal{J}_m}$. Correspondingly, in the continuous-time data generating process for market return (2), for each jump time $\tau \in \mathcal{T}$, we also find the index j such that the jump $\kappa_{0,\tau}$ occurs in the interval $((j-1)\Delta, j\Delta]$,

$$\mathcal{J} = \{j : 1 \le j \le m, \tau \in ((j-1)\Delta, j\Delta] \text{ for } \tau \in \mathcal{T}\}. \tag{6}$$

An important result from Li et al. (2016) is that, under some general regularity conditions, the probability that the set $\mathcal{J}_m$ coincides with $\mathcal{J}$ converges to one as $\Delta \to 0$. This is formally stated in Proposition 1 in Li et al. (2016) that we recall for convenience.

Proposition 1. *Under certain regularity assumptions, as $\Delta \to 0$, we have*

(a) $\mathbb{P}(\mathcal{J}_m = \mathcal{J}) \to 1$;

(b) $((j-1)\Delta, r_{0,j\Delta})_{j\in\mathcal{J}_m} \xrightarrow{\mathbb{P}} (\tau, \kappa_{0,\tau})_{\tau\in\mathcal{T}}$.

Note that Proposition 1(a) implies that the number of elements in the set $\mathcal{J}_m$ consistently estimates the number of jumps in the process $r_{0,t}$; while 1(b) states that as $\Delta \to 0$, the starting point of the interval $(j-1)\Delta$, consistently estimates the jump time τ, and the interval return $r_{0,j\Delta}$ consistently estimates the jump size $\kappa_{0,\tau}$. The asymptotic results in Proposition 1 provides a powerful tool of linking the discrete-time return observations to the unobservable jumps and jump times in the continuous time. We use the discrete-time return observations to estimate the jump regression (3) and obtain consistent estimator of the jump beta β_i^d.[7]

C. Estimating Jump Beta

In accordance with model (3), the discrete-time jump regression model has the form

$$r_{i,j\Delta} = \beta_i^d \, r_{0,j\Delta} + \epsilon_{i,\tau}, \qquad j \in \mathcal{J}_m, \quad i = 1, 2, \ldots, N. \tag{7}$$

Hence a naïve consistent estimator of β_i^d is the analog of the OLS estimator,

$$\tilde{\beta}_i^d = \frac{\sum_{j \in \mathcal{J}_m} r_{i,j\Delta} \cdot r_{0,j\Delta}}{\sum_{j \in \mathcal{J}_m} (r_{0,j\Delta})^2}, \qquad i = 1, 2, \ldots, N. \tag{8}$$

Li et al. (2016) propose an efficient estimator for β_i^d in (7). It is an optimal weighted estimator in the sense that it minimizes the conditional asymptotic variance among all weighting schemes. The optimal weight w_j^* is a function of the preliminary consistent estimator $\tilde{\beta}_i^d$, and the approximated pre-jump and post-jump spot covariance matrices $\hat{C}_{j-}$ and $\hat{C}_{j+}$:

$$w_j^* = \frac{2}{(-\tilde{\beta}_i^d, 1)(\hat{C}_{j-} + \hat{C}_{j+})(-\tilde{\beta}_i^d, 1)'}, \qquad \text{for} \quad j \in \bar{\mathcal{J}}_m, \tag{9}$$

where $\bar{\mathcal{J}}_m = \{j \in \mathcal{J}_m : k_m + 1 \leq j \leq m - k_m\}$, and k_m is an integer such that $k_m \to \infty$ and $k_m \cdot \Delta \to 0$ as $\Delta \to 0$. The spot covariance matrices are estimated in the following manner. We construct the truncation threshold for the vector $\boldsymbol{r}_{j\Delta} \equiv (r_{0,j\Delta}, r_{i,j\Delta})$ jointly in the same way

[7] In our analysis we focus on the 5-min frequency but we evaluate our results for a range of frequencies. To minimise the effect of microstructure noise, we also apply pre-averaging method proposed in Mykland and Zhang (2016). See Alexeev et al. (2016) for details.

as in (4), and denote it as $\boldsymbol{u}_m \equiv (u_{0,m}, u_{i,m})$. For any $j \in \bar{\mathcal{J}}_m$, we have

$$\hat{C}_{j-} = \frac{1}{k_m \cdot \Delta} \sum_{l=1}^{k_m} \boldsymbol{r}'_{(j+l-k_m-1)\Delta} \cdot \boldsymbol{r}_{(j+l-k_m-1)\Delta} \cdot \mathbb{1}_{\{|\boldsymbol{r}_{(j+l-k_m-1)\Delta}| \leq \boldsymbol{u}_m\}}, \tag{10}$$

$$\hat{C}_{j+} = \frac{1}{k_m \cdot \Delta} \sum_{l=1}^{k_m} \boldsymbol{r}'_{(j+l)\Delta} \cdot \boldsymbol{r}_{(j+l)\Delta} \cdot \mathbb{1}_{\{|\boldsymbol{r}_{(j+l)\Delta}| \leq \boldsymbol{u}_m\}}, \tag{11}$$

as the approximated pre-jump and post-jump spot covariance matrices, respectively.

Given any weighting function w_j, the class of weighted estimators $\hat{\beta}_j^d$ can be represented as

$$\hat{\beta}_i^d = \frac{\sum_{j \in \bar{\mathcal{J}}_m} w_j \cdot r_{i,j\Delta} \cdot r_{0,j\Delta}}{\sum_{j \in \bar{\mathcal{J}}_m} w_j \cdot (r_{0,j\Delta})^2}, \qquad i = 1, 2, \ldots, N. \tag{12}$$

In Theorem 2, Li et al. (2016) show that the weighting function in (9) combined with the estimator (12) produces the most efficient estimate of the jump beta β_i^d. The standard errors and subsequently the confidence intervals of the estimators can be constructed using the bootstrap procedure outlined in Li et al. (2015). However, the authors restrict the negative and positive betas to be the same. In the next section, we consider the positive jump and negative jump separately and allow for the difference in impact of the two tail activities.

D. Asymmetric Jump Beta

In developing Modern Portfolio Theory in 1959, Henry Markowitz recognized that since only downside deviation is relevant to investors, using downside deviation to measure risk would be more appropriate than using standard deviation (Markowitz, 1971). Ang et al. (2006) explore the asset pricing implications of the downside risk without, however, separately considering extreme events or jumps. In this section, instead of pooling jumps at both positive and negative ends together, we examine the jump covariation between individual asset (or portfolio) and the market index at times of positive market jumps and negative market jumps separately. In this way we could accommodate separate risk premia for these two components.

Although we focus on the negative jump in the market portfolio and the negative jump beta associated with it, our modelling approach naturally gives rise to a similar definition of the positive jump beta. The naive estimators of the two asymmetric betas $\tilde{\beta}_i^{d+}$ and $\tilde{\beta}_i^{d-}$ are as follows:

$$\tilde{\beta}_i^{d-} = \frac{\sum_{j \in \mathcal{J}_m} r_{i,j\Delta} \cdot r_{0,j\Delta} \cdot \mathbb{1}_{\{r_{0,j\Delta}<0\}}}{\sum_{j \in \mathcal{J}_m} (r_{0,j\Delta})^2 \cdot \mathbb{1}_{\{r_{0,j\Delta}<0\}}}, \tag{13}$$

$$\tilde{\beta}_i^{d+} = \frac{\sum_{j \in \mathcal{J}_m} r_{i,j\Delta} \cdot r_{0,j\Delta} \cdot \mathbb{1}_{\{r_{0,j\Delta}>0\}}}{\sum_{j \in \mathcal{J}_m} (r_{0,j\Delta})^2 \cdot \mathbb{1}_{\{r_{0,j\Delta}>0\}}}, \tag{14}$$

for $i = 1, 2, \ldots, N$. When calculating the weighted estimators, the weighting function (9) would differ for the positive and negative jump betas:

$$w_j^- = \frac{2}{(-\tilde{\beta}_i^{d-}, 1)(\hat{C}_{j-} + \hat{C}_{j+})(-\tilde{\beta}_i^{d-}, 1)'}, \quad \text{for} \quad j \in \bar{\mathcal{J}}_m \quad \text{and} \quad r_{0,j\Delta} < 0, \tag{15}$$

$$w_j^+ = \frac{2}{(-\tilde{\beta}_i^{d+}, 1)(\hat{C}_{j-} + \hat{C}_{j+})(-\tilde{\beta}_i^{d+}, 1)'}, \quad \text{for} \quad j \in \bar{\mathcal{J}}_m \quad \text{and} \quad r_{0,j\Delta} > 0. \tag{16}$$

Here we assume that before and after the jumps, the spot covariance matrices are the same for positive and negative jumps. These lead to the formation of the weighted estimators of the asymmetric betas:

$$\hat{\beta}_j^{d-} = \frac{\sum_{j \in \bar{\mathcal{J}}_m} w_j^- \cdot r_{i,j\Delta} \cdot r_{0,j\Delta} \cdot \mathbb{1}_{\{r_{0,j\Delta}<0\}}}{\sum_{j \in \bar{\mathcal{J}}_m} w_j^- \cdot (r_{0,j\Delta})^2 \cdot \mathbb{1}_{\{r_{0,j\Delta}<0\}}}, \tag{17}$$

$$\hat{\beta}_j^{d+} = \frac{\sum_{j \in \bar{\mathcal{J}}_m} w_j^+ \cdot r_{i,j\Delta} \cdot r_{0,j\Delta} \cdot \mathbb{1}_{\{r_{0,j\Delta}>0\}}}{\sum_{j \in \bar{\mathcal{J}}_m} w_j^+ \cdot (r_{0,j\Delta})^2 \cdot \mathbb{1}_{\{r_{0,j\Delta}>0\}}}, \tag{18}$$

for $i = 1, 2, \ldots, N$.

In what follows, we will use real data to test the asymmetry in the jump betas and its implications for portfolio risk management.

III. Data

We investigate the behavior of the β_i^{d+} and β_i^{d-} estimates over the period from January 2, 2003 to December 30, 2011. This period includes the financial crisis associated with the bankruptcy of Lehman Brothers in September 2008 and the subsequent period of turmoil in US and international financial markets. The underlying data are equidistant 5-minute price observations on 501 stocks drawn from the constituent list of the S&P500 index during our sample period, obtained from Thomson Reuters Tick History through Securities Industry Research Centre of Asia-Pacific (SIRCA). This dataset was constructed by Dungey et al. (2012) and does not intend to cover all stocks listed on the S&P500 index, but includes those with sufficient coverage and data availability for high-frequency time series analysis of this type.

A. Data Processing and Preparation

The original dataset consists of over 900 stocks taken from the 0#.SPX mnemonic Reuters Identification Code (RIC) code provided by SIRCA for the S&P500 index historical constituents list. This included a number of stocks which trade OTC and on alternative exchanges, as well as some which altered currency of trade during the period; these stocks were excluded. We adjusted the dataset for changes in RIC codes during the sample period to account for mergers and acquisitions, stock splits, and trading halts. We also removed stocks with insufficient observations during the sample period. The data handling process is fully documented in the web-appendix to Dungey et al. (2012). The final data set contains 501 individual stocks, hence $N = 501$.

The intra-day price and return data start at 9:30 am and end at 4:00 pm, observations with time stamps outside this window and overnight returns are removed. Missing 5-minute price observations are filled forward resulting, in such cases, in zero inter-interval returns. In the case where the first observations of the day are missing, we use the first non-zero price observation on that day to fill backwards. "Bounce back" outliers as defined in Aït-Sahalia et al. (2011) are also removed. Thus, we have 77 intra-day observations for 2262 trading days.

The 5-minute sampling frequency is chosen as relatively conventional in the high-frequency literature, especially for univariate estimation (see, for example, Andersen et al., 2007; Lahaye et al., 2011), and for some sensitivity to alternatives, see Dungey et al. (2009). Optimal sampling frequency is an area of ongoing research, and despite the univariate work by Bandi and Russell (2006), this issue is highly contentious, especially when analyzing multiple series with varying degrees of liquidity. The 5-minute frequency is much finer than those employed in Todorov and Bollerslev (2010); Bollerslev et al. (2008), both of which use 22.5-minute data. Lower sampling frequencies are generally employed due to concerns over the Epps effect (Epps, 1979); however, as the quality of high-frequency data and market liquidity have improved in many ways, finer sampling does not threaten the robustness of our results.[8]

Estimates of β_i^{d+} and β_i^{d-} are computed on an annual basis. High-frequency data permits the use of 1-year non-overlapping windows to analyse the dynamics of our systematic risk estimates. Li et al. (2016) also finds in their empirical application using US equity market data that the positive or negative jump beta remains constant over a year most of the time. Given the 5-minute sampling frequency, not all stocks in the S&P 500 list have sufficient

[8]Investigating continuous and discontinuous betas in the cross section of expected returns, Bollerslev et al. (2016) favour a 75-minute sampling frequency to overcome the lack of liquidity across their sample of stocks, whilst we use a 5-minute sampling frequency which gives us better properties for the in-fill asymptotics for the statistical procedures we employ. We conducted Monte Carlo experiments demonstrating the small sample properties to support our contention that the statistics work well in our scenario. In addition, we investigated the stability of our continuous and discontinuous beta estimates using 15-min, 30-min, and 75-min sampling frequencies and find no substantive deviations from results based on the 5-min frequency employed in this paper. These results are available upon request.

data coverage to perform the analysis. Thus, each year, we consider a subset of stocks that have at least 75% of the entire 1-year window as non-zero 5-min return data. Depending on the year, this filter removes 30-40 stocks from the 501 available. We construct an equally weighted portfolio of all remaining stocks in each estimation window as the benchmark market portfolio. We side with Bollerslev et al. (2008) and use equally weighted portfolios rather than value weighted ones to avoid situations where the weight on one stock is disproportionately large relative to other portfolio constituents. This issue will become especially pertinent in the portfolio simulation section later in the paper.[9]

B. Parameter Values

In our empirical application we normalize each trading day to be one unit in time. Given the number of observations in each day $m = 77$, the sampling frequency is $\Delta = 1/77$.

Parameters in the truncation threshold (4) are chosen as follows. The constant $\varpi = 0.49$.[10] Taking into account the time-varying volatility $\sigma_{i,t}$, we set α to be a function of the estimated daily continuous volatility component for each individual asset. In finite sampling, the continuous volatility is consistently estimated by the bipower variation (Barndorff-Nielsen and Shephard, 2004, 2006):

$$BV_i = \left(\frac{\pi}{2}\right) \cdot \sum_{j=1}^{m-1} |r_{i,j\Delta}|\,|r_{i,(j+1)\Delta}| \xrightarrow{\mathbb{P}} \int_0^T \sigma_{i,t}^2\,\mathrm{d}t \quad \text{as} \quad \Delta \to 0, \qquad i = 0, 1, \ldots, N. \tag{19}$$

We set $\alpha_i = a\sqrt{BV_i}$, where $a = 3, 4, 5$. This leads to the threshold $u_{i,m} = a\sqrt{BV_i} \cdot (1/77)^{0.49}$. The choice of using a multiple of the estimated continuous volatility is relatively standard in the literature for disentangling jumps from the continuous price movements. It serves the

[9]See Fisher (1966) for the discussion of "Fisher's Arithmetic Index", an equally weighted average of the returns on all listed stocks.

[10]In most empirical literature, typical values for ϖ are between 0.45 and 0.49. See Jacod and Protter (2012, p. 248) for a discussion.

purpose of controlling for the possibly time-varying spot volatility automatically in jump detection. In our empirical application, BV_i is calculated daily resulting in a different threshold for each trading day.[11] We find both positive and negative jumps in each estimation window (year) in the market portfolio, and hence β_i^{d+} and β_i^{d-} can be estimated for each year.

IV. Empirical Analysis

In this section we analyse some statistical properties of betas estimated based on the overall market returns, as well as based on the upside and downside market returns separately. We then use the two-stage regression framework proposed by Fama and MacBeth (1973) to estimate the risk premia on the risk factors of interest.

A. Market Volatility and Jumps

Figure 1 plots the square root of the daily bipower variation of the equally weighted market index on the left axis, and the number of positive (blue) and negative (red) jumps for each year in the 2003-2011 sample period on the right axis. The subsample before mid-2007 is much less volatile than the second half of the sample which includes the global financial crisis (GFC). Market volatility has increased considerably since mid-2007, which is usually regarded as the initial emergence of the GFC, and peaked in late 2008 during the few months after the bankruptcy of Lehman Brothers, the bailout of AIG and the announcement of the TARP (Troubled Asset Relief Program). Other highly volatile periods include mid-2010 during the Greek debt crisis, and late 2011 during the European sovereign debt crisis with the deterioration of economic conditions in the Eurozone as a whole. The two peak values of market volatility after the GFC correspond to the May 6, 2010 flash crash and August 9, 2011. On August 5, 2011 Standard & Poor's downgraded America's credit rating for the first

[11] Alternatively, if higher frequency data is available, the threshold can be estimated on hourly basis to incorporate the intra-day volatility pattern which would make the identification of jumps more efficient.

time in history, followed with short-selling ban by Greece on August 8, 2011, and other 4 EU countries on August 11, 2011.

The choice of the threshold level in jump identification tests has important implications for our analysis. Using lower thresholds results in identifying too many jumps. For example, for threshold $\alpha_i = 3\sqrt{BV_i}$, the number of negative and positive jumps identified often exceeds 100 per year, which implies that the jumps occur almost every other day. Jumps are rare events and should not happen that often. The number of jumps identified using thresholds $\alpha_i = 4\sqrt{BV_i}$ and $\alpha_i = 5\sqrt{BV_i}$ appear more realistic.[12] Notably, Figure 1 shows that the volatile period during the GFC corresponds to fewer jumps in both directions. In particular, in 2008, using the most conservative threshold, $\alpha_i = 5\sqrt{BV_i}$, we do not observe more than 5 jumps (positive or negative). This result is expected as the market volatility is generally higher during crisis than during calmer period, the threshold of detecting jump observations will be elevated accordingly to prevent mistaking volatility bursts from real jumps. Black et al. (2012) also observe that the stock market has fewer jumps during crisis periods. Since only the jump observations are taken to estimate the non-weighted asymmetric beta in (13) and (14), low number of observations could certainly affect the quality of the estimates. Hence, it is necessary to use the weighted estimators, (17) and (18), in order to reduce the small sample size effect.

B. Estimation Results

In addition to the weighted estimators of the overall jump beta (12) and the asymmetric betas (17) and (18), we also calculate the continuous beta obtained using the OLS regression in the spirit of Andersen et al. (2006)'s realized beta

[12] When using $\alpha_i = 6\sqrt{BV_i}$ as our threshold, we failed to identified any jumps in a number of years.

Figure 1: Daily bipower variation of equally weighted market index (left axis) versus the identified number of positive (blue) and negative (red) jumps (right axis). Jumps where identified using the following thresholds: $\alpha_i = 3\sqrt{BV_i}$ (top panel), $\alpha_i = 4\sqrt{BV_i}$ (middle panel) and $\alpha_i = 5\sqrt{BV_i}$ (bottom panel).

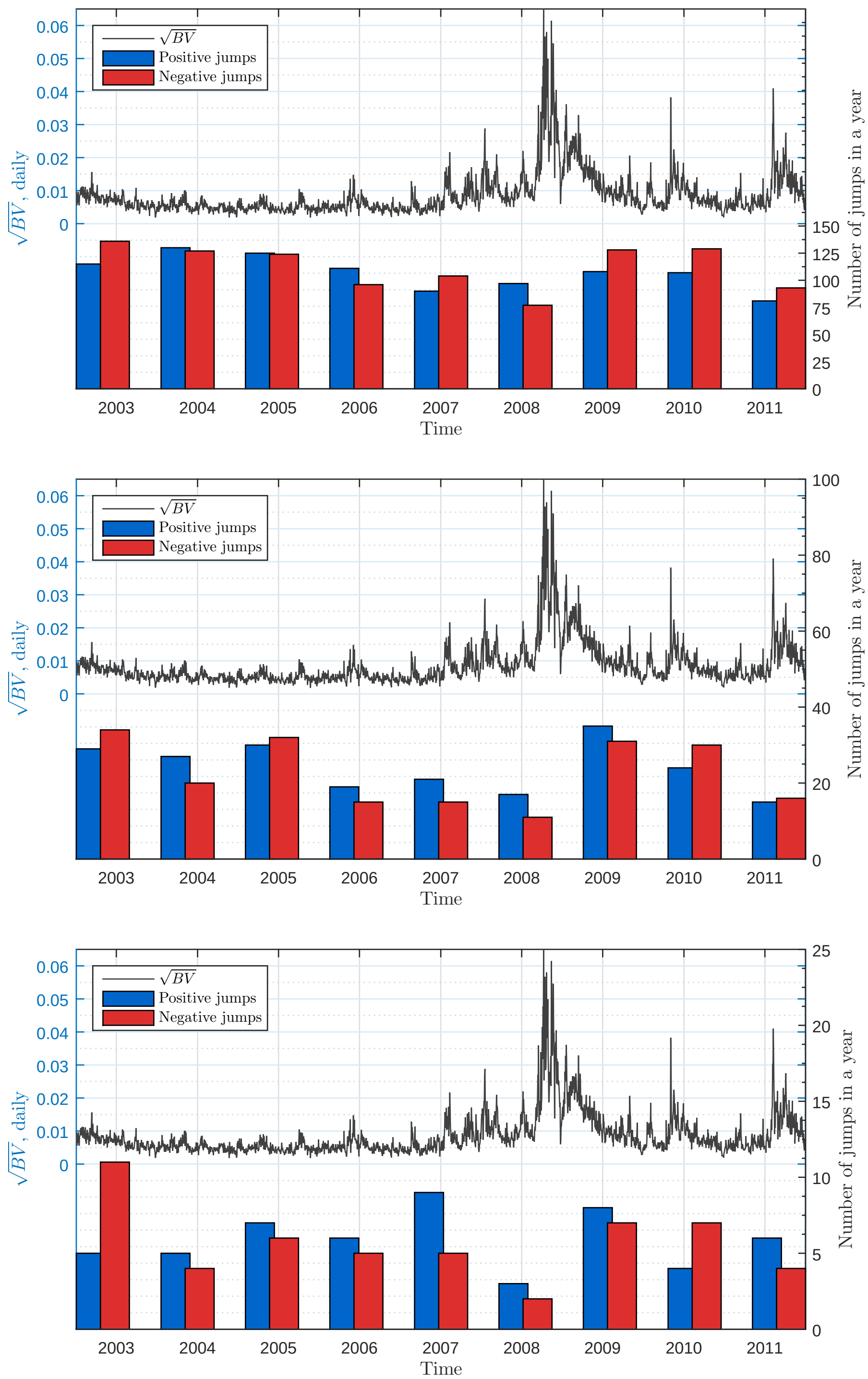


Table 1: Summary Statistics‡

	$\hat{\beta}^c$	$\hat{\beta}^d$	$\hat{\beta}^{d+}$	$\hat{\beta}^{d-}$	$\bar{r}$ (%)
Panel A: Descriptive Statistics					
25%	0.7576	0.5007	0.3155	0.2744	-4.8019
Mean	0.9966	0.9882	0.9957	0.9933	0.2817
75%	1.1828	1.3848	1.5476	1.6067	5.9589
Std.Dev.	0.3464	0.8910	2.1804	2.1674	12.9613
Skew.	0.8650	0.9346	-0.4632	0.7260	-168.9742
Panel B: Correlation Table					
	$\hat{\beta}_i^c$	$\hat{\beta}_i^d$	$\hat{\beta}_i^{d+}$	$\hat{\beta}_i^{d-}$	$\bar{r}_i$
$\hat{\beta}_i^c$	1.0000				
$\hat{\beta}_i^d$	0.3687	1.0000			
$\hat{\beta}_i^{d+}$	0.1738	0.4193	1.0000		
$\hat{\beta}_i^{d-}$	0.1459	0.3889	0.0389	1.0000	
$\bar{r}_i$	-0.0114	-0.0050	0.0104	0.0082	1.0000

‡ This table shows the descriptive statistics (Panel A) and time-series means of pairwise correlations (Panel B) for individual firm continuous betas, β^c, jump beta, β^d, positive and negative jump betas, β^{d+} and β^{d-}, respectively, as well as the monthly excess returns, $\bar{r}$. All beta estimates are obtained using 5-minute data from previous 12-month using $\alpha_i = 5\sqrt{BV_i}$ thresholds, whereas $\bar{r}$ correspond to one-month-ahead excess returns. The estimates for all individual stocks and all calendar months are pooled together in calculating the statistics in this table.

$$\hat{\beta}_i^c = \frac{\sum_{j=1}^m r_{i,j\Delta} \cdot r_{0,j\Delta} \cdot \mathbb{1}_{\{|r_{0,j\Delta}| \leq u_{0,m}\}}}{\sum_{j=1}^m (r_{0,j\Delta})^2 \cdot \mathbb{1}_{\{|r_{0,j\Delta}| \leq u_{0,m}\}}}, \qquad i = 1, 2, \ldots, N, \tag{20}$$

where, in contrast, only the 5-minute return observations below the threshold are used to construct $\hat{\beta}_i^c$. The continuous beta would be able to incorporate the impact of co-movements in the continuous component of individual asset (or portfolio) and the market index.

In Table 1 we present the descriptive statistics and the correlations between the beta estimates as well as the average excess monthly returns of each asset. The means of all four estimated beta measures are very close to one. The distributions of the estimates are positively skewed with exception of β^{d+}. The average continuous beta is statistically different from zero, whereas the three jump beta estimates are statistically insignificant, possibly due to the high heterogeneity among different assets. Monthly excess stock returns are very dispersed and negatively skewed.

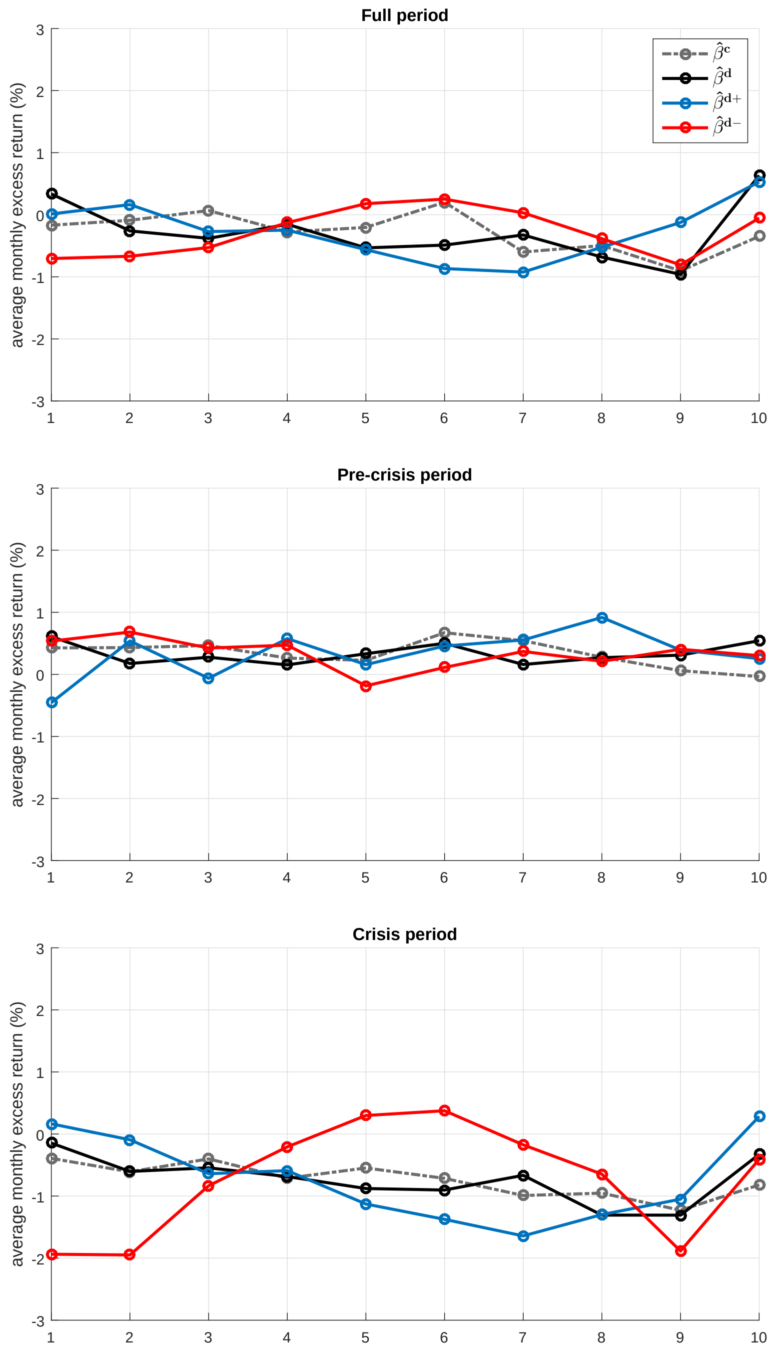
We sort stocks in terms of the estimated jump betas, and divide them into percentile portfolios. This is analogues to Ang et al. (2006) and Bollerslev et al. (2016) where the authors tabulate the returns among quintile portfolios sorted according to different betas. Instead, we choose to divide the stocks into decile portfolios, and present our results in Figure 2 to highlight the difference between portfolio sorts based on β^{d+} and β^{d-} in different time periods. We estimate the different betas based on the past 12 months returns while recording the returns for the following month. This predictive single-sorted portfolio method is of much practical value.[13]

Although our result does not agree with the finding by Ang et al. (2006) that higher downside beta is associated with higher return, it should be noted that Ang et al. (2006) do not separate jump and diffusive components, and that their result could be attributed either to diffusive component of risk, jump component, or both. We separate positive and negative jumps and investigate whether positive and negative jump risk are priced risk factors.

The general finding in Figure 2 is that there is strong inverse relationship in predictive return pattern among stocks sorted by β^{d+} and β^{d-}. This is particularly pronounced during the crisis period (bottom panel). It is interesting to note that this relationship is non-linear: the highest expected return is observed among stocks from the 4th to the 7th decile, which is around the mean of β^{d-}. From Table 1, the average value of β^{d-} is approximately equals to 1, indicating the subset of stocks that react proportionally to extreme market downturns. In contrast, stocks with defensive and highly sensitive characteristics towards extreme negative market movements are expected to earn returns that are magnitudes lower than stocks with values of β^{d-} around 1. It appears that during this highly volatile period, investors "follow the herd" and value the stocks that move in unison with the broad market index the most.

[13]We also estimate the contemporaneous portfolio sorts where return and betas are estimated over the same holding period. The results, available upon request from the authors, avail no particular difference or change in pattern when compared with the predictive single-sorted portfolios.

Figure 2: The figure presents the average monthly excess returns by beta deciles. The sample consists of 501 individual stocks included in the S&P index over the full sample period: Jan 2003-Dec 2011 (top panel), pre-crisis period: Jan 2003-June 2007 (middle panel), and crisis period: July 2007-Dec 2011 (bottom panel). Jumps are identified using the $\alpha_i = 5\sqrt{BV_i}$ thresholds. At the end of each month, stocks are sorted into percentiles according to betas computed from previous 12-month returns. The average monthly excess returns (on the y-axis) are the average one-month-ahead excess returns of each decile portfolio (on the x-axis), where decile 1 and decile 10 portfolios are equally-weighted portfolios with the lowest and highest betas respectively.



Protection from extreme positive market swings appears to carry some value to investors (indicated by the 1st decile portfolio on the blue line that represent expected returns for β^{d+}-sorted portfolios).

This is consistent with the result in Cremers et al. (2015) for symmetric jump risk, and applies to our case for the negative jump betas. In fact, our results show a negative market price of "negative jump" risk. This implies that stocks with high sensitivities to negative market jumps should earn low returns. Cremers et al. (2015) argue that this makes sense economically, as such stocks provide useful hedging opportunities for risk-averse investors, who dislike high systematic negative jump risk. On the contrary, with a positive risk premia for "positive jump" risk, stocks earn higher return.

Results in this section point to viability of negative and positive jump risks as market-wide factors. In the next section we estimate the market risk premia associated with these factors.

C. Risk Premia of the Asymmetric Jump Betas

We use the two-stage regression by Fama and MacBeth (1973) to estimate the risk premia for both the symmetric and asymmetric jump betas. The beta estimates obtained above are taken as the explanatory variables in the cross-sectional regression

$$\bar{r}_{i,s} = \gamma_s \hat{\boldsymbol{\beta}}_{i,s-1} + \epsilon_{i,s}, \qquad i = 1, 2, \ldots, N, \tag{21}$$

where $\bar{r}_{i,s}$ denotes the average monthly excess return for stock i from month $s-1$ to s and $\hat{\boldsymbol{\beta}}_{i,s-1}$ represents a vector of betas measured at the end of month $s-1$ using 5-minute data from the preceding 12 months.[14] The vector of coefficients γ_s represent the estimated risk

[14]We choose a 12-month window to be able to identify at least one jump in the market index to ensure the existence of β^d. Although the length of the window is relatively flexible under thresholds with $\alpha = 3$ or 4, it becomes rigid under a more conservative threshold of $\alpha = 5$.

Table 2: Fama-MacBeth regressions and risk premia (γ) associated with the estimated risk factors (betas)‡

Model	1		2		
	$\hat{\beta}^c$	$\hat{\beta}^d$	$\hat{\beta}^c$	$\hat{\beta}^{d+}$	$\hat{\beta}^{d-}$
2004	-1.1504	0.2774†	-1.2219	0.2224**	0.0568
2005	0.2974	-0.3074	0.1874	0.1178	-0.4423†
2006	-0.7129	0.1485	-0.6774	0.0647	0.0456
2007	0.6267	-0.2373	0.4671	-0.1790	-0.0784
2008	-4.6394	-0.76291†	-4.5674	-1.1281**	0.2019
2009	2.0067	0.6535*	0.0382	0.3873†	0.2778*
2010	1.1788	-0.0969	1.1341	0.1664	-0.2908
2011	-2.8637	0.0311	-2.9609	0.0651	-0.0568
Pre-crisis period	-0.2205	0.0353	-0.2477	0.1572**	-0.1301
Crisis period	-0.9967	-0.0928	-0.0105	-0.1852	-0.0376
Full period	-0.6571	-0.0368	-0.7001	-0.0354	-0.0358

‡ This table reports the cross-sectional pricing of the continuous and jump risks (both symmetric and asymmetric). The sample period is from January 2003 to December 2011. We run Fama-MacBeth cross-sectional regressions for one-month-ahead excess returns on betas computed from previous 12-month returns. Significance levels: $\dagger$: 10%, $*$: 5%, $**$: 1% (calculated using heteroskedasticity and auto-correlation consistent standard errors).

premium awarded to each risk factor in $\hat{\boldsymbol{\beta}}_{i,s-1}$.

Table 2 displays the estimated risk premia γ for each year from 2004 to 2011, as well as pre-crisis, crisis, and the full sample periods. There are two different model specifications under consideration: model 1 decomposes the overall systematic risk into the continuous and discontinuous factors without taking into account asymmetry, while model 2 separate the positive and negative discontinuous betas. Contrasting these two models helps us detect the existence and degree of asymmetry in the jump risk. Significance levels in Table 2 are calculated using heteroskedasticity and auto-correlation consistent standard errors.

Table 2 shows that under both model specifications the continuous beta consistently receives a negative but insignificant risk premium in the pre-crisis, crisis, and overall periods. In the pre-crisis period, the symmetric jump beta, β^d carries a positive but insignificant risk premium. On the other hand, the risk premium for positive jump beta is significantly positive, while the negative jump beta has insignificant but negative risk premium. The γ over the entire sample period for all beta estimates are statistically insignificant. In model (2), the positive jump betas have positive and significant risk premia in the pre-crisis period, while the symmetric jump beta during the same period is insignificant. These mixed results may provide additional support to the non-linear relationship between the expected returns and the systematic jump risk factors. Furthermore, it strengthens the point in Bajgrowicz et al. (2016) that argue that in the absence of co-jumps, jump risk is diversifiable and thus with no compensation for investors.

In this section, we show that individual stock betas can vary greatly. Experienced investors will always consider allocating wealth in a number of assets rather than one single security. As the number of holdings increases, the range of portfolios betas will become more limited compared to betas of individual securities and eventually converges to unity for an equi-weighted market benchmark. Given that investing in all securities listed on a broad market index may not be informationally or cost effective, finding the optimal number of holdings in a portfolio to mitigate most jump risk (positive or negative) is of significant importance. We discuss this in the next section.

V. Portfolio Simulation

The concept of portfolio diversification is straightforward: the level of portfolio risk falls as the number of holdings in a portfolio increases.[15] In the previous section we explored the behaviour and pricing implications of continuous and jump risks for individual stocks. In this section we explore how fast these systematic risks dissipate in portfolios.

In the last decade, the availability of high-frequency data allowed new insights into portfolio diversification. For example, Silvapulle and Granger (2001) investigate asset correlations at the tails of return distributions and discusses the implications for portfolios of stocks. Bollerslev et al. (2013) examine the relationship between jumps in individual stocks

[15] Artzner et al. (1997, 1999) shows that this holds for any coherent risk measure.

and jumps in a market index. The authors find that jumps occur more than three times as often at the individual stock level compared to jump occurrence in an aggregate equi-weighted index constructed from the same stocks. This may point to the fact that jumps are diversifiable. In fact, using daily data, Pukthuanthong and Roll (2014) consider implications of jumps for international diversification. Studies directly investigating optimal portfolio size using high-frequency data have only recently started to emerge (e.g., Alexeev and Dungey, 2015; Alexeev et al., 2016).

Using large-scale portfolio simulation, we contrast the variability in portfolio jump betas as the number of holdings in these portfolios changes. We analyse the spread of estimated betas in equi-weighted randomly constructed portfolios of different sizes, focusing on the difference in convergence between negative and positive jump betas as the number of holdings in portfolios increases. For investors, the knowledge that individual stocks respond differently to the positive and negative extreme events is likely to be valuable in developing portfolio risk management strategies. However, investors with several stocks may be rightfully concerned with the overall exposure of their portfolios to systematic jump risk. Moreover, investors exhibit different attitudes towards extreme gains and extreme losses. We assert that if an asset tends to move downwards in a declining market more than it moves upward in a rising market, such asset is unattractive to hold, especially during market downturns when wealth of investors is low.

Using a 12-month estimation window, for each year from 2003 to 2011 we construct 5,000 random equally-weighted portfolios with the number of holdings in each portfolio ranging from 1 to 200. For each of these portfolios we estimate several systematic discontinuous risks — symmetric jump beta, β^{d}, as well as β^{d+} and β^{d-}. We assess the stability of the systematic portfolio risks by analyzing the inter-quartile ranges of the beta distributions as the number of stocks in portfolio increases. Defined as the difference between two percentiles, 75% and

25%, the inter-quartile range (IQR) is a stable measure that is robust to outliers. That is,

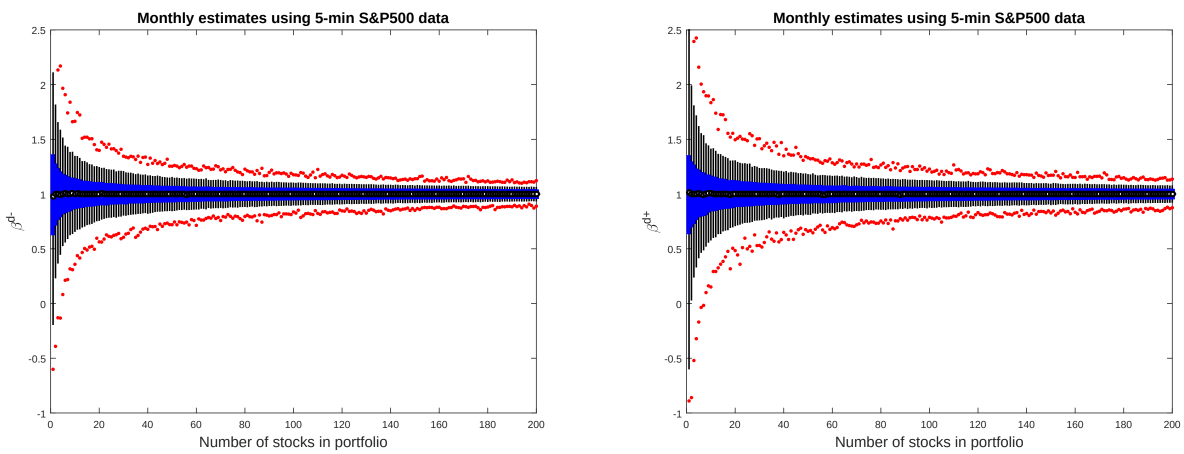
$$IQR_{(n)} \equiv EDF_{(n)}^{-1}(.75) - EDF_{(n)}^{-1}(.25), \tag{22}$$

where $EDF_{(n)}$ is the empirical distribution function of the estimated betas (β^{d}, β^{d+} or β^{d-}) for randomly drawn n-stock portfolios.

Figure 3 depicts the typical distributions of β^{d-} and β^{d+} for equally weighted randomly drawn portfolios of $n = 1, \ldots, 200$ stocks for the year of 2008.[16] Since these central ranges are dependent on the particular time period, for each year in our analysis, we normalize the IQR for the n-stock portfolios and represent it as a fraction of the IQR of the single-stock portfolio. This approach was first introduced by Alexeev et al. (2016). The normalized IQRs, or $IQR_{(n)}/IQR_{(1)}$, of $\hat{\beta}^{d}$, $\hat{\beta}^{d+}$ and $\hat{\beta}^{d-}$ for year 2008 are depicted in Figure 4 for $n = 1, \ldots, 200$. Since the market index is an equally-weighted portfolio consisting of all investible stocks, and is thus unique, it has $IQR_{(N)} = 0$. As a result, the normalized IQRs in Figure 4 are bounded between 0 and 1. We find that the difference among the normalized IQRs for the three different betas are more pronounced during periods of high volatility and for more extreme events (consider the top versus the bottom panel in Figure 4). Figure 4 shows that the IQR of portfolio jump betas decrease substantially as the number of stocks in the portfolio, n, increases. The use of the normalized IQRs enables us to contrast the required portfolio sizes at different periods of time, in order to achieve the same proportional reduction in IQRs of beta for these portfolios relative to the beta spreads of individual securities. In discussing our findings, we consider a fivefold reduction in the spread of portfolio jump risk component relative to the jump beta spread of individual securities (represented by the horizontal line

[16] Results for other years are omitted for brevity because they exhibit similar patterns. These results are available upon request.

Figure 3: Distribution of β^{d-} (top panel) and β^{d+} (bottom panel) across portfolio sizes. Red points represent maxima and minima, black lines represent interpercentaile range from 2.5% to 97.5%, blue lines denote interguantile range and the black circles are the medians of the distributions. We use 2008 data in estimating the results in this figure.



at 0.2 in Figure 4).

Table 3 outlines the required portfolio sizes in order to reduce the normalized IQR five fold. We examine each year separately and consider several threshold levels for jump identification. It is evident that during periods of market distress characterised by high volatility, the number of stocks required to reduce the IQR and stabilise the negative jump beta is considerably higher than during the less volatile periods. During periods of normal market activity, there is little distinction between negative and positive jump risk and, consequently, the recommended portfolio sizes. The difference in the number of stocks required in order to achieve the same proportionate reduction increases substantially in 2008 and 2011. When we consider only negative market jumps, our recommended portfolio sizes are greater than when both negative and positive jumps are considered (the symmetric jump risk). This discrepancy is more pronounced during periods of increased market volatility. This asymmetry effect is most vividly observed when we consider only the most extreme events (last three columns of Table 3). For example, when the threshold is $\alpha_i = 5\sqrt{BV_i}$, during normal market

Figure 4: Normalized IQR of betas across portfolio sizes. Both panels display results based on year 2008. The top panel displays results based on a threshold $\alpha_i = 3\sqrt{BV_i}$ and the bottom panel is based on $\alpha_i = 5\sqrt{BV_i}$. As can be observed from the figures below, the asymmetry in signed betas is more pronounced when more extreme events are considered. The optimal number of holdings in a portfolio is determined at the intersection of the normalised IQR curve (red, blue and black) with the desired level of variability reduction (in this case 0.2 denoted by horizonal purple line).

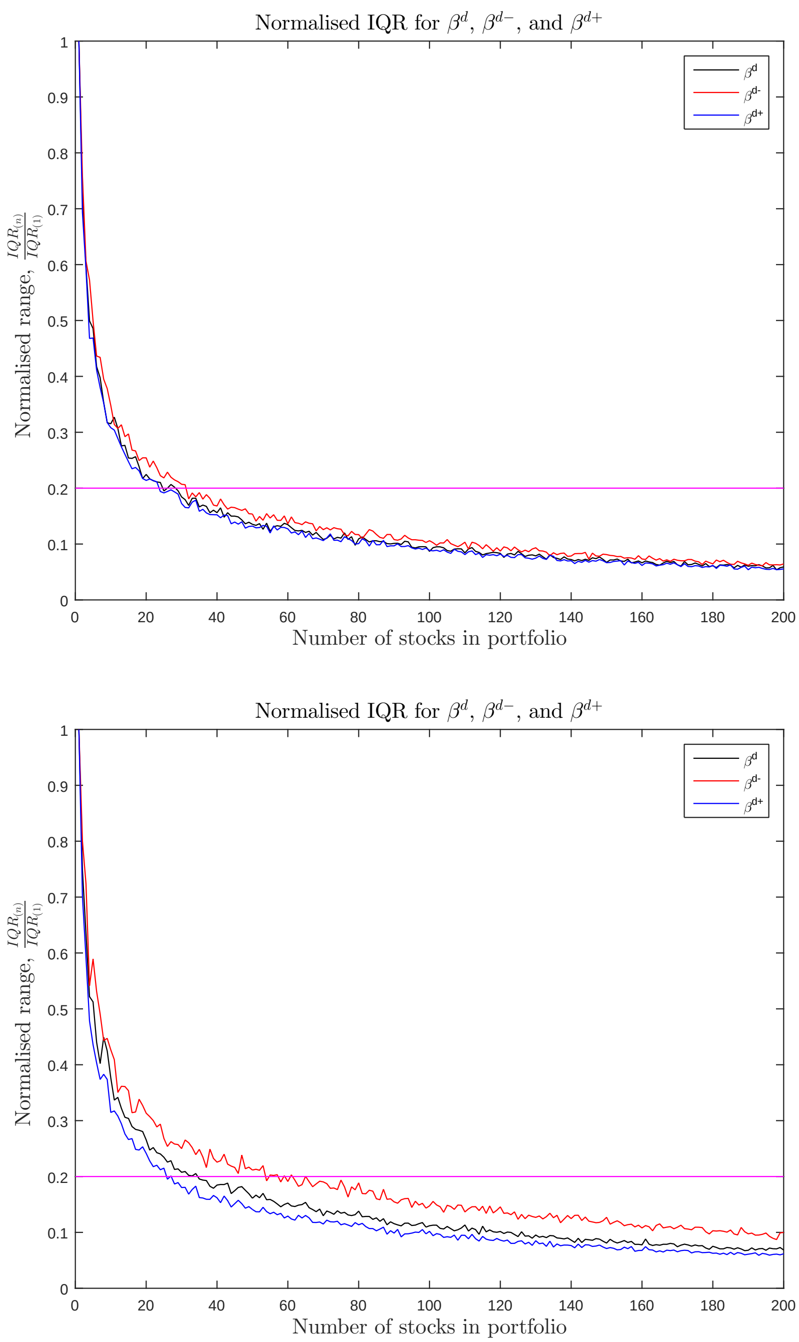


Table 3: Portfolio sizes, n, required to reduce normalised IQR, or $IQR_{(n)}/IQR_{(1)}$, to 0.2.‡

Year	$\alpha_i = 3\sqrt{BV_i}$			$\alpha_i = 4\sqrt{BV_i}$			$\alpha_i = 5\sqrt{BV_i}$		
	β^d	β^{d-}	β^{d+}	β^d	β^{d-}	β^{d+}	β^d	β^{d-}	β^{d+}
2003	30	31	30	36	38	35	36	39	35
2004	31	33	31	36	37	36	34	40	33
2005	30	31	29	35	36	36	35	40	33
2006	30	30	29	34	35	34	33	39	32
2007	28	29	28	32	33	32	32	39	30
2008	25	32	24	29	36	25	35	54	26
2009	30	31	28	34	36	33	34	41	31
2010	31	31	31	35	37	34	35	42	32
2011	24	30	23	29	40	28	34	51	27
2003-2011	29	31	28	34	36	32	35	41	29

‡ This table shows the number of holdings in a portfolio required to stabilize portfolio betas, that is β^d, β^{d-} or β^{d+}. We include the results for a number of severities of extreme events (threshold α used to identify jumps). We consider level of $IQR_{(n)}/IQR_{(1)} = 0.2$ as appropriate: the IQR of individual stock betas can be reduced fivefold if a portfolio is constructed with at least a number of stocks outlined in the table. For example, in 2003, for $\alpha_i = 5\sqrt{BV_i}$, 39 stocks are required to reduce portfolio sensitivity to negative jumps by a factor of 5 compared to when a typical single-stock portfolio is held. To get the same reduction in sensitivity to positive shocks, 35 stocks will suffice. This is in contrast with year 2008, where as many as 54 stocks are required to reduce sensitivity to negative events with only 26 stocks needed to get the same reduction in sensitivity of portfolio to positive events. Note that the asymmetry in results is more pronounced for more extreme events (e.g., larger threshold level α).

times this discrepancy ranges from 3 to 7 stocks, while in 2008 and 2011 there are 17 to 19 stocks' difference. Ignoring the asymmetry in sensitivities to negative versus positive market jumps lead to under-diversification of portfolios, and hence increased exposure to extreme negative market shifts. Without considering asymmetric betas, the optimal portfolio sizes are 35 and 34 in 2008 and 2011 for threshold $\alpha_i = 5\sqrt{BV_i}$, respectively. However, if investor is concerned with extreme negative shifts in the market, it would be advisable to hold 54 and 51 stocks instead, to reduce the sensitivity of portfolio returns to extreme negative market shifts compared to a single stock portfolio.

VI. Conclusion

In this paper, we studied jump dependence between processes using high-frequency observations concentrating only on segments of data around a few outlying observations that are informative for the jump inference. In particular, we studied the relationship between jumps of a process for an asset (or portfolio of assets) and an aggregate market factor, and analysed the co-movement of the jumps in these two processes. We focused on a linear relationship between the jumps and assessed the sensitivity of jumps in (portfolios of) assets to jumps in the market.

We assert that if an asset tends to move downwards in a declining market more than it moves upward in a rising market, such asset is unattractive to hold, especially during market downturns when the wealth of investors is low. We estimated jump betas for the negative and positive market shifts separately, and investigated the implications for portfolio risk management using upside and downside jump betas. Using high-frequency data, we showed that investors care differently about downside losses as opposed to upside gains, but do not demand additional compensation for holding stocks with high sensitivities to extreme market movements. In the absence of co-jumps, this suggests that jump risk is diversifiable and thus with no compensation for investors.

In the context of asset portfolios, we investigated to what extend the downside and upside jump risk can be stabilised. The idea is similar to tracking error in index funds but focusing on the systematic extreme even risks. This has important implications for the pricing of jump risk, and can have a direct impact on investors' decision-making. We found that ignoring the asymmetry in sensitivities to negative versus positive market jumps results in an under-diversification of portfolios and increased exposure to extreme negative market shifts. Our empirical findings for the S&P500 constituents suggest holding 54 stocks is advisable to reduce the IQR of the portfolio sensitivity to extreme market downturn. This number is more than twice the number of stocks (26 stocks) required to induce similar reduction in portfolio sensitivity to positive market shifts. Moreover, if the asymmetry is ignored, the recommended portfolio size is 35 stocks which is substantially lower than the case of diversifying against extreme negative events. This is based on our estimates for 2008, however similar pattern can be noticed for 2011. Thus we conclude that such patterns are more pronounced for the most extreme events (with high jump identification thresholds) occurring during periods of very high market volatility.

Findings in this paper suggest some interesting future developments. First, it would be interesting to investigate behaviour of jump betas during flash crashes, namely short unpredictable periods between an extreme "low" and extreme "high". Second, based on our empirical finding of non-linear relationship between expected returns and asymmetric jump betas during the crisis, we may explore a regime-switching behaviour in sensitivities to extreme market shifts. This refinement will provide further evidence of the discrepancy between the asymmetric jump betas. This is part of the ongoing research agenda.

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