On Discrete-Time Greek Hedging of Longevity Risk

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Agenda

Standardization of Longevity Risk

- Standardization could resolve the misalignment of incentives between longevity hedgers and capital market investors.
- Capital markets could provide sufficient supply for acceptance of longevity risk.
- Capital market investors could enjoy diversification benefits and risk premiums.
- High liquidity and symmetric information could be achieved through standardized longevity-linked securities.

Greek Hedging for Longevity Risk

- Greeks measure the sensitivity of the value of a security to changes in certain underlying parameters on which the value depends.
- Longevity risk is embedded in mortality liabilities that depend on multiple ages over multiple years.
- Greek hedging for longevity requires a mortality model that combines the multiple underlying forces into a handful of period effects.
- Longevity Greeks are calculated based on the underlying period effects.

Greek Hedging in Longevity Literature

- Cairns (2011) developed a dynamic hedging strategy considering delta-only hedges.
- Luciano et al. (2012) developed a delta-gamma hedging strategy in the context of continuous-time modelling.
- Cairns (2013) developed the concept of nuga-hedging to mitigate the recalibration risk of period effects.

Our Objectives

1. The discrete-time Lee-Carter model with conditional heteroscedasticity (Chen et al., 2015; Wang et al., 2015).
2. Multiple Greeks hedging with the consideration of 'vega'.
3. Properties and patterns of longevity Greeks.
4. Suggestions on the selection of hedging instruments.
5. Empirical analysis with the England and Wale mortality data.

The Lee-Carter Model

$$\ln(m_{x,t}) = a_x + b_x k_t$$

where k_t is the period effect following a random-walk process

$$k_t = k_{t-1} + \theta + \epsilon_t.$$

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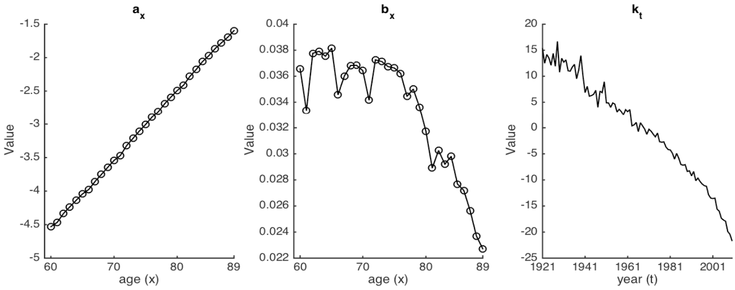

The GARCH Model

$$k_t = k_{t-1} + \theta + \epsilon_t$$

where ϵ_t is the error term following a GARCH process

$$\epsilon_t = \sqrt{h_t}\eta_t$$
$$h_t = \omega + \alpha\epsilon_{t-1}^2 + \beta h_{t-1}$$

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GARCH(1,1) Conditional Variance Model:
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Conditional Probability Distribution: Gaussian
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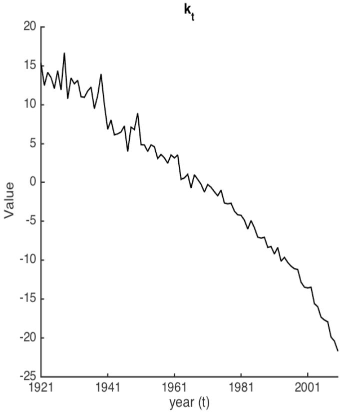
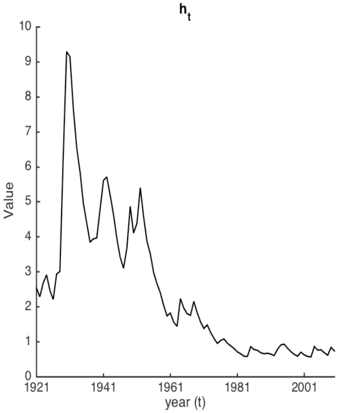
Parameter	Value	Standard Error	t Statistic
Constant	0.0369668	0.0522337	0.707719
GARCH{1}	0.834315	0.0728195	11.4573
ARCH{1}	0.135123	0.0632534	2.13621

The Conditional Volatility of the Period Effect





Survival Rate

Survival rate:

$$S_{x,t}(T) = e^{-\sum_{s=1}^{T} m_{x+s-1,t+s}} = e^{-\sum_{s=1}^{T} e^{a_{x+s-1}+b_{x+s-1}k_{t+s}}}$$
$$= e^{-\sum_{s=1}^{T} e^{Y_{x,t}(s)}} = e^{-W_{x,t}(T)}$$

where

$$k_{t+s} = k_t + s\theta + \sum_{i=1}^{s} \epsilon_{t+i}$$
$$\epsilon_{t+i} = \eta_{t+i}\sqrt{h_{t+i}}$$
$$h_{t+i} = \omega + \omega \sum_{v=1}^{i-1} \prod_{n=1}^{v} (\alpha\eta^2_{t+i-n} + \beta) + (\alpha\epsilon^2_t + \beta h_t) \prod_{n=1}^{i-1} (\alpha\eta^2_{t+i-n} + \beta).$$

Survival Probability

Survival probability:

$$\begin{aligned} p_{x,t}(T, k_t, h_t) :&= \mathrm{E}[S_{x,t}(T) \mid k_t, h_t] \\ &= \mathrm{E}\left[e^{-W_{x,t}(T)} \,\middle|\, k_t, h_t \right] \\ &= \mathrm{E}\left[e^{-\sum_{s=1}^{T} e^{Y_{x,t}(s)}} \,\middle|\, k_t, h_t \right]. \end{aligned}$$

Annuity Liability:

$$L = \sum_{s=1}^{\infty} (1+r)^{-s} p_{x,t}(s, k_t, h_t).$$

q-forwards:

$$Q = (1+r)^{-T^*} (p_{x_f, t+T^*-1}(1, k_t, h_t) - (1 - q^f)).$$

where r is the constant interest rate and q^f is the predetermined forward mortality rate.

Calculation of Delta

Annuity liability:

$$\Delta_{x,t}(T) = \frac{\partial}{\partial k_t} p_{x,t}(T, k_t, h_t) = -\sum_{s=1}^{T} b_{x+s-1}\, \mathrm{E}\left[e^{Y_{x,t}(s) - W_{x,t}(T)} \,\middle|\, k_t, h_t \right]$$

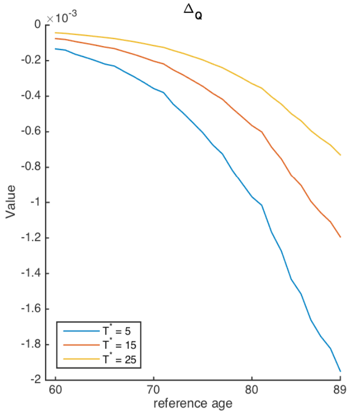
$$\Delta_L = \frac{\partial}{\partial k_t} \sum_{s=1}^{\infty} (1+r)^{-s} p_{x,t}(s, k_t, h_t) = \sum_{s=1}^{\infty} (1+r)^{-s} \Delta_{x,t}(s).$$

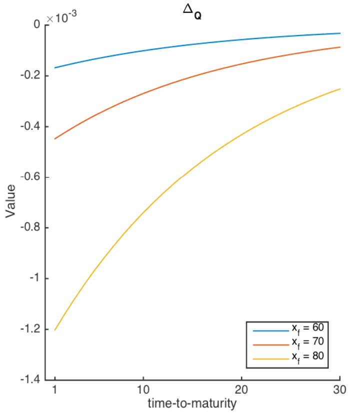
q-forwards:

$$\Delta_{x_f, t+T^*-1}(1) = \frac{\partial}{\partial k_t} p_{x_f, t+T^*-1}(1, k_t, h_t)$$

$$\Delta_Q = \frac{\partial}{\partial k_t} (1+r)^{-T^*} p_{x_f, t+T^*-1}(1, k_t, h_t) = (1+r)^{-T^*} \Delta_{x_f, t+T^*-1}(1).$$

Delta of q-forwards





Calculation of Gamma

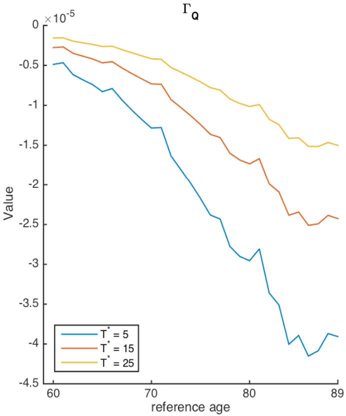
Annuity liability:

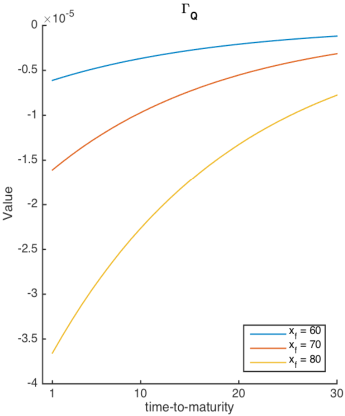
$$\begin{aligned}\Gamma_{x,t}(T) &= \frac{\partial^2 p_{x,t}(T, k_t, h_t)}{\partial k_t^2} \\ &= \mathrm{E}\left[e^{-W_{x,t}(T)} \left(\left(\sum_{s=1}^{T} b_{x+s-1} e^{Y_{x,t}(s)} \right)^2 - \sum_{s=1}^{T} b_{x+s-1}^2 e^{Y_{x,t}(s)} \right) \,\middle|\, k_t, h_t \right]\end{aligned}$$

q-forwards:

$$\Gamma_{x_f, t+T^*-1}(1) = b_{x_f}^2 \, \mathrm{E}\left[e^{Y_{x_f,t+T^*-1}(1) - e^{Y_{x_f,t+T^*-1}(1)}} \left(e^{Y_{x_f,t+T^*-1}(1)} - 1 \right) \,\middle|\, k_t, h_t \right]$$

Gamma of q-forwards





Calculation of Vega

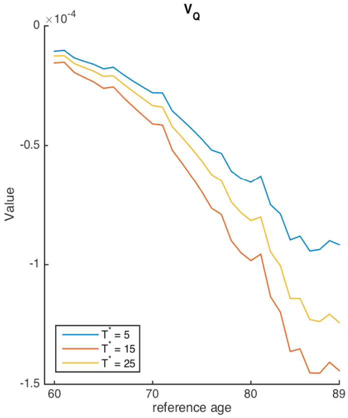
Annuity liability:

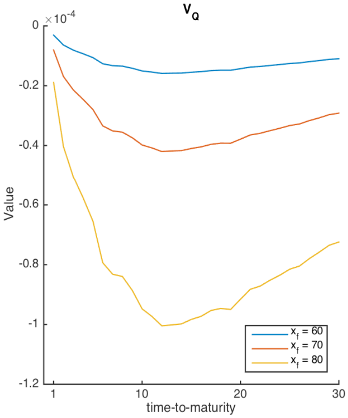
$$V_{x,t}(T) = \frac{\partial p_{x,t}(T, k_t, h_t)}{\partial h_t}$$
$$= -\sum_{s=1}^{T} b_{x+s-1}\,\mathrm{E}\left[e^{Y_{x,t}(s) - W_{x,t}(T)} \left(\sum_{i=1}^{s} \frac{\beta \epsilon_{t+i}}{2h_{t+i}} \prod_{n=1}^{i-1} (\alpha \eta^2_{t+i-n} + \beta) \right) \,\middle|\, k_t, h_t \right]$$

q-forwards:

$$V_{x_f, t+T^*-1}(1) = b_{x_f}\,\mathrm{E}\left[e^{Y_{x_f, t+T^*-1} - e^{Y_{x_f, t+T^*-1}}} \left(\sum_{i=1}^{T^*} \frac{\beta \epsilon_{t+i}}{2h_{t+i}} \prod_{n=1}^{i-1} (\alpha \eta^2_{t+i-n} + \beta) \right) \,\middle|\, k_t, h_t \right].$$

Vega of q-forwards





Hedge Ratios

Hedge ratios (u) are determined by matching the Greeks of the liabilities and q-forwards.

- Delta-only:

$$u_1 \Delta_{Q1} = \Delta_L$$

- Delta-Gamma:

$$\begin{bmatrix} \Delta_{Q1} & \Delta_{Q2} \\ \Gamma_{Q1} & \Gamma_{Q2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \Delta_L \\ \Gamma_L \end{bmatrix}$$

- Delta-Gamma-Vega:

$$\begin{bmatrix} \Delta_{Q1} & \Delta_{Q2} & \Delta_{Q3} \\ \Gamma_{Q1} & \Gamma_{Q2} & \Gamma_{Q3} \\ V_{Q1} & V_{Q2} & V_{Q3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \Delta_L \\ \Gamma_L \\ V_L \end{bmatrix}$$

Example

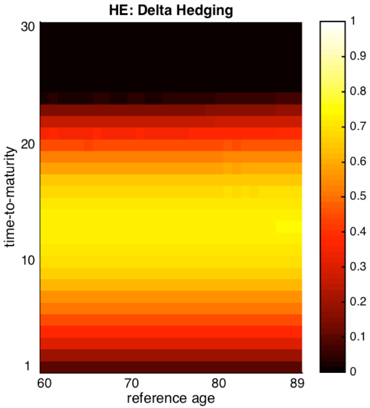
- A 30-year temporary annuity sold to a male individual aged 60 at time 0.

Δ_L	Γ_L	V_L
-0.067485	-0.0016493	-0.0078816

- q-forwards with reference age from age 60 to 89 and time-to-maturity from 1 to 30 years.
- Delta-only, vega-only, delta-gamma and delta-vega hedges are considered.
- Hedges are evaluated at time 30, where the hedge effectiveness is

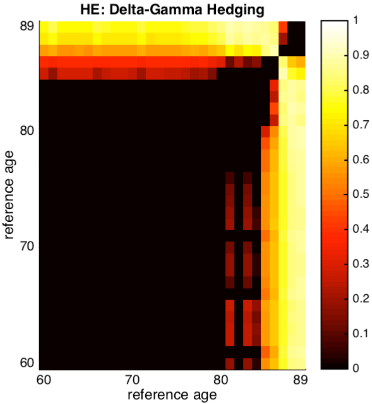
$$HE = 1 - \frac{\text{var}(PH - PL)}{\text{var}(PL)}$$

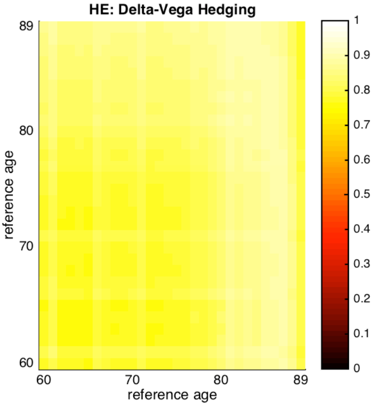
Single Greek Hedging



Heat maps for the hedge effectiveness of delta-only (left) and vega-only (right) hedges.

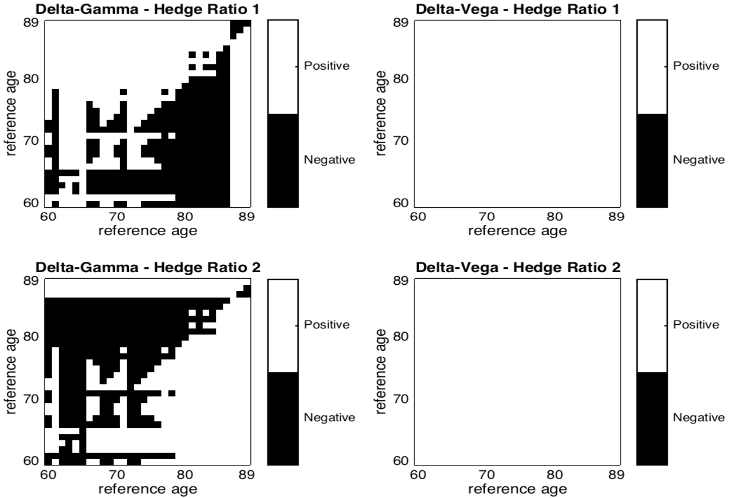
Multiple Greeks Hedging





Heat maps for the hedge effectiveness of delta-gamma (left) and delta-vega (right) hedges.

Multiple Greeks Hedge Ratios



Multiple Greeks Hedge Ratios

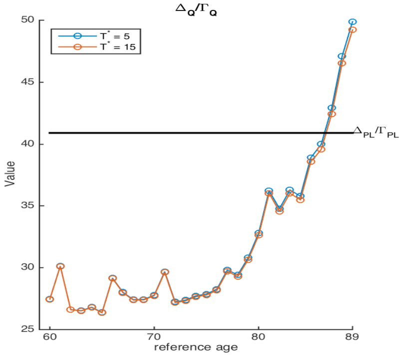
For multiple Greek hedging with q-forwards, hedge ratios need to be all positive. Recall that the hedge ratios are calculated as

$$\begin{bmatrix} \Delta_{Q1} & \Delta_{Q2} \\ \Gamma_{Q1} & \Gamma_{Q2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \Delta_L \\ \Gamma_L \end{bmatrix} \quad \Rightarrow \quad \begin{aligned} u_1 &= \frac{\Delta_L \Gamma_{Q2} - \Gamma_L \Delta_{Q2}}{\Delta_{Q1}\Gamma_{Q2} - \Delta_{Q2}\Gamma_{Q1}} \\ u_2 &= \frac{\Gamma_L \Delta_{Q1} - \Delta_L \Gamma_{Q1}}{\Delta_{Q1}\Gamma_{Q2} - \Delta_{Q2}\Gamma_{Q1}}. \end{aligned}$$

To have positive u_1 and u_2, we want the following inequality to hold:

$$\frac{\Delta_{Q1}}{\Gamma_{Q1}} > \frac{\Delta_L}{\Gamma_L} > \frac{\Delta_{Q2}}{\Gamma_{Q2}}.$$

Ratios of Delta, Gamma and Vega



Ratios of delta over gamma (left) and delta over vega (right) for q-forwards with different reference ages. The horizontal Bold line is the ratio for the liability.

Conclusion

- Greek hedging can be an useful hedging tool in mitigating longevity risk.
- The choice of reference ages and time-to-maturities of the hedging instrument used is important.
- Single Greek hedging is robust to the choice of reference ages, but not to the choice of time-to-maturities.
- Multiple Greeks hedging can further improve the hedge effectiveness, but careful calibrations are required.
- With limited choices of reference ages, delta-vega hedging is more robust than delta-gamma hedging.

Further Remarks

- The empirical results heavily depend on the assumptions underlying the Lee-Carter model.
- Other mortality models and their model estimates, especially non-parametric models, should be considered to verify the results.
- The robustness of Greek hedging relative to other risks such as population basis risk deserves further studies.
- Other benefits of using multiple Greeks such as reduction of left-tail risk should be further investigated.