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Abstract

This paper proposes copula-based tests for testing cross-sectional independence of panel models.

JEL *Classification*: C13; C33

Keywords: Copulas; Panel data; Cross-sectional independence

1 Introduction

This paper considers tests of cross-sectional dependence using copulas in panel models. It is important to test the cross-sectional dependence in panel models because the existence of cross-sectional dependence will invalidate conventional tests such as t-tests and F-tests which use standard covariance estimators of parameter estimators. Moreover, the choice of estimation methods may depend upon whether there exists cross-sectional dependence in the errors of panel models. When the errors are cross-sectionally dependent in panel data models, for example, the computation of MLE and GMM could be rather complicated, and the feasible GLS estimator will be invalid or have to be modified substantially.

Since the pioneering work of Moran (1950), there has been a lot of work on testing for cross-sectional dependence or spatial correlation in the literature, e.g., Cliff and Ord (1973), Burridge (1980), King (1981). For a survey see Anselin and Bera (1997). Moran's test is similar in structure to Durbin-Watson test for serial correlation. Cliff and Ord (1973) generalize Moran's test in order to derive a test for spatial correlation in a linear regression model. King (1981) studies the small sample properties of Cliff-Ord test for spatial correlation. Burridge (1980) shows that Cliff-Ord test is a Lagrange multiplier (LM) test. Brett and Pinkse (1997)

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introduce a nonparametric test for spatial independence based on characteristic function and they showed the proposed test is consistent against a fairly general class of alternative.

There have been some tests for cross-sectional dependence in panel models, e.g., Baltagi *et. al.*. (2003), Pesaran (2004), and Ng (2006). Pesaran (2004) proposes a convenient OLS-based test for cross-sectional dependence by modifying Breusch and Pagan (1980)'s Lagrange multiplier (LM) statistic. However, none of the above literature uses copula method.

Copula method has been widely discussed in literature, e.g., Frees and Valdez (1998), Cherubini *et. al.* (2004), Oaks (1994), Genest *et. al.* (1995), Shih and Louis (1995), Joe and Xu (1996), Patton (2002b), Chen and Fan (2005a, 2006a, 2006b), to name a few. Moreover, copula method was also applied to model correlation structure or test dependence between time series data, e.g., Patton (2002a, b), Chen, Fan, and Patton (2004). Patton (2002a) uses the concept of conditional copula to model the time-varying correlation of exchange rates. Chen, Fan, and Patton (2004) apply integral transform and kernel estimation to test the dependence between financial time series. Nonetheless, there is still no research, as far we know, about using copulas to test the cross-sectional dependence in panel models.

The organization of the paper is as follows. In Section 2, we describe the panel models and copulas. In Section 3 we discuss the copula-based tests. Section 4 presents the conclusion. The introduction of copula families and their parameters under independence are in the Appendix.

2 The Model

Consider the following panel model

$$y_{it} = x_{it}^{'}\beta + \mu_i + \lambda_t + v_{it} \tag{1}$$

$i = 1, ..., n$, and $t = 1, ..., T$, where y_{it} is a scalar, x_{it} is a $p \times 1$ vector of regressors that may contain lagged dependent variables, β is a $p \times 1$ vector of slope parameters, μ_i is the individual effect, λ_t is the time effect, and v_{it} is the error term. We allow for fixed or random effects. The slope parameter β is often of interest and it can be estimated, e.g., by the within estimator

$$\widehat{\beta} = \left[\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{x}_{it}\widetilde{x}_{it}^{'}\right]^{-1}\left[\sum_{i=1}^{n}\sum_{t=1}^{T}\widetilde{x}_{it}\widetilde{y}_{it}^{'}\right] \tag{2}$$

where

$$\widetilde{x}_{it} = x_{it} - \overline{x}_{i\cdot} - \overline{x}_{\cdot t} + \overline{x},$$

$$\overline{x}_{i\cdot} = \frac{1}{T}\sum_{t=1}^{T}x_{it},$$

$$\overline{x}_{\cdot t} = \frac{1}{n}\sum_{i=1}^{n}x_{it},$$

and

$$\overline{x} = \frac{1}{n}\frac{1}{T}\sum_{i=1}^{n}\sum_{t=1}^{T}x_{it}.$$

The variables $\widetilde{y}_{it}$, $\overline{y}_{i\cdot}$, $\overline{y}_{\cdot t}$, and $\overline{y}$, are defined similarly. For interval estimation and hypothesis testing, one often uses the standard covariance estimator

$$\widehat{\Omega}_{\widehat{\beta}} = \widehat{\sigma}_v^2 \left(\sum_{i=1}^{n} \sum_{t=1}^{T} \widetilde{x}_{it} \widetilde{x}_{it}^{'} \right)^{-1}$$

of $\widehat{\beta}$, where $\widehat{\sigma}_v^2$ is an estimator for $\sigma_v^2 = Var\left(v_{it}\right).$ This estimator is valid when $\{v_{it}\}$ in (1) is cross-sectionally uncorrelated, among other things. The existence of cross-sectional dependence of any form, however, will generally invalidate the covariance estimator and related inference. in particular, conventional t and F tests will be misleading.

We are interested in testing whether the error process $\{v_{it}\}$ is cross-sectionally dependent. To test the null hypothesis, we will examine the cross-sectional dependence in the demeaned estimated residual $\widehat{v}_{it} = \widehat{u}_{it} - \widehat{u}_{i\cdot} - \widehat{u}_{\cdot t} + \overline{u}$, where

$$\widehat{u}_{it} = y_{it} - x_{it}^{'} \widehat{\beta},$$

$$\widehat{u}_{i\cdot} = \frac{1}{T} \sum_{t=1}^{T} \widehat{u}_{it},$$

$$\widehat{u}_{\cdot t} = \frac{1}{n} \sum_{i=1}^{n} \widehat{u}_{it},$$

$$\overline{u} = \frac{1}{nT} \sum_{i=1}^{n} \sum_{t=1}^{T} \widehat{u}_{it},$$

and $\widehat{\beta}$ is a consistent estimator for β under the null of no cross-sectional dependence. When $\widehat{\beta}$ is the within estimator in (2), $\widehat{v}_{it}$ is the usual within residual in the literature.

Let $v_t = \left(v_{1t}\cdot, , , , v_{nt}\right)^{'}$. For each t, we assume that $\{v_t\}$ has a continuous joint distribution $H\left(v_{1t}\cdot, , , , v_{nt}\right)$ and continuous marginal distribution $F_i\left(v_i\right)$ for $i = 1, ..., n$. By Sklar's (1959) theorem[1], there exists a unique copula function

$$H\left(v_{1t}, .., v_{nt}\right) = C\left(F_1\left(v_{1t}\right), ..., F_n\left(v_{nt}\right)\right).$$

The essence of copulas is that one can always model any multivariate distribution by modeling its marginal distributions and its copula functions separately, where the copula captures all the scale-free dependence in the multivariate distribution. Thus, a copula is a multivariate distribution function that connects marginal distributions so that to exactly form the joint distribution. A copula thus completely parameterizes the entire dependence structure between two or more random variables. It is important to note that a given distribution function H defines only one set of marginal distribution functions F_i, $i = 1, ..., n$, where given marginal

[1] About the detail description of copula method and its application, please refer to Nelson (1999), Cherubini *et.al.* (2004)

distributions do not determine a unique joint distribution. To connect copulas to likelihood-based model, let h and c be the derivatives of the distributions H and C, respectively. Then

$$\begin{aligned} h\left(v_{1t},...,v_{nt}\right) &= \frac{\partial^n H\left(v_{1t},..,v_{nt}\right)}{\partial v_{1t}...\partial v_{nt}} \\ &= \frac{\partial^n C\left(F_1\left(v_{1t}\right),...,F_n\left(v_{nt}\right)\right)}{\partial v_{1t}...\partial v_{nt}} \\ &= \frac{\partial^n C\left(U_{1t},...,U_{nt}\right)}{\partial U_{1t}...\partial U_{nt}}|_{U_{it}=F_i(v_{it})} \prod_{i=1}^{n} f_i\left(v_{it}\right) \\ &= c\left(F_1\left(v_{1t}\right),...,F_n\left(v_{nt}\right)\right) \prod_{i=1}^{n} f_i\left(v_{it}\right). \end{aligned}$$

That is, the joint density is the product of the copula density and the marginal densities. The hypotheses of interest are

$$\begin{cases} H_0 : c\left(F_1\left(v_{1t}\right),...,F_n\left(v_{nt}\right)\right) = 1 \text{ for all } t \\ H_A : c\left(F_1\left(v_{1t}\right),...,F_n\left(v_{nt}\right)\right) < 1 \text{ for some } t. \end{cases}$$

The alternative hypothesis H_A allows (but not all) the time series to be independent. Then log-likelihood function for (1) under the alternative hypothesis is $l = \sum_{t=1}^{T}\sum_{i=1}^{n}\left[\ln f_i\left(v_{it};\theta\right) + \ln c\left(F_1,...,F_n;\alpha\right)\right]$, where θ is regression parameter in (1), and α is the copula parameter. Under the null hypothesis the log-likelihood function can be reduced to $l = \sum_{t=1}^{T}\sum_{i=1}^{n}\ln f_i\left(v_{it};\theta\right)$.

3 Copula-Based Tests

In the literature, the estimation for copula parameter can be categorized into three types: exact maximum likelihood estimation (MLE), two-step MLE, and semiparametric two-step estimation[2]. In this paper, we use the semiparametric two-step approach.

Let $C^0\left(\bullet;\alpha\right)$ denotes a class of correctly-specified parametric copulas with unknown parameter α. The two-step semiparametric estimator, $\widehat{\alpha}$, is defined as

$$\widehat{\alpha} = \arg\max_{\alpha\in\Theta}\left[\frac{1}{T}\sum_{t=1}^{T}\log c^0\left\{\widetilde{F}_1\left(\widehat{v}_{1t}\right),\ldots,\widetilde{F}_n\left(\widehat{v}_{nt}\right);\alpha\right\}\right]$$

where $\widehat{v}_{it} = v_{it}\left(\widehat{\beta}\right)$, $c^0\left(\bullet;\alpha\right)$ is the density of the parametric copula $C^0\left(\bullet;\alpha\right)$ and $\widetilde{F}_i\left(v\right)$ is the rescaled empirical distribution function of $\widehat{v}_{i1},\ldots,\widehat{v}_{iT}$:

$$\widetilde{F}_i\left(v\right) = \frac{1}{T+1}\sum_{t=1}^{T}\mathbf{I}\left(\widehat{v}_{it}\le v\right),\ i=1,\ldots,n \tag{3}$$

and $\mathbf{I}\left(\bullet\right)$ is an indicator function.

[2] Chap5 in Cherubini *et.al* (2004) provides a thorough introduction about the estimation of copula model.

Notice that we use $\widehat{v}_{it}$ in (3), instead of v_{it}, because v_{it} is not observable. In particular, we are interested in seeing how the asymptotic behavior of $\widetilde{F}_i(v)$ and hence $\widehat{\alpha}$ depend on the estimator $\widehat{\beta}$ of β in $\widehat{v}_{it} = v_{it}\left(\widehat{\beta}\right)$. Let

$$\widehat{F}_i(v) = \frac{1}{T+1}\sum_{t=1}^{T}\mathbf{I}(v_{it} \leq v).$$

Then it can be shown that (e.g., Mammen, 1996, p. 308)

$$\sqrt{T}\left(\widetilde{F}_i(v) - \widehat{F}_i(v)\right) = \frac{1}{\sqrt{T}} f(v)\sum_{t=1}^{T}\left[\widetilde{x}_{it}'\left(\widehat{\beta}-\beta\right)\right] + o_p(1)$$

where $f(v)$ is the density of $F(v)$. Hence, one has to expect that the asymptotics of $\widehat{\alpha}$ will depend on the $\widehat{\beta}-\beta$. However, interestingly and surprisingly, Chen and Fan (2006b, Proposition 3.1) have shown that the asymptotics of $\widehat{\alpha}$ is not affected by the initial estimator $\widehat{\beta}$ in the context of a copula-based multivate GARCH model, Following the similar steps in Chen and Fan (2006b), we can establish that

$$\begin{aligned}\widehat{\alpha} &= \arg\max_{\alpha\in\Theta}\left[\frac{1}{T}\sum_{t=1}^{T}\log c^0\left\{\widetilde{F}_1(\widehat{v}_{1t}),\ldots,\widetilde{F}_n(\widehat{v}_{nt});\alpha\right\}\right] \\ &\approx \arg\max_{\alpha\in\Theta}\left[\frac{1}{T}\sum_{t=1}^{T}\log c^0\left\{\widehat{F}_1(v_{1t}),\ldots,\widehat{F}_n(v_{nt});\alpha\right\}\right]. \qquad (4)\end{aligned}$$

Let $U_t = (U_{1t},\ldots,U_{nt})^{\top}$ with $U_{it} = F_i^0(v_{it})$, $i=1,\ldots,n$; where $F_i^0(\bullet)$ is the true marginal distribution, $l(u_1,\ldots,u_n;\alpha) = \log c^0(u_1,\ldots,u_n;\alpha)$, $l_\alpha(u_1,\ldots,u_n;\alpha) = \frac{\partial}{\partial\alpha}l(u_1,\ldots,u_n;\alpha)$, $l_j(u_1,\ldots,u_n;\alpha) = \frac{\partial}{\partial u_j}l(u_1,\ldots,u_n;\alpha)$, $l_{\alpha\alpha}(u_1,\ldots,u_n;\alpha) = \frac{\partial^2}{\partial\alpha\partial\alpha'}l(u_1,\ldots,u_n;\alpha)$, $l_{\alpha j}(u_1,\ldots,u_n;\alpha) = \frac{\partial^2}{\partial\alpha\partial u_j}l(u_1,\ldots,u_n;\alpha)$, $E^0\{\bullet\}$ is an expectation taken with respect to distribution $C^0\left(u_1,\ldots,u_n;\alpha^0\right)$, $B \equiv -E^0\left\{l_{\alpha\alpha}\left(U_{1t},\ldots,U_{nt};\alpha^0\right)\right\}$ is positive definite, $\Sigma \equiv var^0\left\{l_\alpha\left(U_{1t},\ldots,U_{nt};\alpha^0\right) + \sum_{i=1}^{n} W_i\left(U_{it};\alpha^0\right)\right\}$ is finite, positive definite, and $W_i\left(U_{it};\alpha^0\right) \equiv E^0\left[\{\mathbf{I}(U_{it}\leq U_{is})\}\, l_{\alpha i}\left(U_{1s},\ldots,U_{ns};\alpha^0\right)|U_{it}\right]$.

The asymptotic properties of $\widehat{\alpha}$ in (4) have been discussed by Genest *et.al.* (1995):

Proposition 1 *Under suitable regularity conditions stated in Genest et al. (1995), we have* $\sqrt{T}\left(\widehat{\alpha}-\alpha^0\right) \xrightarrow{d} N\left(0, B^{-1}\Sigma B^{-1}\right)$ *as* $T \longrightarrow \infty$.

The B and Σ in asymptotic variance are not observable; therefore, some consistent estimators must be given. From Genest *et. al.* (1995), we note that B can be consistently estimated by:

$$\widehat{B} = -\frac{1}{T}\sum_{t=1}^{T} l_{\alpha\alpha}\left(\widetilde{U}_t;\widehat{\alpha}\right)$$

where $\widetilde{U}_t = \left(\widetilde{U}_{1t},\ldots,\widetilde{U}_{nt}\right)^{\top}$, $\widetilde{U}_{it} = \widetilde{F}_i(\widehat{v}_{it})$ for $i=1,\ldots,n$, and

$$\widehat{\Sigma} = \frac{1}{T}\sum_{t=1}^{T}\left\{l_\alpha\left(\widetilde{U}_t;\widehat{\alpha}\right) + \sum_{i=1}^{n}\widehat{W}_i\left(\widetilde{U}_{it};\widehat{\alpha}\right)\right\}\left\{l_\alpha\left(\widetilde{U}_t;\widehat{\alpha}\right) + \sum_{i=1}^{n}\widehat{W}_i\left(\widetilde{U}_{it};\widehat{\alpha}\right)\right\}^\top$$

with

$$\widehat{W}_i\left(\widetilde{U}_{it};\widehat{\alpha}\right) = \frac{1}{T}\sum_{s=1,s\neq t}^{T} l_{\alpha i}\left(\widetilde{U}_s;\widehat{\alpha}\right)\left\{I\left(\widetilde{U}_{it} \leq \widetilde{U}_{is}\right)\right\}$$

Then the test of independence in panel models can be stated as:

$$\begin{cases} H_0 : \alpha^0 = \alpha^* \\ H_A : \alpha^0 \neq \alpha^* \end{cases}$$

where α^* is the copula parameter under the null of independence which are discussed in the Appendix, and α^0 is the true copula parameter. Using the asymptotic property of $\widehat{\alpha}$, we can construct a Wald test, for example,

$$\mathrm{W} = \left(\widehat{\alpha} - \alpha^*\right)^\top \left(\frac{1}{T}\widehat{B}^{-1}\widehat{\Sigma}\widehat{B}^{-1}\right)^{-1}\left(\widehat{\alpha} - \alpha^*\right) \tag{5}$$

and it can be shown that W follows a χ^2_k asymptotically under H_0, where k is the dimension of α.

4 Conclusion

This paper presents copula-based tests to detect cross-sectional dependence in panel models. Some commonly used copula families and their related properties are provided in Appendix. By checking respective copula parameter under independence, we can construct tests, e.g., Wald test statistic, to test cross-sectional dependence in panel models.

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A Appendix

A.1 Copula families and parameters under independence

In this appendix, we list the properties of a few widely-used copulas, including copula forms, copula density, and copula parameters under independence. In this section, $C(.)$ denotes copula function, $c(.)$ denotes copula density, and α^* denotes copula parameter under independence which either makes copula function become independent copula or makes copula density equal 1, where independent copula $C(u,v) = uv$. About more detail explanation, please refer to Nelson (1999).

A.1.1 Elliptical copulas

1. Gaussian copula

   Let $\mathbf{R}$ be symmetric, positive definite correlation matrix and $\Phi_{\mathbf{R}}(.,.)$ be the standard bivariate normal distribution with correlation matrix $\mathbf{R}$. The density function of bivariate Gaussian copula is:

   $c(u,v) = \frac{1}{|\mathbf{R}|^{0.5}} \exp\left(-\frac{1}{2}\boldsymbol{\eta}^\top \left(\mathbf{R}^{-1} - \mathbf{I}\right)\boldsymbol{\eta}\right)$

   where, $\boldsymbol{\eta} = \left(\Phi^{-1}(u), \Phi^{-1}(v)\right)^\top$ and $\Phi^{-1}(.)$ is the inverse of the univariate normal CDF. The bivariate Gaussian copula is:

   $C(u,v,\mathbf{R}) = \Phi_{\mathbf{R}}\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$

   Hu (2003) shows the bivariate Gaussian copula can be approximated by Taylor's expansion:

   $C(u,v,\boldsymbol{\rho}) \approx uv + \rho\phi\left(\Phi^{-1}(u)\right)\phi\left(\Phi^{-1}(v)\right)$

   where, $\phi(.)$ is the density function of univariate Gaussian distribution and ρ is the correlation coefficient between $\Phi^{-1}(u), \Phi^{-1}(v)$ It is very trivial that when ρ is 0, this copula is an independent copula. In multivariate case, independence holds when $\mathbf{R}$ is an identity matrix.

A.1.2 Copulas with quardratic-sections

In this family, copula can be represented as:

$C(u_1, u_2, \ldots, u_n) = a(v)u^2 + b(v)u + c(v)$, for appropriate functions a, b, c.

1. Farlie-Gumbel-Morgenstern family:

   $C(u,v,\alpha) = uv(1 + \alpha(1-u)(1-v))$
   $c(u,v,\alpha) = 1 + \alpha - 2\alpha u - 2\alpha v + 4\alpha uv$
   $\alpha \in [-1,1]$
   $\alpha^* = 0$

A.1.3 Archimedean copulas

Archimedean copulas can be constructed by an originator, $\varphi(t)$, via this generator function:

$C(u_1, u_2, \ldots, u_n) = \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n))$

where, $\varphi : \mathbf{I} \to [0, \infty]$, continuous, $\varphi'(t) < 0$ and $\varphi''(t) > 0$, for all $t \in (0, 1)$

Hence, the density of copula can be expressed as:

$c(u_1, u_2, \ldots, u_n) = \frac{\partial^n C(u_1, u_2, \ldots, u_n)}{\partial u_1 \ldots \partial u_n} = \frac{\partial^n \varphi^{-1}(\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_n))}{\partial \varphi(u_1) \ldots \partial \varphi(u_n)} \frac{\partial \varphi(u_1)}{\partial u_1} \cdots \frac{\partial \varphi(u_n)}{\partial u_n}$

Here, we only list bivariate case. They can be easily extended to n-variate case.

1. Joe family:

   $C(u, v, \alpha) = 1 - ((1-u)^\alpha + (1-v)^\alpha - (1-u)^\alpha (1-v)^\alpha)^{1/\alpha}$
   $\alpha \in [1, \infty)$
   $\varphi(t) = -\log(1 - (1-t)^\alpha)$
   $\alpha^* = 1$

2. Ali-Mikhail-Haq family:

   $C(u, v, \alpha) = \frac{uv}{1 - \alpha(1-u)(1-v)}$
   $\alpha \in [-1, 1)$
   $\varphi(t) = \log\left(\frac{1 - \alpha(1-t)}{t}\right)$
   $\alpha^* = 0$

3. Clayton family:

   $$C(u, v, \alpha) = \begin{cases} uv & \alpha = 0 \\ (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha} & \alpha \neq 0 \end{cases}$$
   $\alpha \in [0, \infty)$
   $\varphi(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$
   $\alpha^* = 0$

4. Gumble family:

   $C(u, v, \alpha) = \exp\left[-\left((-\ln u)^\alpha + (-\ln v)^\alpha\right)^{1/\alpha}\right]$
   $\alpha \in [1, \infty)$
   $\varphi(t) = (-\log t)^\alpha$
   $\alpha^* = 1$

5. Frank family:

   $$C(u, v, \alpha) = \begin{cases} uv & \alpha = 0 \\ -\frac{1}{\alpha} \ln\left[1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{e^{-\alpha} - 1}\right] & \alpha \neq 0 \end{cases}$$
   $\alpha \in (-\infty, \infty)$
   $\varphi(t) = -\ln \frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}$
   $\alpha^* = 0$

A.1.4 Others

1. Plackett family:

$$C(u,v,\alpha) = \begin{cases} uv & \alpha = 1 \\ \frac{\left(1+(u+v)(\alpha-1)-\sqrt{(1+(u+v)(\alpha-1))^2-4uv\alpha(\alpha-1)}\right)}{2(\alpha-1)} & \alpha \neq 1 \end{cases}$$

$\alpha \in (0, \infty)$
$\alpha^* = 1$