# Basis risk in static versus dynamic longevity-risk hedging

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**Longevity 11 Conference**
**7-9 September 2015, Lyon**

# Contents

# Motivation

How can Insurance Companies hedge their exposure to *Longevity Risk*?

1. **Static hedging** - using customized, Over-The-Counter, Derivatives written on the actual Portfolio Population.
   - Pros: Perfect Hedge, no need for readjustments
   - Cons: Informational asymmetry
2. **Dynamic hedging** - using Standardized, traded, products written on a Reference Population.
   - Pros: Less Opacity, easier valuation for both counterparts
   - Cons: Non-perfect hedge, requires readjustment over time, basis risk may arise.

# Aim

- How do Static and Dynamic hedging strategies for Longevity risk compare with each other in terms of cost/efficiency?
  1. When hedging error is due to the discrete rebalancing frequency only.
  2. When also basis risk is present.

1. Provide a parsimonious model for basis risk;
2. Evaluate the cost of a dynamic hedging strategy to derive the "acceptable" cost of a static hedge: what would be the cost of the static hedge equivalent to a certain percentile of the hedging error of the dynamic hedging strategy?

# Mortality Intensity/ Reference Population

We model the mortality intensity of a specific generation $x$ belonging to the *Reference Population*, under the risk-neutral measure $\mathbb{Q}$, as a non-mean reverting CIR process:

$$d\lambda_x^{rp}(t) = (a + b\lambda_x^{rp}(t))dt + \sigma\sqrt{\lambda_x^{rp}(t)}dW_x(t), \qquad (1)$$

with $a > 0, b > 0, \sigma > 0,\ \lambda_x(0) = \lambda_0 \in \mathbb{R}^{++}$.

**Properties:**

- If $a \geq \frac{\sigma^2}{2}$, then $\lambda^{rp}(t) > 0$ for every $t \geq 0$.
- Denoting $\tau$ the time to death, then we have a closed form expression for $S_x^{rp}(t,T) = \mathbb{P}\left(\tau \geqslant T \mid \tau > t\right)$.

# Survival Probability/ Reference Population

The conditional Survival Probability at time $t$ for the horizon $T \geq t$ is given by:

$$S_x^{rp}(t,T) = A^{rp}(t,T)e^{-B^{rp}(t,T)\lambda_x^{rp}(t)}, \quad (2)$$

where $A^{rp}(t,T)$ and $B^{rp}(t,T)$ are solutions of an appropriate system of Riccati equations.

$$A^{rp}(t,T) = \left( \frac{2\gamma e^{\frac{1}{2}(\gamma-b)(T-t)}}{(\gamma-b)\left(e^{\gamma(T-t)}-1\right)+2\gamma} \right)^{\frac{2a}{\sigma^2}}, \quad (3)$$

$$B^{rp}(t,T) = \frac{2\left(e^{\gamma(T-t)}-1\right)}{(\gamma-b)\left(e^{\gamma(T-t)}-1\right)+2\gamma}, \quad (4)$$

with $\gamma = \sqrt{b^2+2\sigma^2}$.

# Survival Probability/ Reference Population

**Definition:**

We define the **Longevity Risk Factor** $I(t)$ as the difference between the actual intensity at time $t$ and its forecast made at time 0 (the time-0 forward rate):

$$I(t) = \lambda_x^{rp}(t) - f_x^{rp}(0,t) \tag{5}$$

It turns out that:

$$S_x^{rp}(t,T) = e^{-X^{rp}(t,T)I(t)+Y^{rp}(t,T)}, \tag{6}$$

where

$$\begin{aligned} X^{rp}(t,T) &= B^{rp}(t,T), \\ Y^{rp}(t,T) &= lnA^{rp}(t,T) - B^{rp}(t,T)f_x^{rp}(0,t). \end{aligned}$$

# Mortality Intensity/ Portfolio Population

We assume that the mortality intensity of generation $x$ belonging to the *Portfolio population* follows a **2-factor CIR process** given by:

$$\lambda_x^{pp}(t) = \underbrace{\delta_x \, \lambda_x^{rp}(t)}_{\textbf{Common Factor}} + \underbrace{(1-\delta_x)\,\lambda_x^{'}(t)}_{\textbf{Idiosyncratic Factor}}, \tag{7}$$

with

$$d\lambda_x'(t) = (a' + b'\lambda_x'(t))dt + \sigma'\sqrt{\lambda_x'(t)}dW_x'(t). \tag{8}$$

- $a' > 0$, $\sigma' > 0$, $b' \in \mathbb{R}$, with $a' \geq \frac{(\sigma')^2}{2}$,
- $W_x$ and $W_x^{'}$ are **independent** standard Brownian,
- $0 \leq \delta_x \leq 1$.

# Basis Risk

The weight of the idiosyncratic component $(1 - \delta_x)$ can be interpreted as a measure for Basis Risk:

$$\begin{aligned} \textbf{Reference Population:} \quad & & \lambda_x^{rp}(t) \\ \textbf{Portfolio Population:} \quad & \lambda_x^{pp}(t) = & \delta_x \lambda_x^{rp}(t) + (1 - \delta_x)\lambda_x'(t) \end{aligned}$$

$$Corr_u\left[\lambda_x^{pp}(t), \lambda_x^{rp}(t)\right] = \delta_x \sqrt{\frac{Var_u\left(\lambda_x^{rp}(t)\right)}{Var_u\left(\lambda_x^{pp}(t)\right)}} \quad \in [0,1],$$

1. If $\delta_x = 1 \Rightarrow$ no Basis Risk $\Rightarrow$ Benchmark Case
2. If $0 < \delta_x < 1 \Rightarrow$ Basis Risk $\Rightarrow$ partial coverage possible
3. If $\delta_x = 0 \Rightarrow$ Basis Risk $\Rightarrow$ no partial coverage possible

# Survival Probability/ Portfolio Population

As in the previous case, the conditional Survival Probability of the Portfolio Population can be written in terms of the longevity risk factor $I(t)$:

$$S_x^{pp}(t,T) = e^{-X^{pp}(t,T)\delta_x I(t) - X'(t,T)(1-\delta_x)\lambda'_x(t) + Y^{pp}(t,T)} \qquad (9)$$

- Explicit dependence on $\delta_x$.
- $X^{pp}(t,T), X'(t,T), Y^{pp}(t,T)$ deterministic coefficients.

# Interest rate risk

For the sake of symmetry with the longevity case, we assume that the spot interest rate follows, under the risk-netrual measure, a mean-reverting CIR process of the type:

$$dr(t) = (\bar{a} - \bar{b}r(t))dt + \bar{\sigma}\sqrt{r(t)}d\bar{W}(t), \tag{10}$$

- $\bar{a} > 0, \bar{b} > 0, \bar{\sigma} > 0, r(0) = r_0 \in \mathbb{R}^{++}$,
- $\bar{W}$ independent of $W$ and $W'$.

We assume independence between the Financial and the Longevity markets.

# Interest rate risk

**Definition:**

We define the **Interest Rate Risk Factor** $J(t)$ as the difference between the short rate at time $t$ and the time $t$ forward rate at time 0:

$$J(t) = r(t) - f(0,t) \tag{11}$$

As in the longevity case, the value of a zero-coupon bond $D(t,T)$ can be expressed in terms of $J(t)$ as:

$$D(t,T) = e^{-\bar{X}(t,T)J(t)+\bar{Y}(t,T)}. \tag{12}$$

# The Insurance Portfolio

- We assume that the liabilities of the insurance company are represented by an Annuity contract, with maturity $T$ and annual instalments $R$ paid at year-end, written on an individual belonging to the ***Portfolio Population*** and aged $x$ at time $t = 0$.
- The value of the reserves for such contract at time $t$ is:

$$N^{pp}(t,T) = R\sum_{u=1}^{T-t} D(t,t+u)S_x^{pp}(t,t+u),$$

$$= R\sum_{u=1}^{T-t} e^{-\bar{X}(t,t+u)J(t)+\bar{Y}(t,t+u)} \cdot e^{-X^{pp}(t,t+u)\delta_x I(t)-X^{'}(t,t+u)(1-\delta_x)\lambda_x^{'}(t)+Y^{pp}(t,t+u)}.$$

# Static Hedging

To hedge the unexpected changes in longevity, an alternative for the insurance company is to buy an $S$-Swap or **Longevity Swap**.

**Definition:**

A **Longevity Swap** is a contract in which one party (the Insurer) agrees to pay, at a set of specified dates $T_i$ (e.g. once a year), a fixed amount $K(T_i)$ in exchange for the survivorship of a specific generation $x$ belonging to the Portfolio Population. The contract lasts until the last individual of the generation $x$ is dead (at tine $w$).

# Static Hedging

Under the assumption of no arbitrage and independence between the mortality and interest rate risk, the value at time $t = 0$ of such a contract, from the point of view of the Insurer, is given by:

$$LS(0) =$$

$$= \sum_{T=1}^{w} \mathbb{E}_0\Big[\exp\Big(-\int_0^T \lambda_x^{pp}(s)\,ds\Big) - K(T)\Big]\mathbb{E}_0\Big[\exp\Big(-\int_0^T r(u)\,du\Big)\Big],$$

$$= \sum_{T=1}^{w} \Big[S_x^{pp}(0,T) - K(T)\Big] D(0,T).$$

- $K(T)$ is called the *swap rate* for the period $(T-1, T)$,
- to ensure that the contract is fairly valued, i.e. it has zero value at inception, $\Rightarrow K(T) = S_x^{pp}(0,T)$.

# Static Hedging

The swap entails a cost $C_0$, which is distributed over different tenors at the rate $m$.

$$K'(T) = K(T)(1+m) = S_x^{pp}(0,T)(1+m), \quad (13)$$

where

$$m = \frac{C_0}{\sum_{T=1}^{\omega} S_x^{pp}(0,T)D(t,T)}. \quad (14)$$

# Dynamic Hedging

If the market of such products would exist, an alternative would be to hedge using traded mortality-linked contracts. We consider a dynamic **Delta-Gamma hedge** using synthetic longevity bonds. When Basis Risk is present,

- the market offers Longevity Bonds $M_i^{rp}$, written on generation $x$ of a Reference Population which is different from the Portfolio Population ($\delta_x \neq 1$),
- the payoff at maturity $T_i$ of the longevity bond $M_i^{rp}$ is

$$\exp\left( -\int_t^{T_i} \lambda_x^{rp}(s) ds \right),$$

- the value of $M_i^{rp}$ at any time $0 \leq t \leq T_i$ is

$$\begin{aligned} M_i^{rp}(t) &= D(t, T_i) S_x^{rp}(t, T_i), \\ &= e^{-\bar{X}(t,T_i)J(t)+\bar{Y}(t,T_i)} \cdot e^{-X^{rp}(t,T_i)I(t)+Y^{rp}(t,T_i)}. \end{aligned}$$

# Dynamic Hedging

- Cover, at any rebalancing date, only the changes in the fair value of the reserve (the liabilities) approximated at the first or second order using a portfolio of Longevity Bonds.
- Any gain or loss from the hedging revision is stored or charged in the *Bank Account*.
- Any payment due because of the annuity contract is also taken from the Bank Account.
- The Bank Account accrues or charges the short interest rate $r(t)$.

The absolute value of the Bank Account is the *Hedging Error* of the dynamic hedging strategy.

# Delta-Gamma Hedging with Basis Risk ($\delta_x \neq 1$)

$$N^{pp}(t,T) = R\sum_{u=1}^{T-t} e^{-X^{pp}(t,t+u)\delta_x I(t) + X'(t,t+u)(1-\delta_x)\lambda'_x(t) + Y^{pp}(t,t+u)} \cdot D(t,t+u)$$

$$M_i^{rp}(t) = e^{-X^{rp}(t,T_i) I(t) + Y^{rp}(t,T_i)} \cdot D(t,T_i)$$

A perfect hedge of longevity risk cannot be achieved, even with continuous-time trading.
In our set up, we can identify two sources of risk:

- $I(t)$, that is a *common* longevity risk factor affecting both the Portfolio Population and the Reference Population ⇒ Still Hedgeable.
- $\lambda'_x(t)$, that represents the source of risk that remains unhedgeable.

The Longevity Bonds $M_i^{rp}$ can then be used to perform a $\Delta$-$\Gamma$ hedge against the common longevity risk factor $I(t)$.

# First/Second order changes in the reserves

$$dN^{pp} = \frac{\partial N^{pp}}{\partial t} dt + \frac{\partial N^{pp}}{\partial I} dI + \frac{1}{2} \frac{\partial^2 N^{pp}}{\partial I^2} (dI)^2 + \frac{\partial N^{pp}}{\partial J} dJ + \frac{1}{2} \frac{\partial^2 N^{pp}}{\partial J^2} (dJ)^2 ,$$

where

$$\frac{\partial N^{pp}}{\partial I} = R \sum_{u=1}^{T-t} D(t,t+u) \Delta^M_{pp}(t,t+u),$$

$$\frac{\partial^2 N^{pp}}{\partial I^2} = R \sum_{u=1}^{T-t} D(t,t+u) \Gamma^M_{pp}(t,t+u),$$

$$\frac{\partial N^{pp}}{\partial J} = R \sum_{u=1}^{T-t} \Delta^F(t,t+u) S^{pp}(t,t+u),$$

$$\frac{\partial^2 N^{pp}}{\partial J^2} = R \sum_{u=1}^{T-t} \Gamma^F(t,t+u) S^{pp}(t,t+u),$$

$$\Delta^M_{pp}(t,T) := \frac{\partial S^{pp}(t,T)}{\partial I} = -X^{pp}(t,T)\delta_x S^{pp}(t,T),$$

$$\Gamma^M_{pp}(t,T) := \frac{\partial^2 S^{pp}(t,T)}{\partial I^2} = \left(X^{pp}(t,T)\delta_x\right)^2 S^{pp}(t,T),$$

$$\Delta^F(t,T) := \frac{\partial D(t,T)}{\partial J} = -\bar{X}(t,T) D(t,T) \leq 0,$$

$$\Gamma^F(t,T) := \frac{\partial^2 D(t,T)}{\partial J^2} = \bar{X}(t,T)^2 D(t,T) \geq 0.$$

# First/Second order changes in the Longevity Bond

$$dM_i^{rp} = \frac{\partial M_i^{rp}}{\partial t}dt + \frac{\partial M_i^{rp}}{\partial I}dI + \frac{1}{2}\frac{\partial^2 M_i^{rp}}{\partial I^2}(dI)^2 + \frac{\partial M_i^{rp}}{\partial J}dJ + \frac{1}{2}\frac{\partial^2 M_i^{rp}}{\partial J^2}(dJ)^2,$$

where

$$\frac{\partial M_i^{rp}}{\partial I} = D(t,T_i)\Delta_{rp}^M(t,T_i),$$

$$\frac{\partial^2 M_i^{rp}}{\partial I^2} = D(t,T_i)\Gamma_{rp}^M(t,T_i),$$

$$\frac{\partial M_i^{rp}}{\partial J} = \Delta^F(t,T_i)S_x^{rp}(t,T_i),$$

$$\frac{\partial^2 M_i^{rp}}{\partial J^2} = \Gamma^F(t,T_i)S_x^{rp}(t,T_i),$$

$$\Delta_{rp}^M(t,T_i) := \frac{\partial S_x^{rp}(t,T_i)}{\partial I} = -X^{rp}(t,T_i)S_x^{rp}(t,T_i)$$

$$\Gamma_{rp}^M(t,T_i) := \frac{\partial^2 S_x^{rp}(t,T_i)}{\partial I^2} = X^{rp}(t,T_i)^2 S_x^{rp}(t,T_i).$$

# Hedging Portfolio Composition

- To perform a Self Financing Delta-Gamma hedging strategy, we need at each point in time three Longevity Bonds $M_i^{rp}$, with different maturities $T_1, T_2, T_3$, which we keep constant along the life of the hedge.
- At each rebalancing date, the number of bonds $n_i$, $i = 1, 2, 3$, composing the hedging portfolio is determined solving the following system:

$$-n\frac{\partial N^{pp}(t)}{\partial I}dI + \sum_{i=1}^{3} n_i \frac{\partial M_i^{rp}(t)}{\partial I}dI = 0,$$

$$-n\frac{\partial^2 N^{pp}(t)}{\partial I^2}(dI)^2 + \sum_{i=1}^{3} n_i \frac{\partial^2 M_i^{rp}(t)}{\partial I^2}(dI)^2 = 0,$$

$$-nN^{pp} + \sum_{i=1}^{3} n_i M_i^{rp}(t) = 0.$$

# Delta-Gamma Hedging without Basis Risk ($\delta_x = 1$)

$$N(t,T) = R \sum_{u=1}^{T-t} e^{-X(t,t+u)\,I(t)+Y(t,t+u)} \cdot D(t,t+u)$$

$$M_i(t) = e^{-X(t,T_i)\,I(t)+Y(t,T_i)} \cdot D(t,T_i)$$

- The Reference and Portfolio Population are the same $\Rightarrow$ $\lambda_x^{pp} = \lambda_x^{rp}$.
- Greeks and hedging portfolio can be easily computed as in the previous case.
- Hedging error due only to the discrete rebalancing frequency.

# Observation

$$dN^{pp} = \boxed{\frac{\partial N^{pp}}{\partial t}dt} + \frac{\partial N^{pp}}{\partial I}dI + \frac{1}{2}\frac{\partial^2 N^{pp}}{\partial I^2}(dI)^2 + \frac{\partial N^{pp}}{\partial J}dJ + \frac{1}{2}\frac{\partial^2 N^{pp}}{\partial J^2}(dJ)^2$$

$$dM_i^{rp} = \boxed{\frac{\partial M_i^{rp}}{\partial t}dt} + \frac{\partial M_i^{rp}}{\partial I}dI + \frac{1}{2}\frac{\partial^2 M_i^{rp}}{\partial I^2}(dI)^2 + \frac{\partial M_i^{rp}}{\partial J}dJ + \frac{1}{2}\frac{\partial^2 M_i^{rp}}{\partial J^2}(dJ)^2$$

- The deterministic component of the Annuity and the Longevity Bonds could be easily edged by adding a risk-free zcb to the hedging portfolio.
- This means adding an equation to the previous system and changing the self financing condition but this has no effect on the distribution of the hedging error.

# Calibration

- The calibrations were performed using the data provided by the Human Mortality Database:
  - Reference Population: generation of UK males aged 65 on 31/12/2010.
  - Portfolio Population: generation of Scottish males aged 65 on 31/12/2010.
- We fit our models minimizing the Rooted Mean Squared Error (RMSE) between the model-implied and the observed survival probabilities.
- We performed separate calibrations for the case with and without basis risk:
  - Benchmark Case: single calibration of only the Reference Population's parameters.
  - Basis Risk Case: joint calibration of both the Reference and Portfolio population's parameters.

# Calibration

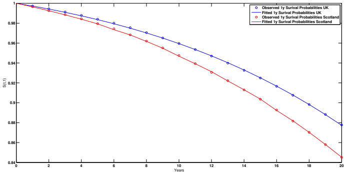


Figure: Observed and fitted survival probabilities for the Reference and the Portfolio Population.

- **Benchmark Case**: RMSE= 0.00006

| $a$ | $b$ | $\sigma$ |
|---|---|---|
| $4.13 \cdot 10^{-5}$ | 0.0709 | 0.0087 |

- **Basis Risk Case**: RMSE= 0.00015

| $a$ | $b$ | $\sigma$ | $\delta_x$ | $a^{'}$ | $b^{'}$ | $\sigma^{'}$ |
|---|---|---|---|---|---|---|
| $3.3357 \cdot 10^{-5}$ | 0.0727 | 0.0082 | 0.9897 | 0.0077 | 0.0155 | $4.4463 \cdot 10^{-08}$ |

# Simulations

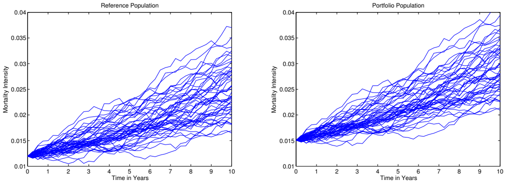


Figure: On the left-hand side, sample paths of the Reference Population intensity process $\lambda_x^{np}(t)$. On the right-hand side, sample paths of the Portfolio Population intensity process $\lambda_x^{pp}(t)$.

- Number of Simulations$= 100,000$
- Time Horizon$=$ 50 years
- Rebalancing frequency: $\Delta t$=3$m$, 6$m$, 1$y$
- Longevity Bonds Maturities: $T_i = 10, 15, 20$ years
- For each case and $\Delta t$, the hedging error is evaluated at $t = 30y$
- For each case and $\Delta t$, the equivalent cost $C_0$ of the static hedge is computed as the 99.5% VaR of the Bank Account at $t = 30y$

# Benchmark Case

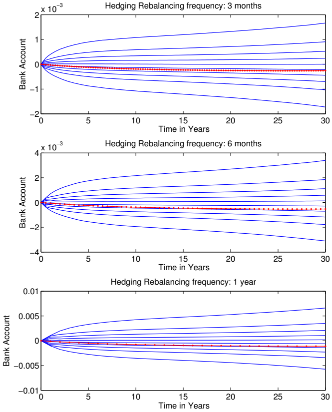


Figure: Simulated 0.05 to 0.95 percentiles of the Bank Account

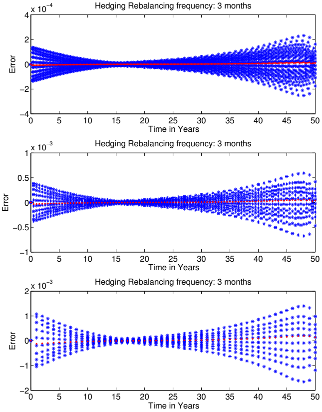


Figure: Simulated 0.05 to 0.95 percentiles of the tracking error.

# Benchmark Case

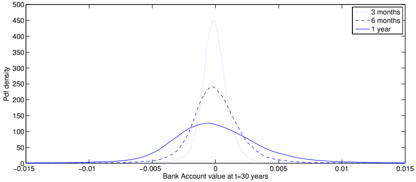


Figure: Distribution of the value of the Bank Account at t=30 years.

Table: Hedging Error's moments.

| | 3 months | 6 months | 1 year |
|---|---|---|---|
| Mean | 0.0008 | 0.0015 | 0.0030 |
| Det. part. | 0.0003 | 0.0005 | 0.0011 |
| Std. Dev. | 0.0007 | 0.0015 | 0.0034 |

Table: Longevity Swap premiums and loadings equivalent to the 99.5% VaR.

| | 3 months | 6 months | 1 year |
|---|---|---|---|
| $C_0$ | 0.0019 | 0.0037 | 0.0083 |
| $m$ | 0.01% | 0.02% | 0.05% |

# Basis Risk Case

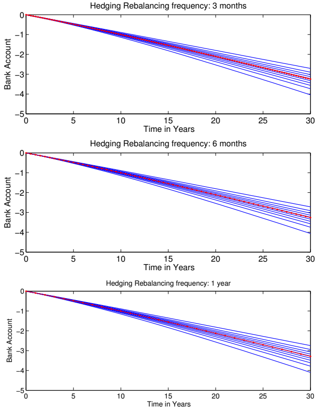


Figure: Simulated 0.05 to 0.95 percentiles with basis risk.

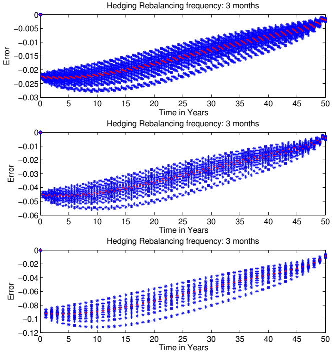


Figure: Simulated 0.05 to 0.95 percentiles of the tracking error with basis risk.

# Basis Risk Case

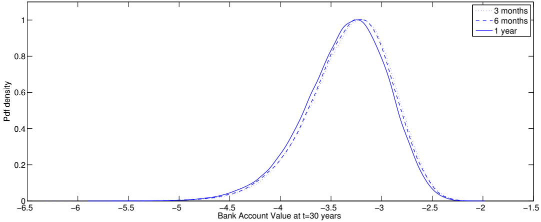


Figure: Distribution of the value of the Bank Account with basis risk.

Table: Hedging Error's moments with basis risk.

| | 3 months | 6 months | 1 year |
|---|---|---|---|
| Mean | 3.3132 | 3.3272 | 3.3593 |
| Det. part. | 3.2471 | 3.2611 | 3.2896 |
| Std. Dev. | 0.4099 | 0.4116 | 0.4146 |

Table: Longevity Swap premiums and loadings equivalent to the 99.5% VaR.

| | 3 months | 6 months | 1 year |
|---|---|---|---|
| $C_0$ | 0.7363 | 0.7409 | 0.7442 |
| $m$ | 4.28% | 4.30% | 4.32% |

# Conclusions

**If no Basis Risk:**

- The average hedging error of the dynamic hedge is moderate.
- The variance and the thickness of the tails of its distribution are decreasing with the rebalancing frequency.
- Equivalent spread over the basic "swap rate" between 0.01% and 0.05%.

**If Basis Risk:**

- Higher average hedging error.
- Basis Risk contributes to the hedging error much more than the error due to discrete-time rebalancing.
- Equivalent spread over the basic "swap rate" between 4.28% to 4.32%.