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The Fractional Merton Model: A New Approach to Credit Risk Pricing

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CEA@Cass Working Paper Series

WP–CEA–06-2008

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March 17, 2008

Abstract

In this paper we develop the theoretical framework of the fractional Merton model, which allows to embed long memory properties of spreads in a straightforward manner in a credit risk pricing model. We carry out an extensive sensitivity analysis exercise and compute the spread sensitivities to the long memory parameter, firm leverage, firm volatility and variance, and risky debt time to maturity. We also compute sensitivities of the equity, risky debt, risk-neutral default probability and option to default to long memory. We show that theoretical spreads of the fractional Merton model are higher than the spreads predicted by the Merton model. This favours the conclusion that fractional Merton model can explain market spreads better than the Merton model.

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Special thanks to Giorgio Consigli, Stephen Hall, John Hatgioannides, Aneel Keswani, Arturo Leccadito, Ian Marsh for providing detailed comments on an earlier version of the paper. Thanks also to partecipants in the "London-Oxford Financial Econometrics Workshop", held at Imperial College on the 23 November 2007, in particular Nour Meddahi, Hashem Pesaran, and Neil Shephard for providing very useful comments and suggestions. However, the usual disclaimer applies.

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Abstract: In this paper we develop the theoretical framework of the fractional Merton model, which allows to embed long memory properties of spreads in a straightforward manner in a credit risk pricing model. We carry out an extensive sensitivity analysis exercise and compute the spread sensitivities to the long memory parameter, firm leverage, firm volatility and variance, and risky debt time to maturity. We also compute sensitivities of the equity, risky debt, risk-neutral default probability and option to default to long memory. We show that theoretical spreads of the fractional Merton model are higher than the spreads predicted by the Merton model. This favours the conclusion that fractional Merton model can explain market spreads better than the Merton model.

Keywords: C14, C22, G13.
J.E.L. Classification Numbers: Credit Risk, Structural Models, Credit Spreads, Fractional Integration.

1 Introduction

The credit spread of corporate bonds has been shown to depend on a number of factors. Elton, Gruber, Agrawal, Mann (2001) show that the magnitude of the spread depends on the probability of default, the loss given default, the tax regime of corporate bonds relative to government bonds, and the systematic risk of corporate bonds.

Models of credit spread combine these factors in different ways. Structural models use an option-pricing approach that follows the original idea of Merton (1974), which defines a stochastic process for the firm value and treats risky debt as a combination between a riskless bond and a short put option on the firm's assets. The put option is an "option to default", giving shareholders the opportunity to sell bondholders the firm value at the debt nominal value. The value of the put takes into account systematic risk, probability of loss and recovery rate. Reduced-form models (such as Jarrow and Turnbull, 1995, and Duffie and Singleton, 1999) assume that exogenous hazard rates (often based on transition probabilities of ratings) and the loss given default drive the spread.

Despite reduced-form models are sufficiently flexible to accommodate relevant market information in the default rate process (e.g., credit rating, firm specific or macroeconomic variables) and to be calibrated to market data, they have a key disadvantage over structural models in that the default rate process is not linked to the value of the firm's assets. This is a strong limitation in order to understand the determinants and dynamics of credit risk. In addition, models based on a direct assumption of the credit spread dynamics usually lack a rigorous treatment of the derivation of a risk-neutral probability from the fundamental assumptions about the underlying process. These models simply assume a risk-adjusted process for the spread.

Structural models, although intuitively appealing, are also prone to a number of criticisms which have made them difficult to implement. First, as pointed out by Jarrow, Lando and Turnbull (1997), the firm's assets are not tradable or easily observable. This is inconsistent with the assumptions that the firm value follows a diffusion process and that firm's assets can be traded in continuous time. In addition, it makes it difficult to estimate the volatility of the firm's assets, which is a key piece of information for risky debt valuation. Second, the capital structure of the firm (which may generate complex priority rules of the payoffs to the firm's liabilities) needs to be fully specified and included in the valuation procedure. This is a particularly difficult task and may even become intractable when applied to coupon-paying bonds, callable bonds or complex capital structures. Third, since this approach does not use credit rating information, it cannot be employed to price credit derivatives whose payoffs depend on credit ratings.

For the aforementioned reasons, empirical studies on credit spread are relatively infrequent. The studies carried out to date show that structural models seem to generate credit spreads that are not consistent with the spreads observed in practice. Jones, Mason and Rosenfeld (1984), Ogden (1987) and Eom, Helwege and Huang (2004) all find that spreads predicted by Merton (1974) are smaller than market spreads. In addition, Eom, Helwege and Huang (2004) investigate alternative models to Merton's and conclude that also Geske (1977), in line with Merton (1974), predicts smaller spreads than observed, whilst the models of Longstaff and Schwartz (1995a) and Leland and Toft (1996) generate larger spreads but also larger errors.

A consequence of the fact that the firm value is modelled as a diffusion process is that the time of default should always be predictable since, under a diffusion process, a sudden drop in the firm value is impossible. As a result, the firm's probability of default on very short-term debt should be zero, which implies that very short-term debt is risk-free and should have zero credit spread. This implication is clearly rejected by the empirical evidence and contributes to explain why structural models predict particularly small spreads for bonds that are near to maturity.

Anderson and Sundaresan (2000) test to what extent structural credit risk models can explain credit spread and find that these models track historical spreads very well during some periods but perform poorly during others.

A more recent literature has however suggested that structural models may predict credit spreads correctly. Gemmill (2002), by using for the first time a dataset of zero-coupon bonds issued by firms with simple capital structures, suggests that spreads predicted by Merton (1974) are consistent with market spreads.

Two real-world features that help justifying the difference between market and model spread are taxation and transaction costs. According to Elton, Gruber, Agrawal and Mann (2001) the taxation of corporate bonds in the US is higher than of government bonds and this should increase the spread. Similarly, Ericsson and Renault (2001) find that there is a liquidity premium in the observed credit spread which increases its size. This finding is in line with De Jong and Driessen (2007) who estimate credit spreads in two different ways, using the Leland and Toft (1996) model and Moody's historical default and recovery rates[1].

In a companion paper (Della Ratta and Urga, 2007) we investigated the long memory properties of credit spreads[2] and we found clear evidence that spreads are long memory nonstationary processes, with long memory parameter d not statistically different from unity in most cases. This conclusion is

[1] Other empirical work has focused on the term structure of the credit spread. Although structural models such as Merton (1974) and Longstaff and Schwartz (1995a) and reduced-form models such as Jarrow, Lando and Turnbull (1997) predict that the term structure of credit spread is upward sloping for high grade bonds and downward sloping for low grade bonds (supporting the intuition that low grade bonds have a chance to upgrade should they not default), empirical results are not consistent. Helwege and Turner (1999), using pairs of bonds from the same company, confirm that B and BB rated bonds have an upward sloping spread term structure. This is in contrast to earlier cross-sectional studies of Sarig and Warga (1989) and Fons (1987). Using a similar approach, He, Hu and Lang (2000) confirm the upward sloping credit spread term structure for high grade bonds (B or better) and a downward sloping structure for low grade bonds (CC or worse).

[2] We used 30-year Historical Treasury Constant Maturity Yields and Moody's Aaa, Aa, A and Baa Long-Term Corporate Bond Yield Averages. The data covers the period from December 1992 to November 2003, for 2703 observations.

supported by the results of standard unit root (DF, ADF and PP) and stationarity (KPSS, R/S and modified R/S) tests, as well as results of parametric and semi-parametric estimation techniques for long-range dependence (GPH, Robinson, ARFIMA). Our results are robust to heteroskedasticity, choice of sample frequency and choice of sample period. We also find that the bivariate systems of Treasury and corporate yields are fractionally cointegrated[3].

The finding of long memory nonstationarity of spreads does not reconcile with the reduced-form model of Duffie and Singleton (1999), which assumes a stationary risk-free rate process, implying stationarity for the credit spread. This also applies, for instance, to the reduced-form models of Das and Tufano (1996) and Jarrow, Lando and Turnbull (1997).

Our evidence of fractional cointegration in the bivariate systems of yields implies that there still exists a long run equilibrium relationship between yields, and deviations from the fractionally cointegrating relationship are mean reverting, so that a shock to the system will eventually die out. Given that for all systems the fractional difference parameter assumes values $d < 1$ but close to unity, shocks may take several years to dissipate. Fractional cointegration implies a link between yields, which is not captured, for instance, in the risky debt pricing models of Merton (1974), Kim, Ramaswamy and Sundaresan (1993) and Longstaff and Schwartz (1995a), as well as the reduced-form models of Duffee (1999) and Duffie and Singleton (1999) and the option pricing models of Das (1995) and Longstaff and Schwartz (1995b).

Our findings have profound implications for modelling credit spreads in a way which accommodates their long memory characteristics. This can be achieved in both the structural form and the reduced form approach to credit risk modelling.

In the reduced form approach, the credit spread s can be assumed to evolve as a fractional Geometric Brownian motion:

$$ds = \mu_s \, sdt + \sigma_s \, sdB_H \tag{1}$$

where μ_s is the instantaneous expected rate of return on the spread, σ_s is the instantaneous standard deviation of return on the firm and dB_H is the increment of a fractional Brownian motion.

In the structural form approach, the evolution of the firm value rather than the spread can be modelled as a Geometric fractional Brownian motion as follows:

$$dV = \mu V dt + \sigma V dB_H \tag{2}$$

where μ is the instantaneous expected rate of return on the firm and σ is the instantaneous standard deviation of return on the firm.

In this paper we propose a structural form approach for risky debt pricing which takes into account the long memory properties of credit spread. Our work aims at exploring alternative parameterisations of structural models to

[3] Yields and spreads are found to be long memory nonstationary processes also in a companion study by Leccadito and Urga (2007) using Lehman Brothers Eurodollar Aaa, Aa, A and Baa Indices and U.S. Global Treasury Index. The data covers the period from June 1996 to July 2006, for 2613 observations.

better explain the empirical credit spread. Specifically, we embed fractional integration in the Merton (1974) model by generalising the Brownian motion of the diffusion term of the firm value process to a fractional Brownian motion. We call this model fractional Merton model.

Merton's model assumes that the firm value follows a Geometric Brownian motion. This implies that the firm value is a lognormally distributed random variable and log returns of the firm value are normally distributed. Log returns are also assumed independent over time. The independency property implies that $H = 1/2$, i.e. returns do not have long memory properties. Independence is a key property of all processes driven by a Brownian motion. By using a fractional Brownian motion instead of a Brownian motion, a dependence structure in returns can be accounted for.

The remaining of the paper is organised as follows. In Section 2 we provide a recap of the seminal Merton (1974) model. In Section 3 we derive a fractional version of the Merton's model, while a detailed sensitivity analysis of the fractional Merton model, reported in Section 4, allows us to assess whether the fractional version is able to predict spreads which are closer to market spreads than those predicted by Merton's models. Section 5 concludes.

2 Merton's Model: A Recap

The option theory based model of Merton (1974) is a seminal paper in risky debt valuation. In order to price defaultable bonds, Merton uses the original idea by Black and Scholes (1973) that the equity E in a firm is equivalent to a call option on the firm's assets V. Since the value of a firm's assets is equal to the value of its debt f plus the value of its equity ($V \equiv E + f$), in the simplest case of a firm with a single issue of zero-coupon debt outstanding, the debt value is equal to the difference between the firm's assets value and the value of a call option on the firm's assets, with strike price equal to the debt nominal value and time to maturity equal to the debt maturity. Equivalently, from the put-call parity, the debt is the combination of a riskless bond and a short put option on the firm's assets. Bondholders in effect buy a safe bond and give shareholders the option to sell them the firm's assets at the debt value. Shareholders will exercise this option upon default.

In the most simplified case of the model, Merton (1974) assumes that the firm's capital structure has one outstanding issue of zero-coupon debt promising D at maturity T, no dividends are paid or shares repurchased prior to the debt maturity, no issuance of new debt of equal or greater seniority is allowed, and the riskless term structure of interest rates is flat and non-stochastic. Default may occur only at T and will occur if the firm value falls short of the debt nominal value. The firm value V is assumed to follow a diffusion process described by the stochastic differential equation (Geometric Brownian motion):

$$dV = \mu V dt + \sigma V dB \tag{3}$$

where dB is the increment of a standard Brownian motion. Under these assumptions, the value at time t of the debt issue, $f(V,\, t)$, is only a function of the firm value and time. By Itô's lemma, $f(V,\, t)$ is also a diffusion process satisfying the partial differential equation:[4]

$$\frac{\partial f}{\partial t} + rV\frac{\partial f}{\partial V} + \frac{1}{2}\sigma^2 V^2 \frac{\partial^2 f}{\partial V^2} = rf \tag{4}$$

with boundary conditions:

$$f(0,t) = 0 \tag{5}$$

$$f(V,t) \leq V\, \forall t \tag{6}$$

$$f(V_T, T) = \min(V_T, D) \tag{7}$$

Condition (5) implies that the debt is worthless if the firm has no assets; condition 6) is a regularity condition stating that the debt is worth no more than the firm's assets. Condition (7) says that the value of debt at maturity is equal to its face value D when $V_T > D$, i.e. when the firm value is high enough for shareholders to meet the promised payment to bondholders, or V_T when $V_T \leq D$, in which case the firm will default.

The debt can be valued either directly, by solving (4) subject to the boundary conditions (5) to (7), or indirectly, valuing the equity first and treating the debt as a residual claim on the firm's assets. Merton uses the latter approach and derives an explicit solution for the debt value by subtracting the value of the equity (call option value) from the value of the firm's assets. The debt value is:

$$f(V,t) = V\,\left[1 - N(d_1)\right] + D\,e^{-rT} N(d_2) \tag{8}$$

where

$$d_1 = \frac{\ln\left(\frac{V}{D}\right) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \tag{9}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{10}$$

and $N(\cdot)$ is the cumulative standard normal distribution function.

The debt value can also be rewritten as:

$$f(V,t) = D\,e^{-[r+s(V,\,t)]\,T} \tag{11}$$

where $s(V,t)$ is the spread above the risk-free rate at which the debt trades at time t. This is equal to:

$$s(V,t) = -\frac{1}{T}\ln\left\{\frac{V}{D\,e^{-rT}}\left[1 - N(d_1)\right] + N(d_2)\right\} \tag{12}$$

from which it is possible to predict the term structure of credit spread.

Merton (1974) carries out a comparative static analysis of changes in the spread and concludes, amongst others, that the spread is an increasing function of the firm leverage $L = De^{-rT}/V$ and the firm variance σ^2. The spread is

[4] See, for example, Black and Scholes (1973) or Merton (1973).

also a decreasing function of time to maturity for $L \geq 1$, while for $L < 1$ the spread first increases and then decreases with time to maturity. Lee (1981) and Pitts and Selby (1983) point out that Figure 3 in Merton (1974) showing the relationship between spread and time to maturity is incorrect.

In the Sections **3** and **4**, we generalise Merton's models to its fractional equivalent and extend Merton's sensitivity analysis to $H \neq 1/2$.

3 The Fractional Merton Model

Over the last few years the academic literature has extended the Black-Scholes model to account for dependency in returns. This literature has proposed to resolve the issue of dependency of returns by replacing the Brownian motion with the fractional Brownian motion B_H, which captures the long range dependency property measured by H. An alternative option pricing model, called the fractional Black-Scholes model, has been developed which is driven by a stochastic differential equation based on a fractional Brownian motion. Recent papers include Sottinen and Valkeila (2001, 2003), Biagini, Øksendal and Sulem (2002), Benth (2003), Hu and Øksendal (2003), Elliot and van der Hoek (2003) and Biagini and Øksendal (2004). Björk and Hult (2005) raise serious issues concerning the definition of a self-financing portfolio in Elliot and van der Hoek (2003), and the definition of the value process in Hu and Øksendal (2003). They provide simple examples showing that these definitions conflict with the usual economically intuitive definitions because of the Wick products. The no-arbitrage fractional Black-Scholes model therefore appeasr to be obtained bu redefining standard accounting notions of the budget constraint and value.

Given that Merton's model is based on the Black-Scholes framework, we can easily exploit the notion of fractional Brownian motion to generalise Merton's model to the 'fractional Merton model'.

As for $H \neq 1/2$ the fractional Brownian motion is neither a Markov process nor a semimartingale, the conventional Itô theory cannot be used to define stochastic integrals with respect to the fractional Brownian motion.

This has important consequences in terms of arbitrage. Absence of arbitrage opportunities is the main axiom of mathematical finance. The fundamental theorem of asset pricing states that no arbitrage implies the existence of an equivalent martingale measure. As a consequence, non-semimartingales are ruled out as models for assets. Given that the fractional Brownian motion is not a semimartingale, the Geometric fractional Brownian motion cannot be a semimartingale. Rogers (1997), Dasgupta and Kallianpur (2000) and Shiryaev (1998) prove the existence of arbitrage with the Geometric fractional Brownian motion in continuous time. Cheridito (2001) proves the existence of arbitrage in discrete time.

Arbitrage opportunities therefore seem to rule out the Geometric fractional Brownian motion as a model in finance. Mathematically, these opportunities are based on the path-wise Riemann-Stieltjes stochastic integral with respect to the fractional Brownian motion. The Riemann-Stieltjes integral of a process u_t

is defined as:

$$\int_0^T u_t dB_H \tag{13}$$

A path-wise integration theory for the fractional Brownian motion was developed by Lin (1995) and Decreusefond and Üstünel (1999). In order to avoid arbitrage opportunities, one can try and use a different definition of integral which avoid possibilities.

Several authors have suggested the use of divergence integrals to avoid arbitrage opportunities. The divergence integral (also called Wick-Itô-Skorohod integral[5]) of a process u_t with respect to the fractional Brownian motion is defined as:

$$\int_0^T u_t \delta B_H \tag{14}$$

The Wick-Itô-Skorohod integral can be considered as the limit of Riemann-Stieltjes sums if the ordinary product is replaced by the Wick product. Duncan, Hu and Pasik-Duncan (2000) develop stochastic calculus for the fractional Brownian motion based on Skorohod integration. The use of Skorohod integration in finance was proposed by Hu and Øksendal (2003).

We therefore have two pricing approaches in the fractional Merton model. The firm value dynamics can be expressed as a fractional Geometric Brownian motion in terms of the Riemann-Stieltjes equation

$$dV = \mu V dt + \sigma V dB_H \tag{15}$$

or the Skorohod equation

$$\delta V = \nu V dt + \sigma V \delta B_H \tag{16}$$

As shown by Sottinen and Valkeila (2003) for the fractional Black-Scholes model, by using the appropriate Itô formulas the solutions to (15) and (16) are, respectively:

$$V_t = V_0 e^{\int_0^t \mu(s)ds + \sigma B_H} \tag{17}$$

and

$$V_t = V_0 e^{\int_0^t \left(\nu(s) - \sigma^2 H s^{2H-1}\right)ds + \sigma B_H} \tag{18}$$

We conclude that the stochastic differential equations (15) and (16) imply the same pricing model for the firm value if and only if:

$$\mu(t) = \nu(t) - \sigma^2 H t^{2H-1} \tag{19}$$

Although the fractional Merton model does not have an equivalent martingale measure, it can be shown that there is a unique measure such that the solution to (15) or (16) is given by the Geometric fractional Brownian motion

$$V_t = V_0 e^{\mu t - \frac{1}{2}\sigma^2 t^{2H} + \sigma B_H} \tag{20}$$

[5] Also called Wick integrals or Skorohod integrals. This definition was introduced by Sottinen and Valkeila (2003).

This is called generalised solution. The path-wise Riemann-Stieltjes solution to (13) is instead given by:

$$V_t = V_0 e^{\mu t + \sigma B_H} \tag{21}$$

Hu and Øksendal (2003) show that with the generalised solution (20) the fractional pricing model is arbitrage-free and complete, in the sense that there is a fractional analogue of Itô's formula. When using path-wise solutions and under continuous trading, arbitrage cannot however be ruled out, as shown, for example, by Shiryaev (1998) and Dasgupta (1998). Cheridito (2001) shows that the model is arbitrage-free if trading is restricted such that there is some non-random minimal time interval between successive transactions. However, this is a very restrictive condition.

Despite the lack of the equivalent martingale measure in the path-wise fractional model, it is possible to compute the solution to the fractional Merton model by using the so-called 'weak pricing principle'. This solution is the same as the one obtained in the generalised model of Hu and Øksendal (2003).

In the fractional Merton model, the value E of the firm's equity is given by:

$$E = VN\left(d_1\left(H\right)\right) - De^{-rT}N\left(d_2\left(H\right)\right) \tag{22}$$

with

$$d_1\left(H\right) = \frac{\ln\frac{V}{D} + rT + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{23}$$

$$d_2\left(H\right) = \frac{\ln\frac{V}{D} + rT - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} = d_1\left(H\right) - \sigma T^H \tag{24}$$

For $H = 1/2$ equations (22)-(24) give the Merton's solution to the firm's equity. This is equivalent to the Black-Scholes price of a call option on a non-dividend paying stock.

In the classical Black-Scholes-Merton framework the risk-neutral measure Q, equivalent to the real world measure P, is characterised by the fact that the firm value grows at the risk-free rate:

$$E_Q\left[\frac{V_T}{V_0}\right] = e^{rT} \tag{25}$$

In a fractional setting the martingale property does not hold. However, the Girsanov theorem for fractional Brownian motion provides a unique probability measure, equivalent to P, such that (25) holds. This measure Q is called the average risk-neutral measure and (25) holds in a fractional setting if the process:

$$\tilde{B}_H\left(t\right) = B_H\left(t\right) + \frac{\mu - r}{\sigma}t + \frac{\sigma}{2}t^{2H} \tag{26}$$

is a fractional Brownian motion under Q. The value of the firm equity is given by:

$$E = E_Q\left[E_T e^{-rT}\right] \tag{27}$$

from which we obtain (22). From (22) and the equivalence $V \equiv E + f$, we can derive the value of the firm's debt as:

$$f = V\left[1 - N\left(d_1\left(H\right)\right)\right] + De^{-rT} N\left(d_2\left(H\right)\right) \tag{28}$$

From (11) and (28) we can calculate the value of the spread:

$$s = -\frac{1}{T}\ln\left\{\frac{V}{D\,e^{-rT}}\left[1 - N\left(d_1\left(H\right)\right)\right] + N\left(d_2\left(H\right)\right)\right\} \tag{29}$$

The risk neutral probability of default is:

$$P_{DEF} = 1 - N\left(d_2\left(H\right)\right) \tag{30}$$

Finally, the value of the option to default is equal to:

$$\begin{aligned} O_{DEF} &= -V\left[1 - N\left(d_1\left(H\right)\right)\right] + De^{-rT}\left[1 - N\left(d_2\left(H\right)\right)\right] \\ &= De^{-rT} - f \end{aligned} \tag{31}$$

This is the value of the put option in the fractional Merton model and is equal to the difference between the risk-free debt value and the risky debt value.

4 Sensitivity Analysis

4.1 Sensitivity Analysis Design

One of the strongest critiques to the Merton model is that equation (12) predicts credit spreads that are lower than the spreads observed in the market. This is because the time of default is predictable with increasing accuracy and therefore very short term risky debt carries zero spread when $V > D$ (the put option is out of the money). In this section, we perform a sensitivity analysis exercise of the fractional Merton models. We want to assess whether the fractional Merton model and specifically equation (29) is able to predict credit spreads which are closer to market spreads than those predicted by Merton's model.

We design our sensitivity analysis as in Merton (1974). First, we re-write the fractional Merton model by using the firm leverage:

$$L = \frac{De^{-rT}}{V} \tag{32}$$

According to the Modigliani-Miller theorem, the firm value is invariant to its capital structure (i.e. $V \equiv E + f$). As a consequence, the firm leverage is the only relevant variable. The value E of the firm's equity is[6]:

$$E = V\left[N\left(d_1\right) - LN\left(d_2\right)\right] \tag{33}$$

[6]In what follow we will omit the dependency of the variables on H.

and

$$d_1 = \frac{-\ln L + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{34}$$

$$d_2 = \frac{-\ln L - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} = d_1 - \sigma T^H \tag{35}$$

The value of the firm's debt is:

$$f = V\left[1 - N(d_1) + LN(d_2)\right] = De^{-(r+s)T} \tag{36}$$

the value of the spread:

$$s = -\frac{1}{T}\ln\left\{\frac{1 - N(d_1)}{L} + N(d_2)\right\} \tag{37}$$

and the value of the option to default:

$$O_{DEF} = -V\left[1 - N(d_1)\right] + LV\left[1 - N(d_2)\right]. \tag{38}$$

Expression (30) for the risk neutral probability of default does not change.

Merton (1974) defines leverage as the "quasi" debt-to-firm value ratio and assesses, amongst others, how the spread moves to changes in leverage, firm variance and debt time to maturity. In our basic sensitivity analysis exercise we extend Merton's comparative static analysis to values of $H \neq 1/2$. This will allow us to assess the sensitivity of the spread to the long memory parameter and directly compare the spread predicted by the Merton model (with $H = 1/2$) with the spread predicted by the fractional Merton model for different values of $H \neq 1/2$.

In addition, we examine how the value of the equity (call option), the risky debt, the option to default (put option), and the risk-neutral probability of default respond to changes in H.

4.2 Sensitivity Analysis Results

4.2.1 Spread sensitivity to long memory

Figures 1 and 2 show how the spread reacts to changes in the long memory parameter. For $T>1$ (Figure 1), we notice that the spread monotonically increases with an increase in the long memory parameter. This is true regardless of the firm's leverage. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 0.0493 for $H = 0.5$ and to 0.1685 for $H = 1.5$. This is more than three times higher. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical spread is equal to 0.0207 for $H = 0.5$ and to 0.1338 for $H = 1.5$. This is six and a half times higher.

[Insert Figures 1-2 somewhere here]

This result is extremely interesting and shows that, when we take into account long memory in spreads, the theoretical spread is significantly higher than the spread predicted by Merton's model, i.e. in absence of long memory. The magnitude of the spread is an indicator of the relative riskiness of corporate bonds. In presence of long memory corporate bonds are riskier than in absence of long memory.

From our empirical analysis on credit spreads, we concluded that spreads are long memory processes with fractional integration parameter $d = 1$ ($H = 1.5$). In light of this result, we claim that spreads predicted by Merton's model underestimate market spreads because Merton's model does not assume long memory in spreads. Our results show that long memory is a fundamental factor affecting the magnitude of spreads and spreads predicted by the fractional Merton model with $H = 1.5$ can potentially better explain market spreads.

Although this conclusion is true for $T > 1$, it does not hold for $T \leq 1$. For $T = 1$, the spread remains constant for changes in H. Mathematically, this results from the fact that, for $T = 1$, the theoretical value of the spread in equation (37) does not depend on H any longer. For $T < 1$ (Figure 2), the spread decreases for increases in H. For example, for $T = 6$ months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 0.1156 for $H = 0.5$ and to 0.0577 for $H = 1.5$, approximately half the value. For a firm leverage of 80%, the theoretical spread is equal to 0.0078 for $H = 0.5$ and to $3.98 \cdot 10^{-5}$ for $H = 1.5$. In this case the theoretical difference in spreads can be hardly observed in the market given that spreads are very close to zero.

This result can be explained by the fact that for short maturities the fractional Merton model is affected by the same fundamental issue of Merton's model: given that default can be predicted with increasing accuracy over time, as we approach the debt time to maturity theoretical spreads tend to zero and therefore systematically underestimate market spreads. This issue remains even when spreads are assumed to have long memory properties.

The spread sensitivity to long memory is equal to:

$$\frac{\partial s}{\partial H} = \sigma T^{H-1} \ln T n\left(d_2\right) e^{sT} \tag{39}$$

where:

$$n\left(d_2\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \tag{40}$$

is the standard normal probability distribution function. Equation (39) is consistent with Figures 1 and 2: the spread sensitivity to long memory is positive for $T > 1$, equal to zero for $T = 1$ and negative for $T < 1$.[7]

4.2.2 Spread sensitivity to firm leverage

Figures 3 and 4 show how the spread changes to changes in the firm's leverage. We notice that the spread monotonically increases with an increase in leverage

[7] The full set of proofs of the sensitivities discussed in this Section are given in the Appendix.

for any value of $0 \leq H \leq 1.5$ and $T \geq 0$. This is an expected result and reflects, ceteris paribus, the firm's greater probability of default for higher levels of leverage, which affects the yield demanded by the market on the firm's debt in terms of higher spread. Consistently with our previous results on the spread sensitivity to long memory, we also notice that, for $T > 1$, the higher H, the higher the spread for any level of leverage. The relationship is reversed for $T < 1$. Figures 3 and 4 generalise Figure 1 in Merton (1974) for $H \neq 1/2$.

[**Insert Figures 3-4 somewhere here**]

The spread sensitivity to leverage is equal to:

$$\frac{\partial s}{\partial L} = \frac{1 - N(d_1)}{L^2 T} e^{sT} = \frac{1 - N(d_2)\, e^{sT}}{LT} \tag{41}$$

which is positive.

4.2.3 Spread sensitivity to firm volatility

Figures 5 and 6 show how the spread changes to change in the firm's volatility. As expected, the spread monotonically increases with an increase in volatility for any value of $0 \leq H \leq 1.5$ and $T \geq 0$, reflecting the firm's greater probability of default for higher levels of volatility. We also notice that, for $T > 1$, the higher H, the higher the spread for any level of volatility. For example, for T=3 years, a firm leverage of 100%, a risk-free interest rate of 3% and a firm volatility of 20%, the theoretical spread is equal to 0.049 for $H = 0.5$ and to 0.168 for $H = 1.5$, approximately three and a half times higher. If we change the firm volatility to 50% and keep the other variables constant, the theoretical spread is equal to 0.136 for $H = 0.5$ and to 0.547 for $H = 1.5$, about four times higher. For $T \leq 1$, opposite conclusions apply. For $T = 1$, although the spread increases with volatility, its sensitivity to volatility does not change for changes in H. For $T < 1$, the spread still increases with volatility, however, the relationship between spread and H is reversed: the lower H, the higher the spread for any level of volatility.

The spread sensitivity to the firm volatility is equal to:

$$\frac{\partial s}{\partial \sigma} = n(d_2)\, T^{H-1} e^{sT} \tag{42}$$

which is positive. It interesting to notice that the spread vega and the spread sensitivity to long memory differ by a factor $\sigma \ln T$. Specifically:

$$\frac{\partial s}{\partial H} = \sigma \ln T \frac{\partial s}{\partial \sigma}. \tag{43}$$

[**Insert Figures 5-6 somewhere here**]

4.2.4 Spread sensitivity to firm variance

Merton (1974) plots the changes in the spread against the changes in the firm variance (i.e. volatility squared) rather than volatility.[8] We carry out the same exercise and show the results in Figures 7a and 8a. As expected, the spread increases with the firm variance for any value of H. More specifically, the spread is a monotonically increasing function of the firm variance. In addition, the function is initially convex and subsequently concave. This result is consistent with Figure 2 of Merton (1974). However, in our plots the spread function turns out to be convex only for very small values of the variance and therefore cannot be noticed. Figures 7b and 8b show magnified sections of Figures 7a and 8a for very small values of the variance (<0.00001). These instances are associated with extremely small values of the spread (<0.003) which cannot be empirically observed. Figure 2 of Merton (1974), although theoretically correct, cannot therefore always be empirically tested in its convex region.
The spread sensitivity to the firm variance is equal to:

$$\frac{\partial s}{\partial \sigma^2} = n\left(d_2\right) \frac{T^{H-1}}{2\sigma} e^{sT} \tag{44}$$

which is positive. Its second derivative is:

$$\frac{\partial^2 s}{\left(\partial \sigma^2\right)^2} = \frac{T^{2H-1}}{4\sigma^2} n\left(d_2\right) e^{sT} \left[\frac{2\ln L}{\sigma^3 T^{3H}} - \frac{\sigma T^H}{4} - \frac{1}{\sigma T^H} + n\left(d_2\right) e^{sT} \right] \tag{45}$$

The theoretical value of the spread where the function changes from convex to concave is equal to:

$$s = \frac{1}{T} \ln \left[\frac{1}{n\left(d_2\right)} \left(\frac{\sigma T^H}{4} + \frac{1}{\sigma T^H} - \frac{2\ln L}{\sigma^3 T^{3H}} \right) \right] \tag{46}$$

which can be computed by setting $\frac{\partial^2 s}{\left(\partial \sigma^2\right)^2} = 0$.

[Insert Figures 7-8 somewhere here]

4.2.5 Spread sensitivity to risky debt time to maturity

Figures 9 to 12 show how the spread changes to changes in the debt time to maturity. Spread can increase or decrease with time to maturity. This depends on the specific combination of leverage L and long memory parameter H.

[Insert Figures 9-12 somewhere here]

For $H = 0$ (Figure 9), the spread monotonically decreases with an increase in time to maturity, for any value of L. Also, the spread increases unbounded as time to maturity approaches zero. This result is in contrast with Merton's model (Figure 10), where for $L < 100, s \to 0$ as $T \to 0$ and only for $L \geq 100$.

[8] See Figure 2 in Merton (1974).

$s \to \infty$ as $T \to 0$. For $H = 0.5$, Figure 10 also confirms that Figure 3 in Merton (1974) is incorrect, as pointed out by Lee (1981) and Pitts and Selby (1983). As H increases, we notice two interesting trends. First, spreads do not decrease monotonically any longer with the increase in time to maturity. For $H = 0.5$, they are monotonically decreasing only for $L \geq 100$,for$H = 1$ and $H = 1.5$ they are monotonically increasing for $L \leq 100$, whilst for $L > 100$ spreads decrease first and increase subsequently.Second, spreads do not increase unbounded any longer as $T \to 0$.Specifically, while for $H = 0.5$ spreads increase unbounded when $L \geq 100$, for $H = 1.5$ spreads increase unbounded only for $L > 100$.Results shown have been computed for a firm volatility value of 10% and a risk-free rate of 3%. Our conclusions are robust to changes in these variables. We can visualise the same results by comparing how the spread changes with time to maturity for different value of H**.** We show this in Figures 13 to 17.

[Insert Figures 13-17 somewhere here]

First, we notice that the higher the leverage, at any maturity, the higher the spread. This reflects the market expectation that firms with high leverage are riskier than firms with low leverage and therefore be ara higher spread. Second, the spread is invariant to changes in H for a one-year maturity. Third, for any $T > 1$, spreads for higher values of H are greater than spreads for lower values of H. Again, this result confirms our claim that the fractional Merton model can predict theoretical spreads better than Merton's model when $T > 1$. Fourth, for any $T < 1$, spreads for higher values of H are smaller than spreads for lower values of H. Fifth, as $T \to 0$, all spreads tend to zero or are unbounded with the exception of the spread calculated with $L = 100$ and $H = 1$. In this case the spread has a minimum value of 0.04.

The spread sensitivity to time to maturity is equal to:

$$\frac{\partial s}{\partial T} = \frac{1}{T^2}\left[H\sigma T^H n\left(d_2\right)e^{sT} - sT\right] \tag{47}$$

4.2.6 Equity sensitivity to long memory

Figures 18 and 19 show how the value of the firm equity changes with long memory. For T>1 (Figure 18), the equity value increases with an increase in the long memory parameter, for any value of the leverage. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical equity value is equal to 12.57 for $H = 0.5$ and to 36.26 for $H = 1.5$. This is almost three times higher. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical equity value is equal to 28.36 for $H = 0.5$ and to 53.07 for $H = 1.5$, which is almost twice as much.

This result shows that in the presence of long memory, the theoretical equity value is significantly higher than the equity value predicted by Merton's model. This result is linked to the result for the risky debt sensitivity to long memory, which has an economic interpretation (see next paragraph).

This conclusion does not hold true for $T \leq 1$. For $T = 1$, the equity value as per equation (33) remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 19), the spread decreases for increases in H. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical spread is equal to 5.58 for $H = 0.5$ and to 2.81 for $H = 1.5$, approximately half the value. For a firm leverage of 80%, the theoretical spread is equal to 25.01 for $H = 0.5$ and 24.63 for $H = 1.5$.

The equity sensitivity to long memory is equal to:

$$\frac{\partial E}{\partial H} = V\sigma T^{H} \left(\ln T\right) n\left(d_1\right) \tag{48}$$

[Insert Figures 18-19 somewhere here]

4.2.7 Risky debt sensitivity to long memory

For T>1, the risky debt value decreases with an increase in the long memory parameter, for any value of leverage. This is illustrated in Figure 20. For instance, for $T = 3$ years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical debt value is equal to 78.82 for $H = 0.5$ and to 55.13 for $H = 1.5$. When the firm leverage is changed to 80% and the other variables remain constant, the theoretical debt value is equal to 85.88 for $H = 0.5$ and to 61.17 for $H = 1.5$.
Ceteris paribus, long memory reduces the value of the risky debt. This is a direct consequence of the fact that long memory increases the value of the spread. In addition, as the value of risky debt is a decreasing function of long memory, the equity value increases with long memory given that $V \equiv E + f$.

Again, the same conclusion does not hold true for $T \leq 1$. For $T = 1$, the risky debt value as per equation (36) remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 21), the risky debt increases for increases in H. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the theoretical debt value is equal to 92.92 for $H = 0.5$ and to 95.69 for $H = 1.5$. For a firm leverage of 80%, the theoretical debt value is equal to 25.01 for $H = 0.5$ and 24.63 for $H = 1.5$.

Given the identity $V \equiv E + f$, the risky debt sensitivity to long memory is equal to minus the equity sensitivity to long memory:

$$\frac{\partial f}{\partial H} = -V\sigma T^{H} \left(\ln T\right) n\left(d_1\right) = -\frac{\partial E}{\partial H} \tag{49}$$

[Insert Figures 20-21 somewhere here]

4.2.8 Sensitivity of the risk-neutral default probability to long memory

Figures 22 and 23 show how the risk neutral default probability changes with long memory.

For $T > 1$ (Figure 22), default probability increases with long memory for $L \leq 100$,whilst first decreases and then increases for L>100. As one would expect, lower values of leverage are associated with lower probabilities of default. It is interesting to notice that, as the long memory parameter increases, probabilities of default for different values of leverage increase and converge to a common value. This suggests that the firm leverage is not as much relevant a variable as in absence of long memory. The increase in default probability with long memory is a direct consequence of the increase in the spread.
For instance, for T=3 years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the probability of default is equal to 56.88% for $H = 0.5$ and to 69.84% for $H = 1.5$. When the firm leverage is changed to 80% and the other variables remain constant, the probability of default is equal to 31.89% for $H = 0.5$ and to 61.99% for $H = 1.5$.
This conclusion does not hold true for $T \leq 1$. For $T = 1$, the default probability remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 23), the default probability decreases with long memory for $L \leq 100$ and increases for L>100. As the long memory parameter increases, the probability of default only increases for highly leveraged firms. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the probability of default is equal to 52.83% for $H = 0.5$ and to 51.43% for $H = 1.5$. For a firm leverage of 80%, the probability of default is equal to 0.07% for $H = 0.5$ and 0.001% for $H = 1.5$.

The sensitivity of the risk neutral default probability to long memory is equal to:

$$\frac{\partial PD}{\partial H} = -d_2 (\ln T)\, n\,(d_2) \tag{50}$$

[Insert Figures 22-23 somewhere here]

4.2.9 Sensitivity of the option to default to long memory

Figures 24 and 25 show how the value of the option to default changes with long memory.
For $T > 1$ (Figure 24), the option to default increases with long memory for any value of the leverage. For instance, for T=3 years, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the value of the option to default is equal to 12.57 for $H = 0.5$ and to 36.26 for $H = 1.5$, almost three times larger. When the firm leverage is changed to 80% and the other variables remain constant, the value of the option to default is equal to 5.51 for $H = 0.5$ and to 30.22 for $H = 1.5$, almost six times larger.

The increase of the value of the option to default is a direct consequence of the increase in the spread, which reduces the value of the risky debt. The higher the value of the option to default the higher the likelihood of exercise, which is consistent with a higher probability of default.

This conclusion does not hold true for $T \leq 1$. For $T = 1$, the option to default remains constant for changes in H as it does not depend on H any longer. For $T < 1$ (Figure 25), the option to default decreases with long memory for any value of the leverage. For example, for T=6 months, a firm volatility of 20%, a risk-free interest rate of 3% and a firm leverage of 100%, the value of the option to default is equal to 5.58 for $H = 0.5$ and to 2.81 for $H = 1.5$. For a firm leverage of 80%, the value of the option to default is equal to 0.39 for $H = 0.5$ and 0.002 for $H = 1.5$.

The sensitivity of the option to default to long memory is the same as the sensitivity of the equity to long memory:

$$\frac{\partial O_{DEF}}{\partial H} = V\sigma T^{H} (\ln T)\, n\,(d_1) = \frac{\partial E}{\partial H} \tag{51}$$

[Insert Figures 24-25 somewhere here]

5 Conclusions

Our empirical research from companion works (Della Ratta and Urga, 2007) indicates that credit spreads are long memory processes with $H = 1.5$. In this paper we develop the theoretical framework of the fractional Merton model, which allows us to embed long memory in a straightforward manner in a credit risk pricing model.

We carry out a sensitivity analysis exercise and compute the spread sensitivities to the long memory parameter, firm leverage, firm volatility and variance, and risky debt time to maturity. We also compute sensitivities of the equity, risky debt, risk-neutral default probability and option to default to long memory.

We find that, for T>1, the spread monotonically increases with an increase in the long memory parameter. This result is extremely interesting and shows that, when we take into account long memory in spreads, the theoretical spread is significantly higher than the spread predicted by Merton's model. The magnitude of the spread is an indicator of the relative riskiness of corporate bonds. In presence of long memory corporate bonds are riskier than in absence of long memory.

Given that spreads are long memory processes with $H = 1.5$, we claim that spreads predicted by Merton's model (i.e. with $H = 0.5$) typically underestimate market spreads because Merton's model does not assume long memory in spreads. Our results show that long memory is a fundamental factor affecting the magnitude of spreads and spreads predicted by the fractional Merton model with $H = 1.5$ can better explain market spreads.

This result does not hold for $T \leq 1$, where the spread decreases with an increase in long memory. This result can be explained by the fact that, for short maturities the fractional Merton model is affected by the same fundamental issue of Merton's model: given that default can be predicted with increasing accuracy over time, as we approach the debt time to maturity theoretical spreads tend to zero and therefore systematically underestimate market spreads.

We also find that the spread monotonically increases with firm leverage and firm volatility. This is an expected result and reflects the firm's greater probability

of default for higher levels of leverage or volatility, which affects the yield demanded by the market on the firm's debt in terms of higher spread.
The relationship between spread and time to maturity is more complex. For $H = 0$ the spread monotonically decreases with an increase in time to maturity. For $H = 1.5$ the spread monotonically increases with an increase in time to maturity but only for $L \leq 1$. For $L > 1$, the spread decreases initially and increases subsequently. These results are very interesting as show that long memory correctly accounts for the greater riskiness of the firm's debt for longer maturities. This is in contrast with short memory, which incorrectly suggests that longer maturities are associated with lower spreads and therefore a lower default risk.
Regarding the other sensitivities to long memory, we find that, for $T > 1$, long memory reduces the value of the risky debt. This is a direct consequence of the fact that long memory increases the value of the spread. In addition, as the value of risky debt is a decreasing function of long memory, the equity value increases with long memory given that $V \equiv E + f$.
For $T > 1$, the risk neutral default probability increases with long memory for $L \leq 100$,whilst first decreases and then increases for L>100. As one would expect, lower values of leverage are associated with lower probabilities of default. It is interesting to notice that, as the long memory parameter increases, probabilities of default for different values of leverage increase and converge to a common value. This suggests that the firm leverage is not as much relevant a variable as in absence of long memory. The increase in default probability with long memory is a direct consequence of the increase in the spread.
Finally, for $T > 1$ the option to default increases with long memory for any value of the leverage. This is a direct consequence of the increase in the spread, which reduces the value of the risky debt. The higher the value of the option to default the higher the likelihood of exercise, which is consistent with a higher probability of default.

We have shown that theoretical spreads of the fractional Merton model are higher than the spreads predicted by the Merton model. This is especially true for values of the fractional difference parameter in line with our empirical results (Della Ratta and Urga, 2007). This is a promising conclusion in favour of that the fractional Merton model can explain market spreads better than the Merton model. This needs to be complemented by additional and comprehensive empirical exercises to compare our theoretical results with company level spreads data. While we leave this to future research, our initial findings based on using average market leverage Standard and Poor data (Adjusted Key US Industrial Financial Ratios 2000-2002) per rating class to calculate implied H values for different maturities, volatilities and values of the risk-free rate suggest that values of $H = 1.5$ or, more generally, values of $1 < H < 1.5$, are consistent with the observed market leverage for all combination of rating classes and maturities.

APPENDIX:
Proofs of Spread and Long Memory Sensitivities

A. Spread Sensitivities of the Fractional Merton Model

In this section we give the proofs of the spread sensitivities of the fractional Merton model. We repeat the key formulas for easy reference:

$$s = -\frac{1}{T}\ln\left\{\frac{1-N(d_1)}{L} + N(d_2)\right\} \tag{52}$$

$$d_1 = \frac{-\ln L + \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} \tag{53}$$

$$d_2 = \frac{-\ln L - \frac{1}{2}\sigma^2 T^{2H}}{\sigma T^H} = d_1 - \sigma T^H \tag{54}$$

$$n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \tag{55}$$

$$n(d_2) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \tag{56}$$

In order to find the partial derivatives, the following relationships are useful.

Lemma 1

Under the assumptions of the fractional Merton model, the following relationship holds:

$$1 - N(d_1) + LN(d_2) = Le^{-sT} \tag{57}$$

Proof. Immediate from (52).

Lemma 2

Under the assumptions of the fractional Merton model, the following relationship holds:

$$n(d_1) = Ln(d_2) \tag{58}$$

Proof. From (55) and (56) we have:

$$\begin{aligned}\frac{n(d_1)}{n(d_2)} &= e^{\frac{1}{2}\left(d_2^2 - d_1^2\right)} \\ &= e^{\frac{1}{2}2\ln L} \\ &= L\end{aligned}$$

In the following, we give proof of the spread sensitivities to long memory, leverage, volatility, variance and time to maturity. We first prove the expression for the general form of the first derivative of (52) with respect to the variable x.

Theorem 1. General spread sensitivity formula

Let (52) hold. Then the general form of the derivative of (52) with respect to x is:

$$\frac{\partial s}{\partial x} = -sT\frac{\partial\left(-\frac{1}{T}\right)}{\partial x} - \left[n\left(d_2\right)\left(\frac{\partial d_2}{\partial x} - \frac{\partial d_1}{\partial x}\right) - \frac{1-N\left(d_1\right)}{L^2}\frac{\partial L}{\partial x}\right]\frac{e^{sT}}{T} \tag{59}$$

Proof. By applying the derivative of a product rule to (52) we have:

$$\frac{\partial s}{\partial x} = \frac{\partial\left(-\frac{1}{T}\right)}{\partial x}\ln\left(\frac{1-N\left(d_1\right)}{L} + N\left(d_2\right)\right) - \frac{1}{T}\frac{\partial \ln\left(\frac{1-N(d_1)}{L} + N\left(d_2\right)\right)}{\partial x}$$

Now, from Lemma 1:

$$\ln\left(\frac{1-N\left(d_1\right)}{L} + N\left(d_2\right)\right) = -sT$$

By using the derivative of a logarithm of a function rule and lemma 3, we have:

$$\begin{aligned}\frac{\partial \ln\left(\frac{1-N(d_1)}{L} + N\left(d_2\right)\right)}{\partial x} &= \left\{\frac{-n\left(d_1\right)\frac{\partial d_1}{\partial x}L - \left[1-N\left(d_1\right)\right]\frac{\partial L}{\partial x}}{L^2} + n\left(d_2\right)\frac{\partial d_2}{\partial x}\right\}e^{sT} \\ &= \left[n\left(d_2\right)\left(\frac{\partial d_2}{\partial x} - \frac{\partial d_1}{\partial x}\right) - \frac{1-N\left(d_1\right)}{L^2}\frac{\partial L}{\partial x}\right]e^{sT}\end{aligned}$$

which gives (59) after the appropriate substitutions.

Corollary 1. Spread sensitivity to long memory

Let $x = H$ in Theorem 1. Then the following result holds:

$$\frac{\partial s}{\partial H} = \sigma T^{H-1}\ln T n\left(d_2\right)e^{sT} \tag{60}$$

Proof. We have:

$$\begin{aligned}\frac{\partial d_1}{\partial H} &= -\frac{\ln L \ln T}{\sigma T^H} + \frac{1}{2}\ln T \sigma T^H \\ \frac{\partial d_2}{\partial H} &= -\frac{\ln L \ln T}{\sigma T^H} - \frac{1}{2}\ln T \sigma T^H \\ \frac{\partial L}{\partial H} &= \frac{\partial\left(-1/T\right)}{\partial H} = 0\end{aligned}$$

which, after substituting in (59), immediately gives (60).

Corollary 2. Spread sensitivity to firm leverage

Let $x = L$ in Theorem 1. Then the following result holds:

$$\frac{\partial s}{\partial L} = \frac{1 - N(d_1)}{L^2 T} e^{sT} \tag{61}$$

Proof. We have:

$$\frac{\partial d_1}{\partial L} = \frac{\partial d_2}{\partial L} = -\frac{1}{L\sigma T^H}$$

$$\frac{\partial L}{\partial L} = 1$$

$$\frac{\partial(-1/T)}{\partial L} = 0$$

which, after substituting in (59), immediately gives (61). Using Lemma 1, the spread sensitivity to firm leverage can also be written:

$$\frac{\partial s}{\partial L} = \frac{1 - N(d_2)\, e^{sT}}{LT} \tag{62}$$

Corollary 3. Spread sensitivity to firm volatility

Let $x = \sigma$ in Theorem 1. Then the following result holds:

$$\frac{\partial s}{\partial \sigma} = n(d_2)\, T^{H-1} e^{sT} \tag{63}$$

Proof. We have

$$\frac{\partial d_1}{\partial \sigma} = \frac{\ln L}{\sigma^2 T^H} + \frac{1}{2} T^H$$

$$\frac{\partial d_2}{\partial \sigma} = \frac{\ln L}{\sigma^2 T^H} - \frac{1}{2} T^H$$

$$\frac{\partial L}{\partial \sigma} = \frac{\partial(-1/T)}{\partial \sigma} = 0$$

which immediately gives (63) after substituting in (59).

Corollary 4. Spread sensitivity to firm variance

Let $x = \sigma^2$ in Theorem 1. Then the following result holds:

$$\frac{\partial s}{\partial \sigma^2} = n(d_2)\, \frac{T^{H-1}}{2\sigma} e^{sT} \tag{64}$$

Proof. We use the transformation $\sigma^2 = x$, calculate $\frac{\partial d_1}{\partial x}$ and $\frac{\partial d_2}{\partial x}$ and substitute back $x = \sigma^2$. We have:

$$\begin{aligned}\frac{\partial d_1}{\partial \sigma^2} &= \frac{1}{2\sigma^2}\left(\frac{\ln L}{\sigma T^H} + \frac{1}{2}\sigma T^H\right) \\ &= -\frac{1}{2\sigma^2} d_2\end{aligned}$$

$$\frac{\partial d_2}{\partial \sigma^2} = \frac{1}{2\sigma^2}\left(\frac{\ln L}{\sigma T^H} - \frac{1}{2}\sigma T^H\right)$$
$$= -\frac{1}{2\sigma^2} d_1$$

$$\frac{\partial L}{\partial \sigma^2} = \frac{\partial \left(-1/T\right)}{\partial \sigma^2} = 0$$

which immediately gives (64) after substituting in (59).

Corollary 5. Second derivative of spread with respect to firm variance

Let (64) hold. Then the following result holds:

$$\frac{\partial^2 s}{\left(\partial \sigma^2\right)^2} = \frac{T^{2H-1}}{4\sigma^2} n\left(d_2\right) e^{sT}\left[\frac{2\ln L}{\sigma^3 T^{3H}} - \frac{\sigma T^H}{4} - \frac{1}{\sigma T^H} + n\left(d_2\right) e^{sT}\right] \tag{65}$$

Proof. We set $\sigma^2 = x$ and compute the derivative of (64) with respect to x. We have:

$$\frac{\partial^2 s}{\left(\partial x\right)^2} = \frac{T^{H-1}}{2}\left(\frac{\frac{\partial n(d_2)}{\partial x} x^{-\frac{1}{2}} - n\left(d_2\right)\frac{\partial \sqrt{x}}{\partial x}}{x} e^{sT} + \frac{n\left(d_2\right)}{x}\frac{\partial e^{sT}}{\partial x}\right)$$

We have:

$$\frac{\partial n\left(d_2\right)}{\partial x} = \frac{n\left(d_2\right)}{2x}\left(\frac{2\ln L}{xT^{2H}} - \frac{xT^{2H}}{4}\right)$$
$$\frac{\partial \sqrt{x}}{\partial x} = \frac{1}{2\sqrt{x}}$$
$$\frac{\partial e^{sT}}{\partial x} = n\left(d_2\right)\frac{T^H}{2\sqrt{x}} e^{2sT}$$

By subsituting back $x = \sigma^2$ and simplifying, we obtain (65).

Corollary 6. Spread sensitivity to risky debt time to maturity

Let $x = T$ in Theorem 1. Then the following result holds:

$$\frac{\partial s}{\partial T} = \frac{1}{T^2}\left[H\sigma T^H n\left(d_2\right) e^{sT} - sT\right] \tag{66}$$

Proof. We have:

$$\frac{\partial d_1}{\partial T} = \frac{H\ln L}{\sigma T^{H+1}} + \frac{1}{2}\sigma H T^{H-1}$$

$$\frac{\partial d_2}{\partial T} = \frac{H\ln L}{\sigma T^{H+1}} - \frac{1}{2}\sigma H T^{H-1}$$

After calculating the other relevant derivatives and substituting into (A.8) we obtain:

$$\frac{\partial s}{\partial T} = -sT\frac{1}{T^2} - n\left(d_2\right)\left(-H\sigma T^{H-1}\right)\frac{e^{sT}}{T}$$

which gives (66).

B. Long Memory Sensitivities to the Fractional Merton Model

This section gives the proofs of the sensitivities to the long memory parameter H of the fractional Merton model. We repeat the key formulas for easy reference:

$$E = V\left[N\left(d_1\right) - LN\left(d_2\right)\right] \tag{67}$$

$$f = V\left[1 - N\left(d_1\right) + LN\left(d_2\right)\right] = De^{-(r+s)T} \tag{68}$$

$$PD = 1 - N\left(d_2\right) \tag{69}$$

$$O_{DEF} = -V\left[1 - N\left(d_1\right)\right] + LV\left[1 - N\left(d_2\right)\right] \tag{70}$$

In order to find the partial derivatives, the following Lemma is useful.

Lemma 3

Under the assumptions of the fractional Merton model, the following relationship holds:

$$d_2^2 - d_1^2 = 2\ln L \tag{71}$$

Proof. From (53) and (54 we have:

$$d_2 - d_1 = -\sigma T^H \tag{72}$$

and

$$d_2 + d_1 = -\frac{2\ln L}{\sigma T^H} \tag{73}$$

which gives:

$$d_2^2 - d_1^2 = \left(d_2 + d_1\right)\left(d_2 - d_1\right) = 2\ln L$$

Theorem 2. Equity sensitivity to long memory

Let (67) hold. Then the following result holds:

$$\frac{\partial E}{\partial H} = V\sigma T^H\left(\ln T\right)n\left(d_1\right) \tag{74}$$

Proof. From (67) we have:

$$\frac{\partial E}{\partial H} = V\left(n\left(d_1\right)\frac{\partial d_1}{\partial H} - Ln\left(d_2\right)\frac{\partial d_2}{\partial H}\right)$$

From Lemma 2 and the expressions for $\frac{\partial d_1}{\partial H}$and $\frac{\partial d_2}{\partial H}$ in Corollary 1 we obtain (74).

Theorem 3. Risky debt sensitivity to long memory

Let (68) hold. Then the following result holds:

$$\frac{\partial f}{\partial H} = -V\sigma T^{H}\left(\ln T\right)n\left(d_1\right) \tag{75}$$

Proof. From (68) we have:

$$\frac{\partial E}{\partial H} = V\left(-n\left(d_1\right)\frac{\partial d_1}{\partial H} + Ln\left(d_2\right)\frac{\partial d_2}{\partial H}\right)$$

From Lemma 2 and the expressions for $\frac{\partial d_1}{\partial H}$ and $\frac{\partial d_2}{\partial H}$ in Corollary 1 we obtain (75).

Theorem 4. Sensitivity of the risk-neutral probability of default to long memory

Let (69) hold. Then the following result holds:

$$\frac{\partial PD}{\partial H} = -d_2\left(\ln T\right)n\left(d_2\right) \tag{76}$$

Proof. From (69) we have:

$$\frac{\partial PD}{\partial H} = -n\left(d_2\right)\frac{\partial d_2}{\partial H}$$

By using the expression for $\frac{\partial d_2}{\partial H}$ in Corollary 1 and equations (72) and (73) in Lemma 3 we obtain (76).

Theorem 5. Sensitivity of the option to default to long memory

Let (70) hold. Then the following result holds:

$$\frac{\partial O_{DEF}}{\partial H} = V\sigma T^{H}\left(\ln T\right)n\left(d_1\right) \tag{77}$$

Proof. From (70) we have:

$$\frac{\partial O_{DEF}}{\partial H} = V\left(n\left(d_1\right)\frac{\partial d_1}{\partial H} - Ln\left(d_2\right)\frac{\partial d_2}{\partial H}\right)$$

By using Lemma 2 and the expressions for $\frac{\partial d_1}{\partial H}$ and $\frac{\partial d_2}{\partial H}$ in Corollary 1 we obtain (77).

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Figure 1 – Spread Sensitivity to Long Memory (T>1)

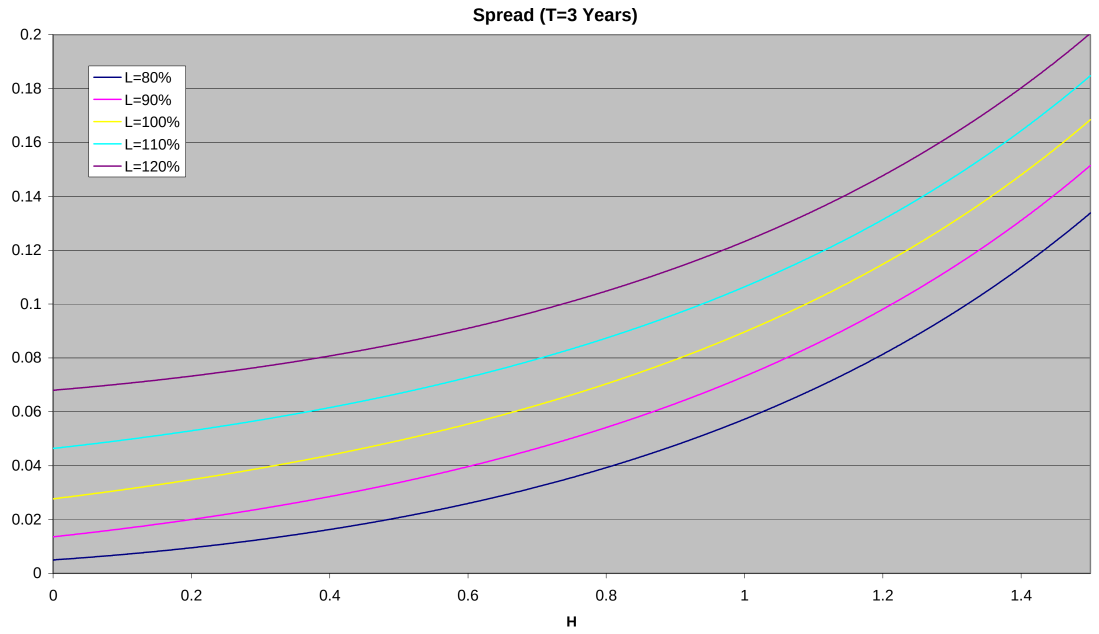

Figure 2 – Spread Sensitivity to Long Memory (T<1)

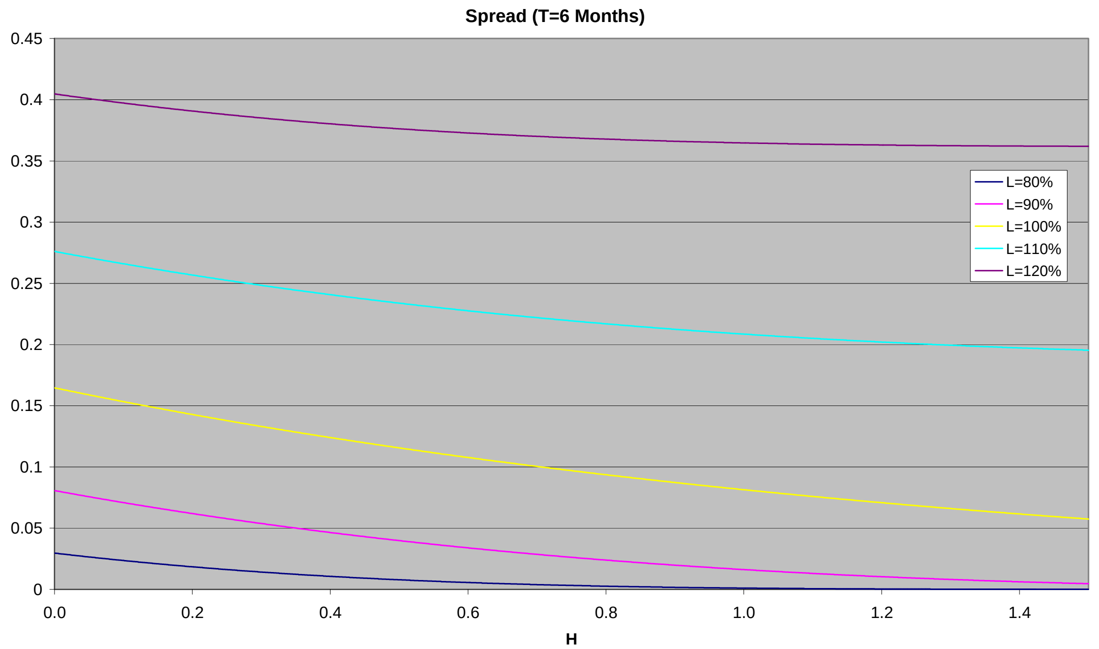

Figure 3 – Spread Sensitivity to Firm Leverage (T>1)

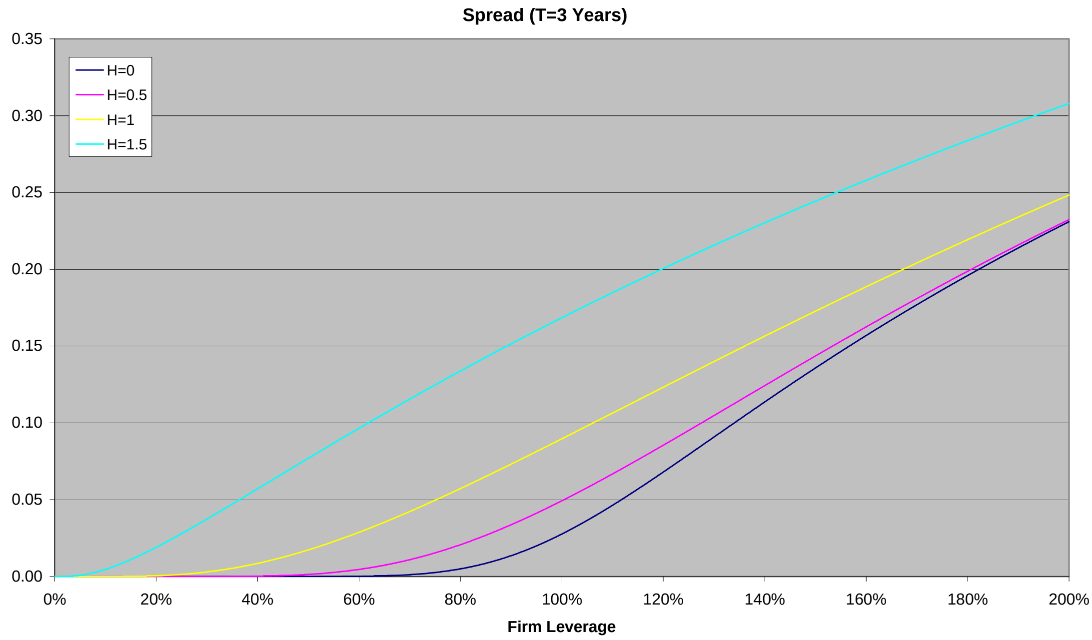

Figure 4 – Spread Sensitivity to Firm Leverage (T<1)

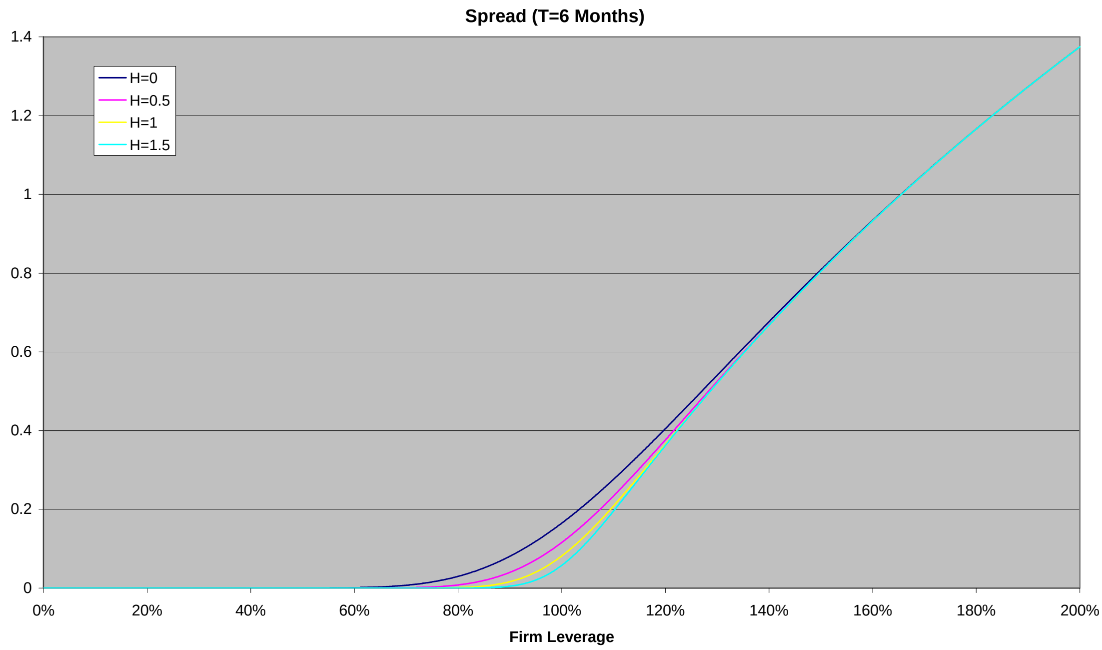

Figure 5 – Spread Sensitivity to Firm Volatility (T>1)

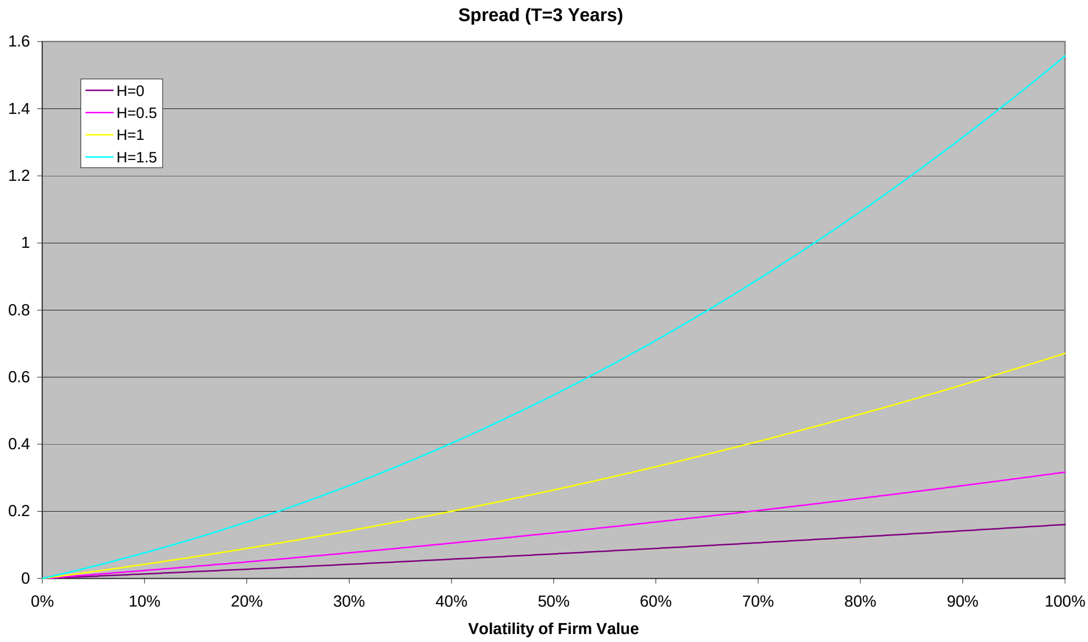

Figure 6 – Spread Sensitivity to Firm Volatility (T<1)

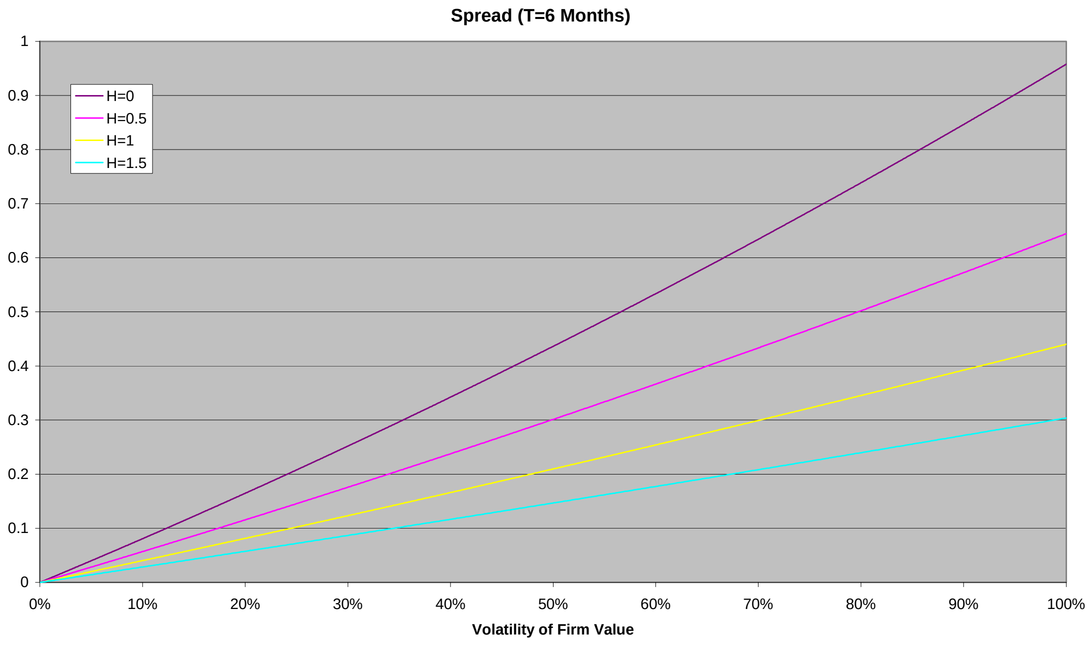

Figure 7a – Spread Sensitivity to Firm Variance (T>1)

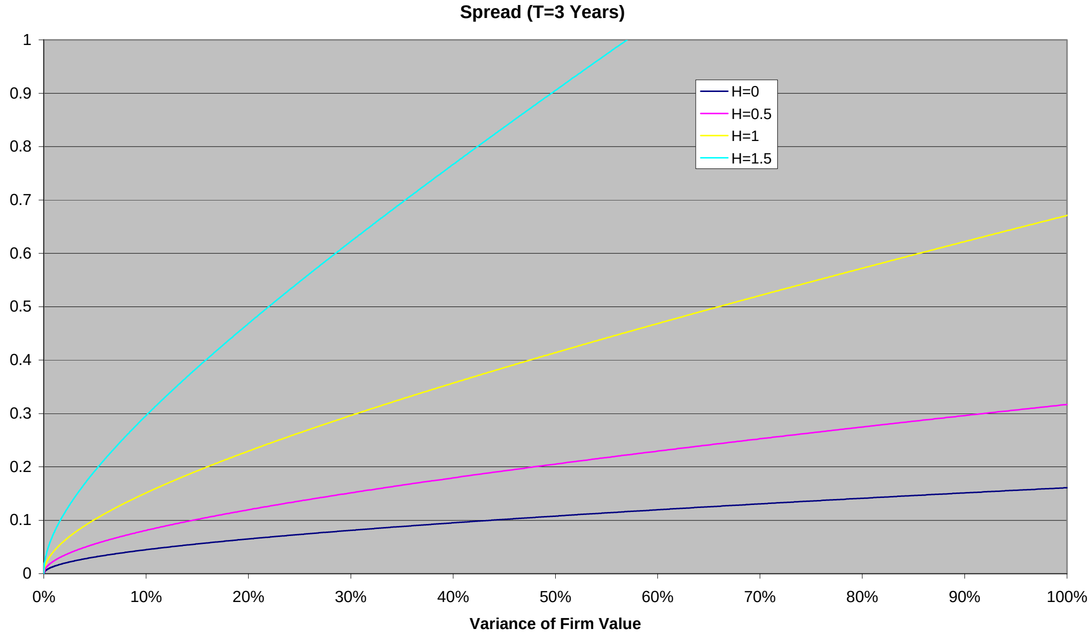

Figure 8a – Spread Sensitivity to Firm Variance (T<1)

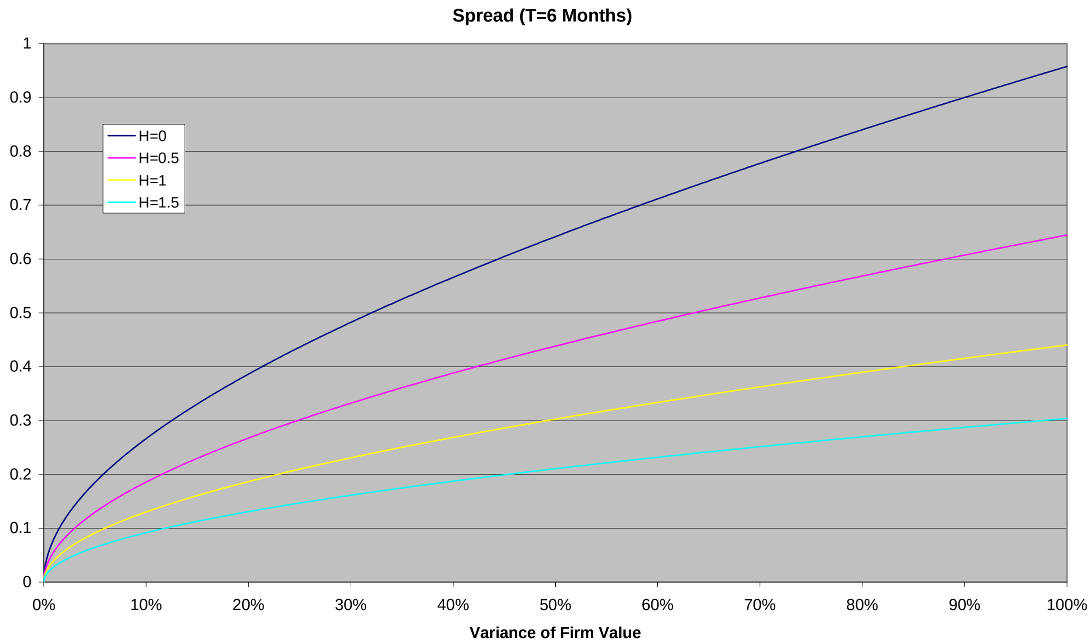

Figure 7b – Spread Sensitivity to Firm Variance (T>1)

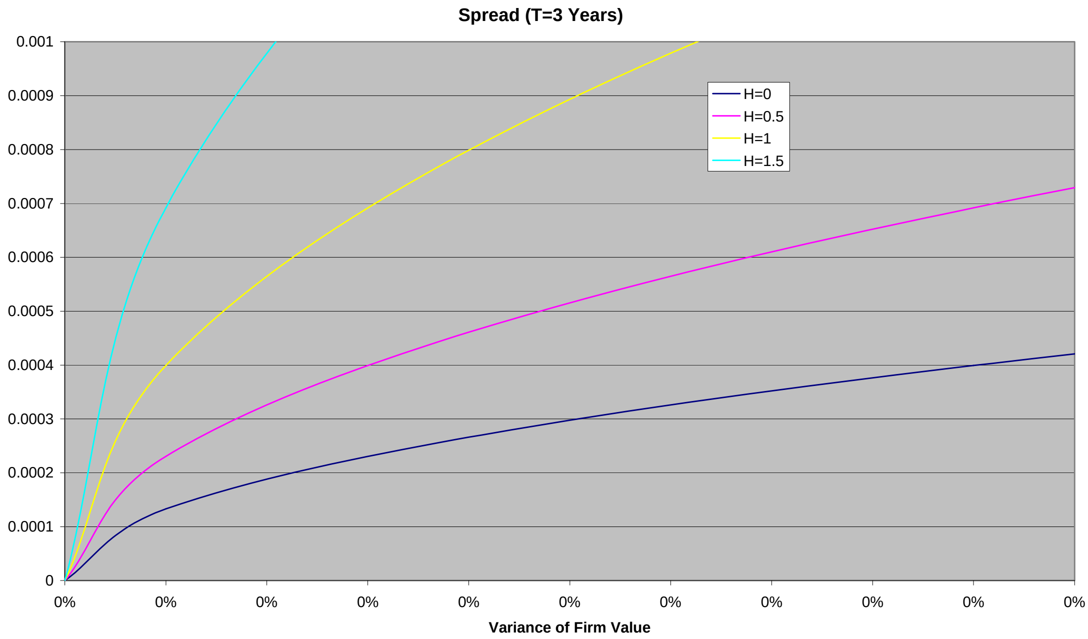

Figure 8b – Spread Sensitivity to Firm Variance (T<1)

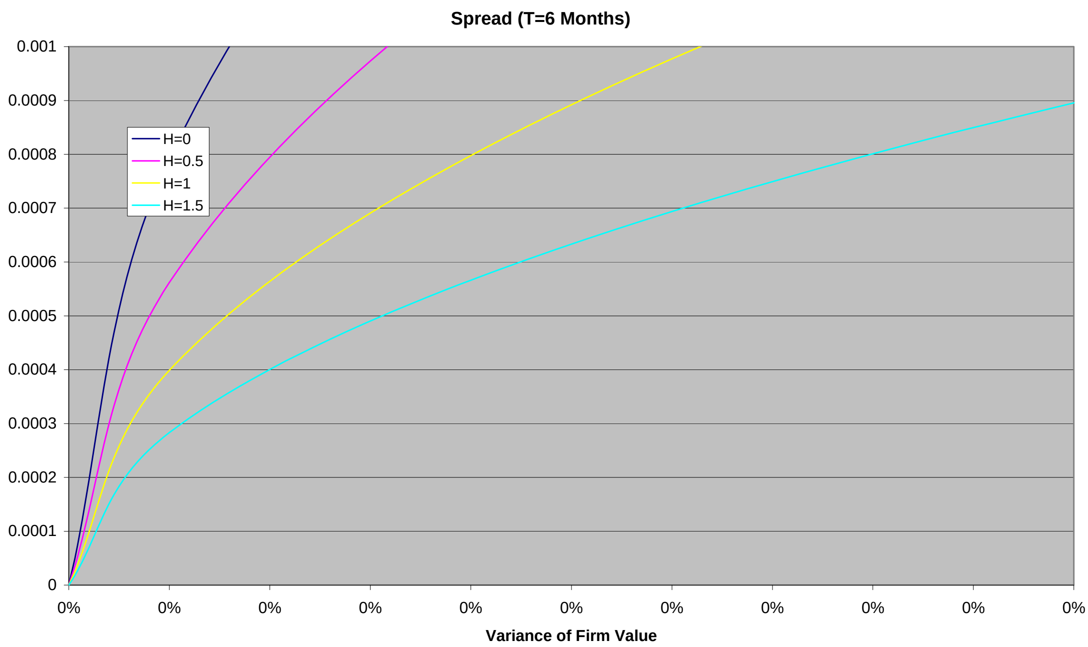

Figure 9 – Spread Sensitivity to Time to Maturity (H=0)

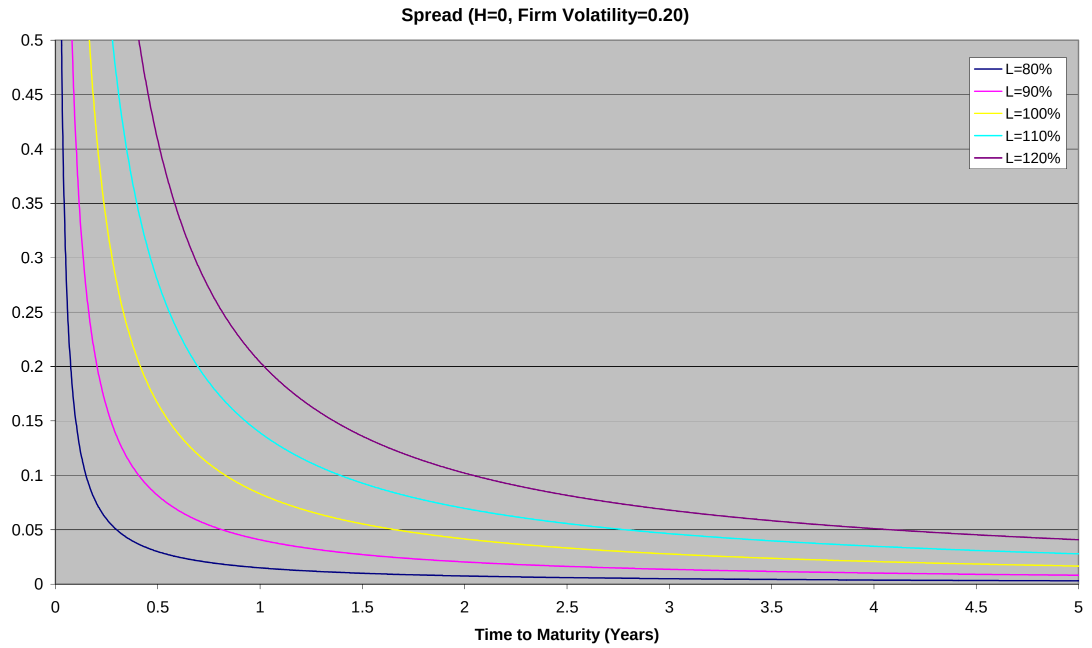

Figure 10 – Spread Sensitivity to Time to Maturity (H=0.5)

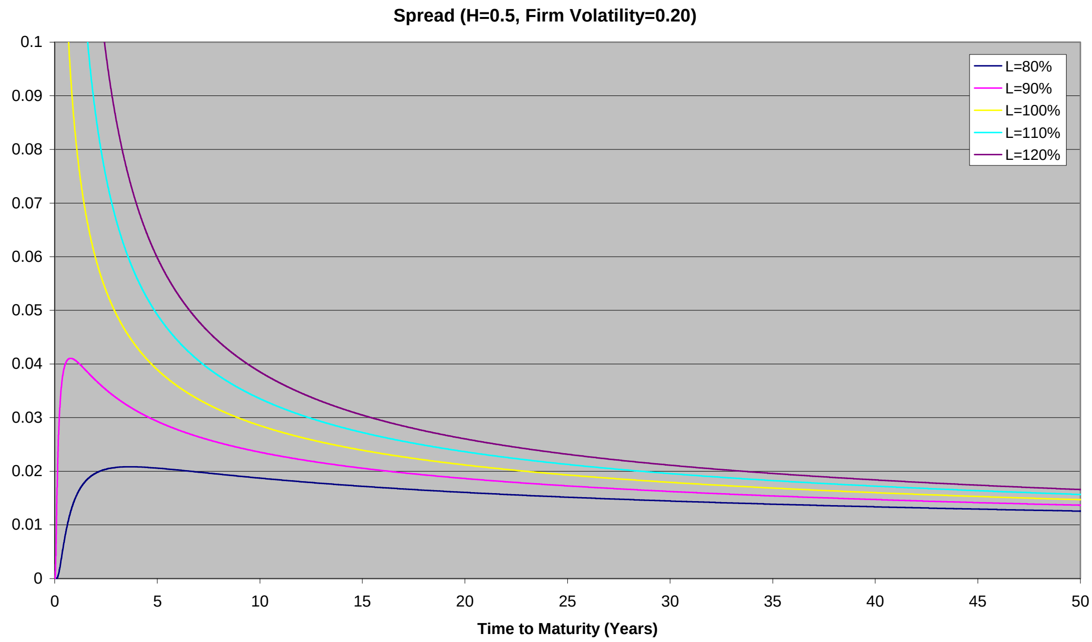

Figure 11 – Spread Sensitivity to Time to Maturity (H=1)

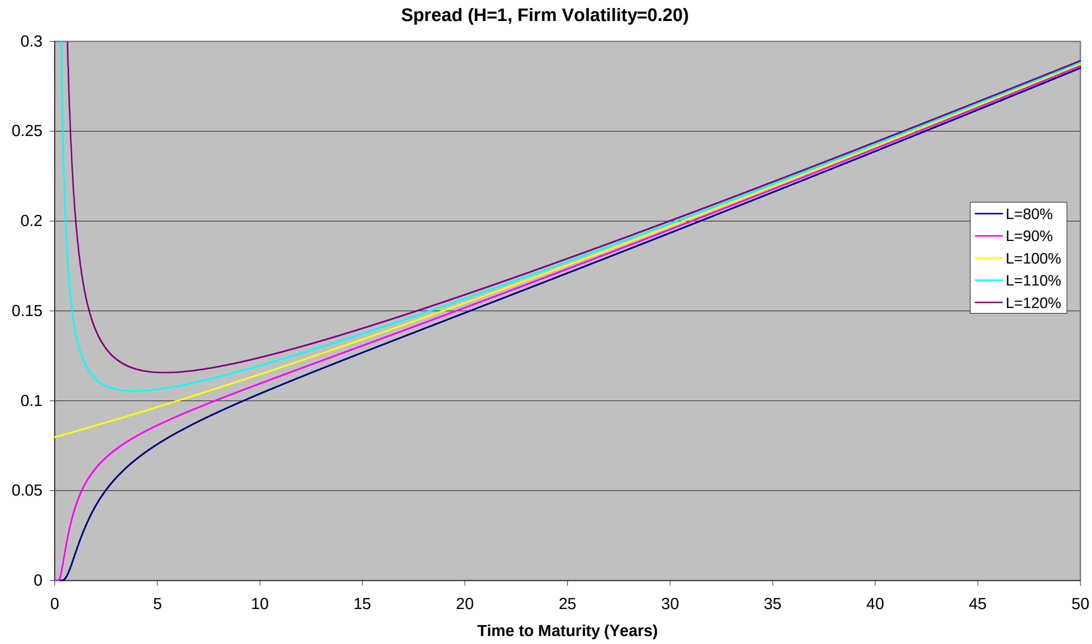

Figure 12 – Spread Sensitivity to Time to Maturity (H=1.5)

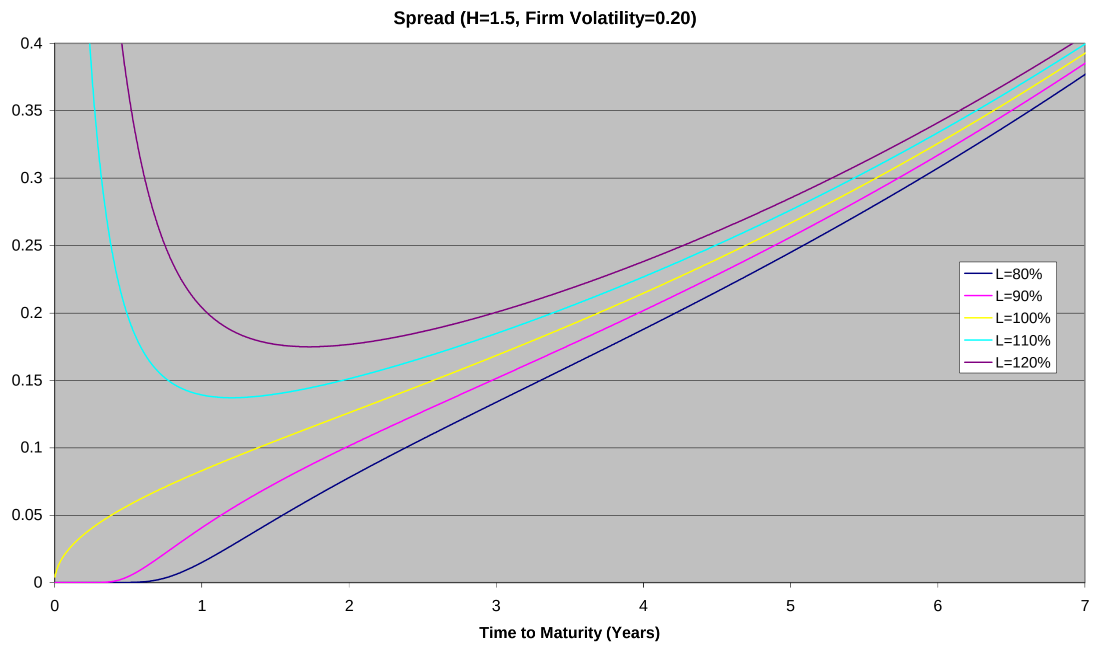

Figure 13 – Spread Sensitivity to Time to Maturity (L=80%)

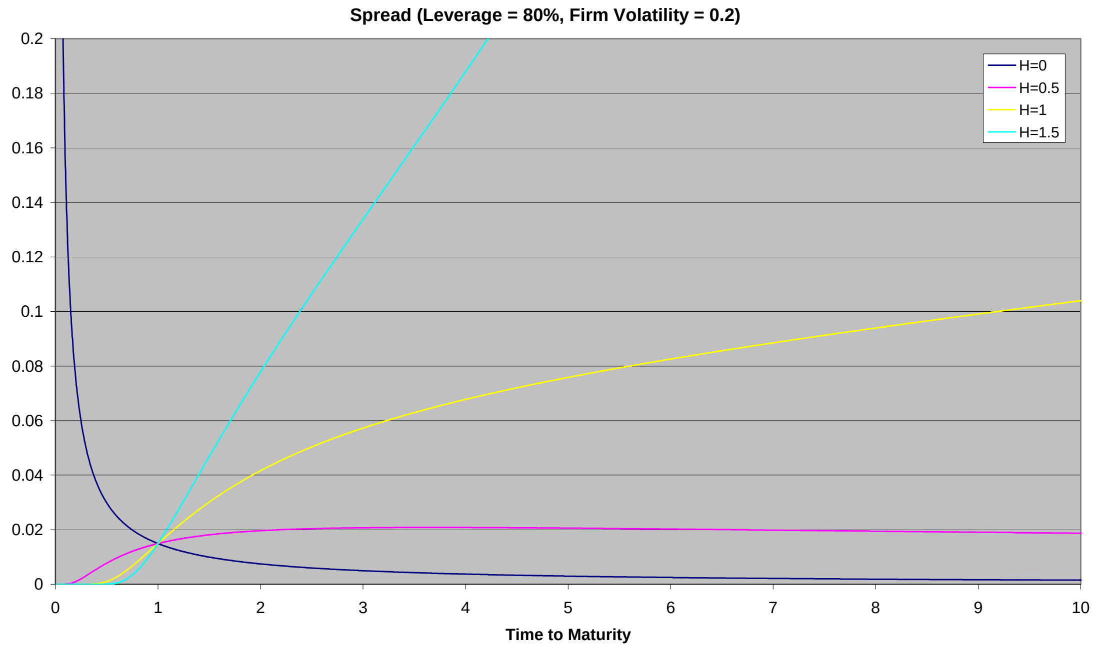

Figure 14 – Spread Sensitivity to Time to Maturity (L=90%)

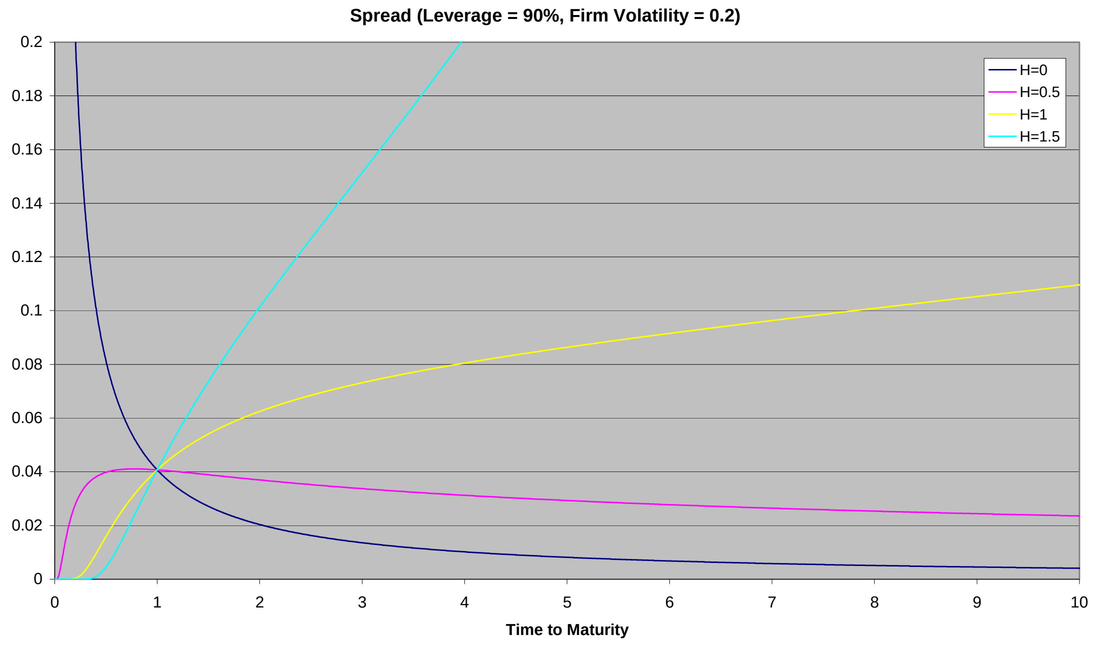

Figure 15 – Spread Sensitivity to Time to Maturity (L=100%)

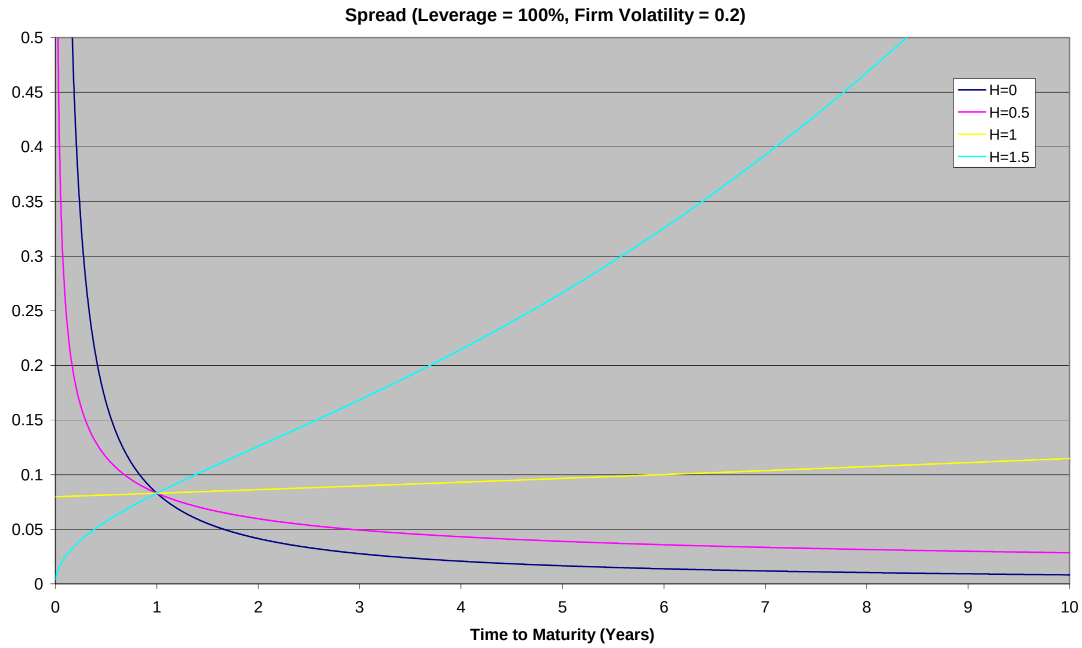

Figure 16 – Spread Sensitivity to Time to Maturity (L=110%)

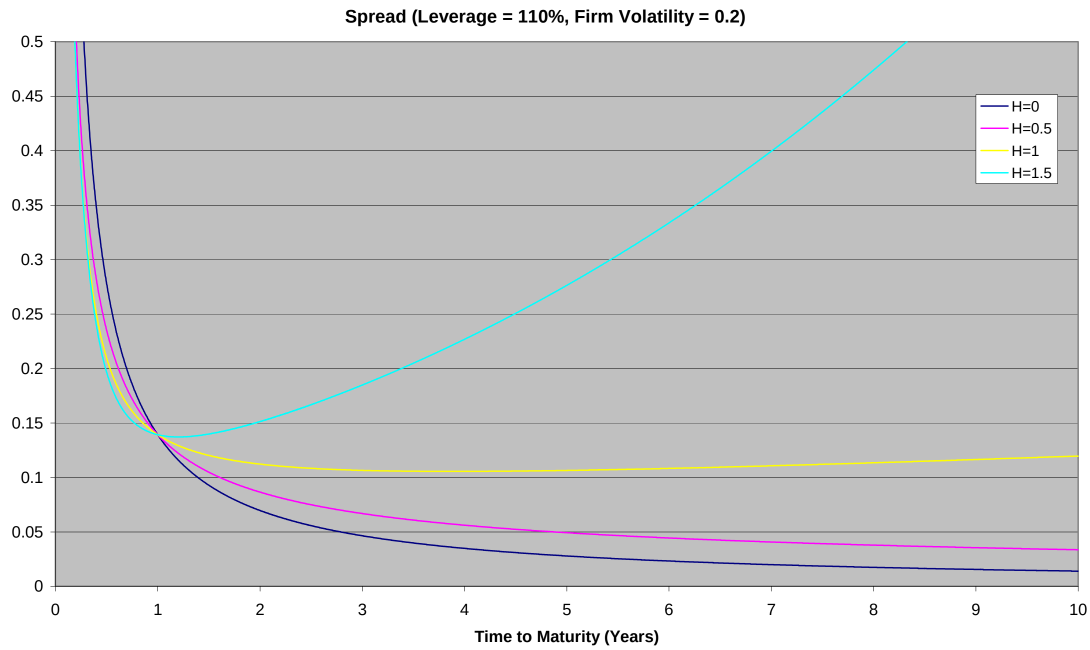

Figure 17 – Spread Sensitivity to Time to Maturity (L=120%)

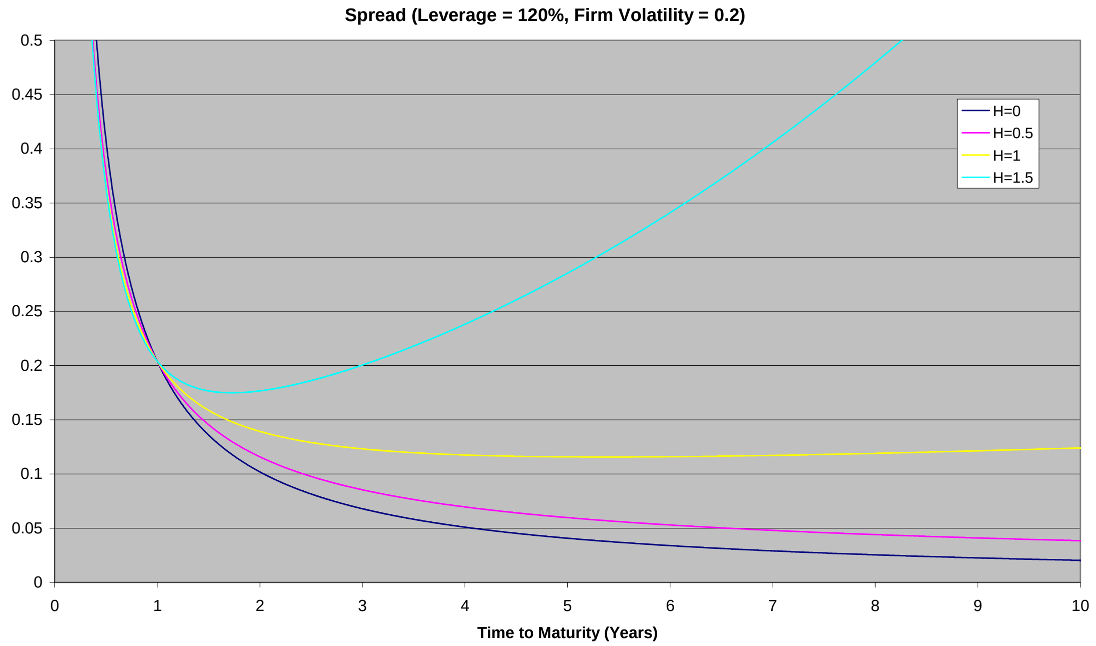

Figure 18 – Firm Equity Sensitivity to Long Memory (T>1)

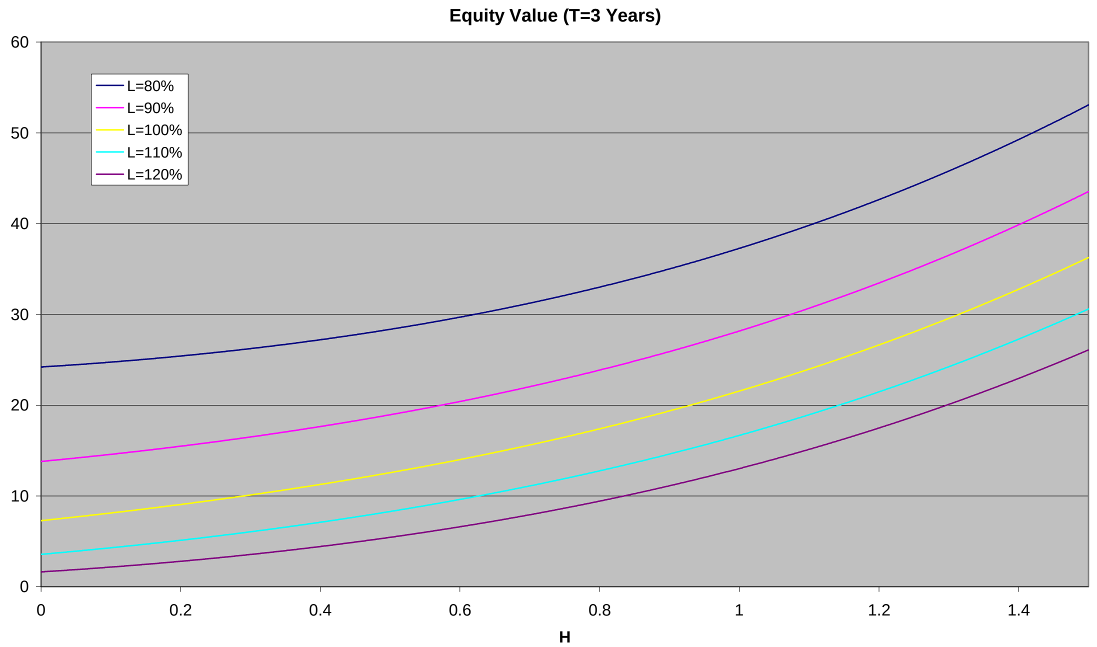

Figure 19. Firm Equity Sensitivity to Long Memory (T<1)

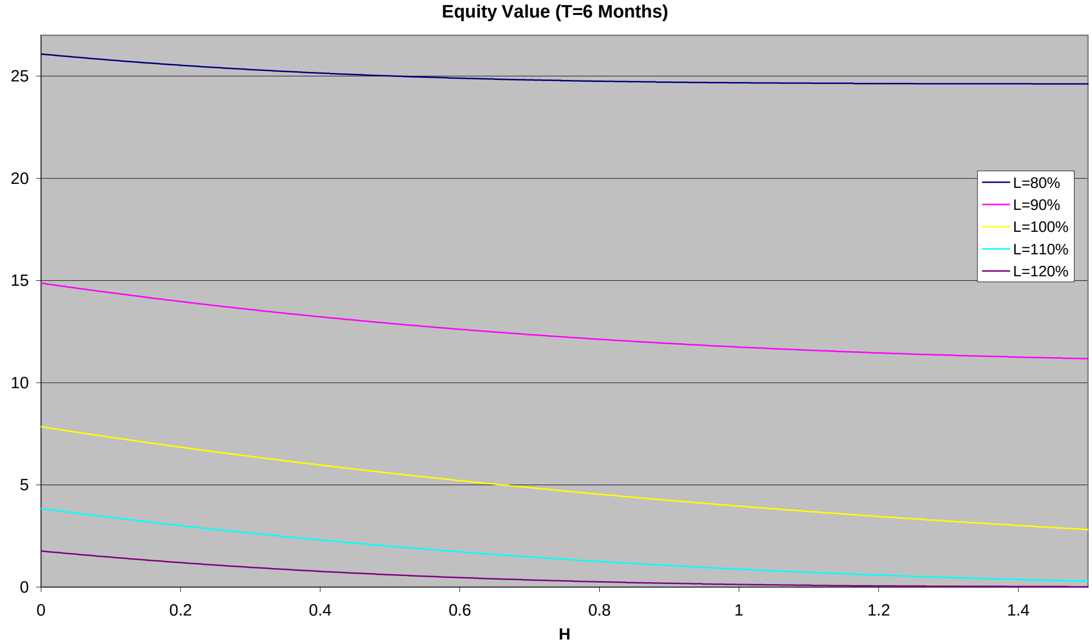

Figure 20. Firm Debt Sensitivity to Long Memory (T>1)

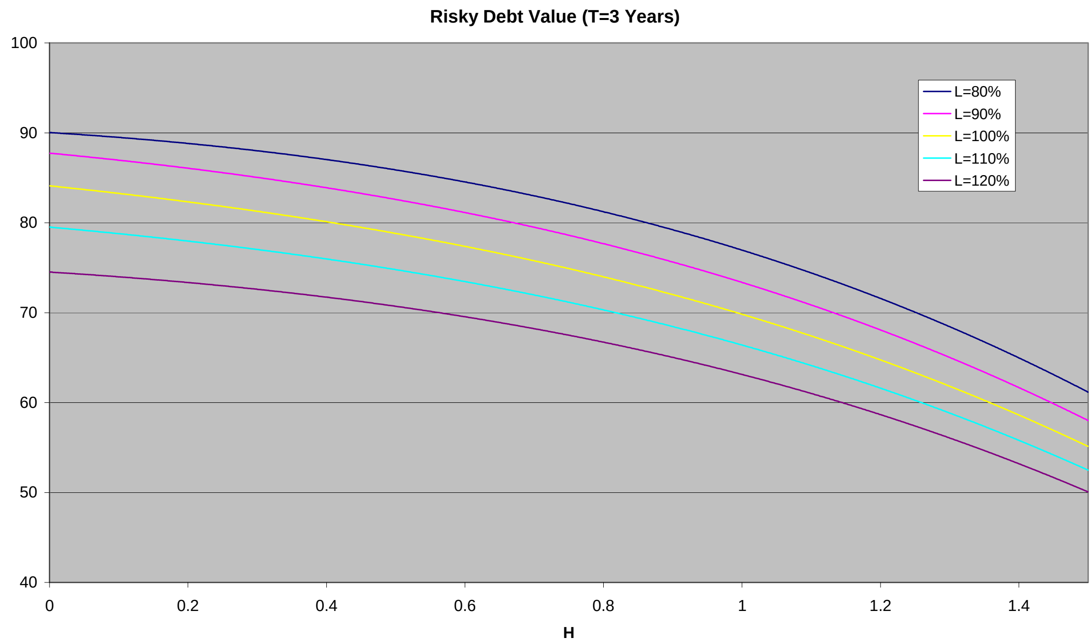

Figure 21. Firm Debt Sensitivity to Long Memory (T<1)

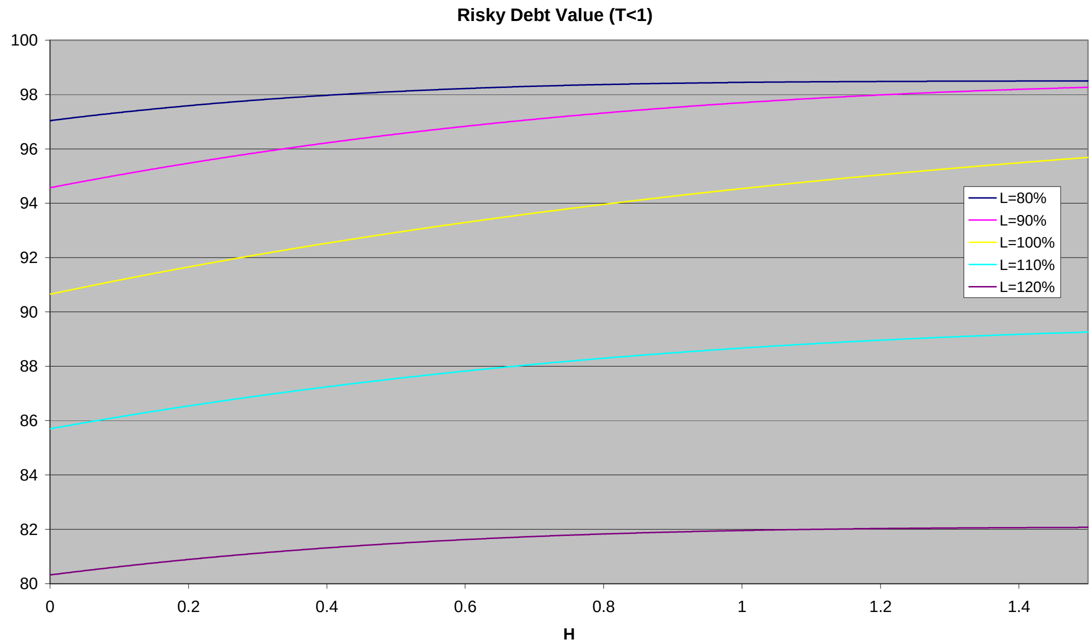

Figure 22. Default Probability Sensitivity to Long Memory (T>1)

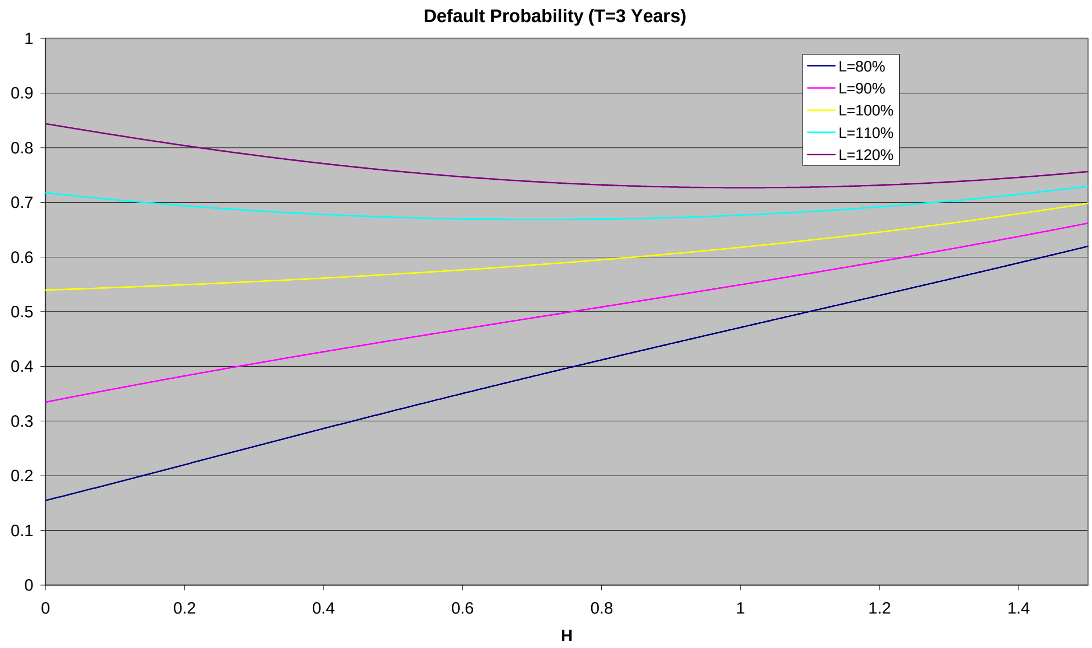

Figure 23. Default Probability Sensitivity to Long Memory (T<1)

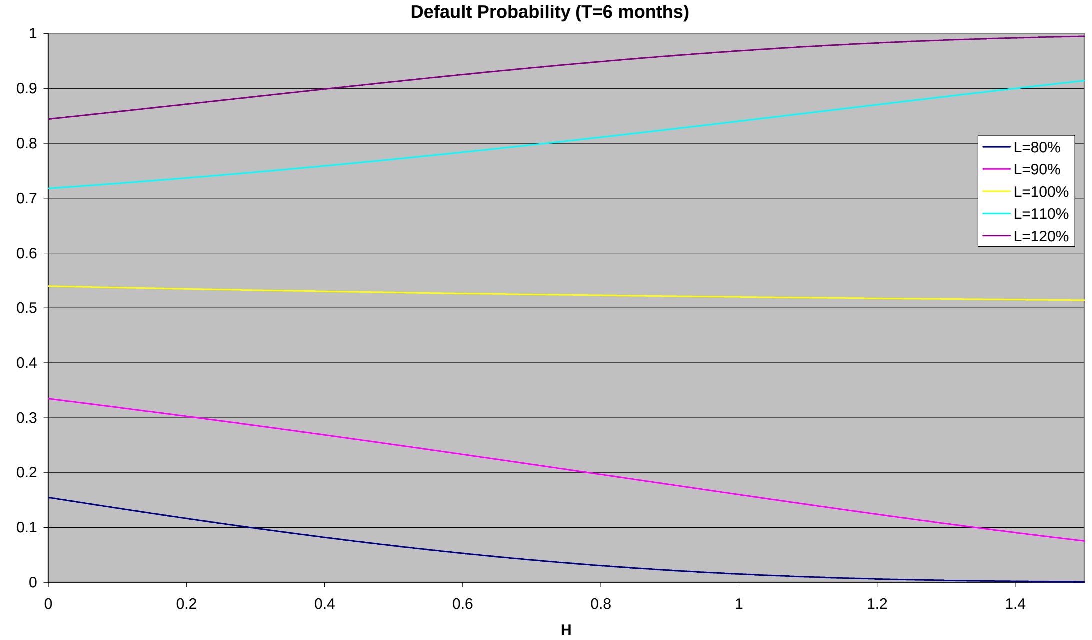

Figure 24. Sensitivity of the Option to Default to Long Memory (T>1)

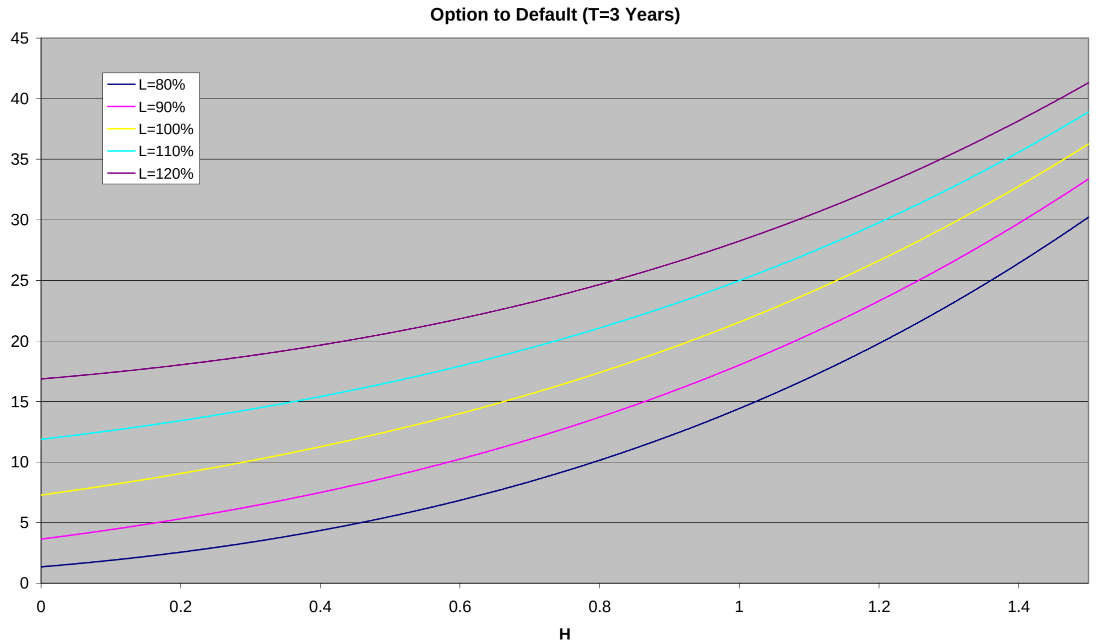

Figure 25. Sensitivity of the Option to Default to Long Memory (T<1)